A (very) brief introduction to
Special Relativity

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These notes are a Calculus II level introduction to the key
ideas of Einstein's special theory of relativity. In particular,
we will study Minkowski spacetime, compare it to Newto-
nian spacetime, and look at some interesting features of the
theory.

Newton's laws of motion and theory of gravity were devised to model the physical observations that could be made at the time ($c.\,1687$). His theory is a very good model for objects moving at low speeds over very small spacetime intervals. Thus it is a good model of physical reality on Earth and in the solar system, which is why it is still taught as "Elementary Physics" today.

It is now known that Newtonian physics breaks down when objects move at high speeds (close to the speed of light), or when we consider very large regions of spacetime. This is where the general theory of relativity takes over.

Nevertheless, we begin our study by considering Newtonian spacetime to develop our intuition and define some key terms that will carry through to special relativity. We will then list a few major problems with Newton's theory. Finally, we introduce Einstein's special relativity *via* Minkowski space, and study some of the main differences between it and Newton's theory.

1 Newtonian Spacetime

Newtonian space is three-dimensional real space $\mathbb{R}^3$ with the usual dot product to measure distances. If P and Q are two points in $\mathbb{R}^3$, and $\vec{u} = \vec{PQ}$, then the distance (squared) between them is $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$, usually written $\langle \vec{u}, \vec{u} \rangle$.

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In other words, if $\vec{u} = \langle x_1, x_2, x_3 \rangle$, then $\langle \vec{u}, \vec{u} \rangle = x_1^2 + x_2^2 + x_3^2$.

Example 1 Find the distance between the points $P(1, -2, 5)$ and $Q(7, 0, -1)$.

Time is represented by another 1-dimensional copy of $\mathbb{R}$. Orthogonally attaching space and time together (so that the time direction is perpendicular to all space directions), we obtain *Newtonian spacetime* $\mathcal{N} = \mathbb{R}^4$. The four-dimensional analog of the dot product (simply called an *inner product*) is defined as follows.

Definition 2 Let $P = (t, x_1, x_2, x_3)$ and $Q = (t', x_1', x_2', x_3')$ be points in $\mathcal{N}$. Then the inner product is given

$$\langle \vec{u}, \vec{u} \rangle = (t - t')^2 + (x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2$$

where $\vec{u} = \vec{PQ}$.

Only time *differences* $s - t$ are significant, not the actual numerical values s and t assigned to certain moments in time. If we consider ourselves as an observer, we usually put ourselves at the origin $(0, 0, 0, 0)$. Then the *past* is all points with negative t coordinates, and the *future* is all points with positive t coordinates.

Example 3 Find the Newtonian spacetime distance between $P(0, 1, 4, 6)$ and $Q(3, 1, 4, 8)$.

This idea of Newtonian "spacetime distance" makes perfect mathematical sense, but it doesn't make much sense physically. Time and distance are measured in different units, so the units of spacetime distance must be some combination of length and time. This is not the case in special relativity, as we will see. In particular, there are *geometric units* (see the project on geometric units).

We now define some terms that we will carry through to special relativity. We write $\mathcal{N} = \mathcal{T} \oplus \mathcal{S}$, where $\mathcal{T}$ is the one-dimensional time direction and $\mathcal{S}$ is the 3-dimensional space directions. We call $\mathcal{T}$ the *universal Newtonian clock* of the spacetime $\mathcal{N}$.

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Definition 4 A *Newtonian particle* is a parametrized curve $\alpha : I \to \mathcal{S}$, where $I = [a, b]$ is a Newtonian time interval. We can write the particle in vector form as

$$\alpha(t) = \langle x_1(t), x_2(t), x_3(t) \rangle.$$

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Definition 5 A *vector field* along a curve α is an assignment of a vector to every point of α.

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The velocity of α at a time t is $\dot{\alpha}(t) = \langle \dot{x}_1(t), \dot{x}_2(t), \dot{x}_3(t) \rangle$. As t varies, $\dot{\alpha}$ determines a *velocity vector* at every point of α, called the *velocity field* of α along α. The *speed* of a particle is defined to be $v(t) = \|\dot{\alpha}(t)\|$. Similarly, $\ddot{a}(t) = \langle \ddot{x}_1(t), \ddot{x}_2(t), \ddot{x}_3(t) \rangle$ is the *acceleration* of α at a time t, and determines a vector field along α, called the *acceleration field* of α along α.

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Example 6 Show that straight lines have zero acceleration in $\mathcal{S}$. Particles with zero acceleration are said to be *freely falling* or *in free fall*.

The *length* of a particle curve is just as we learned in Chapter 9, but with an added dimension. The formula is

$$L(\alpha \mid [a,b]) = \int_a^b \sqrt{(\dot{x}_1)^2 + (\dot{x}_2)^2 + (\dot{x}_3)^2}\, dt.$$

The length of a particle α represents the distance that the particle travels in $\mathcal{S}$ over the time interval $I \subseteq \mathcal{T}$.

Example 7 Find the length of the particle $\alpha(t) = \langle \sin t, \sqrt{8}\, t, 1 - \cos t \rangle$ from 0 to π.

Definition 8 Let $\alpha : I \to \mathcal{S}$ be a Newtonian particle with mass $m = m(t)$. Then

1. The *momentum* of α is the vector field $m\dot{\alpha}$ along c; *scalar momentum* is the function $m\|\dot{\alpha}(t)\|$ on I.

2. The *force* on α is the vector field $\frac{d}{dt}(m\dot{\alpha})$ along α.

3. The *kinetic energy* of α is the function $\frac{1}{2}mv^2$ on I, where $v = \|\dot{\alpha}\|$.

Example 9 If $m(t) = m$ is constant, deduce that number 2 above is Newton's 2nd Law: $F = ma$.

Definition 10 A point in (t, x_1, x_2, x_3) in $\mathcal{N} = \mathcal{T} \oplus \mathcal{S}$ is called an *event*. A *worldline* is the graph of a particle $\alpha : I \to \mathcal{S}$ in $\mathcal{N}$. That is, a worldline W is a collection of events $W = \{(t, x_1(t), x_2(t), x_3(t)) \mid t \in I\}$ such that for every t there is exactly one corresponding $(x_1, x_2, x_3) \in \mathcal{S}$.

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Given a worldline W, the associated particle α is given by the projection map $(t, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)$ of $\mathcal{N}$ onto $\mathcal{S}$.

As mentioned in the introduction, Newtonian spacetime is a simple and intuitive model of the (local) universe that we live in. It gives accurate results for applications on Earth, and provides an accurate model of the solar system. However, the theory gives some disconcerting results that were known before Einstein developed his theory of relativity.

One problem that we will not address in these notes is that "Newtonian relativity" does not agree with Maxwell's theory of electrodynamics. Maxwell's theory was shown to agree with experiments more accurately than Newton's. Therefore Newtonian mechanics are not correct for this context.

Here are some other problems that require less background knowledge in physics. These are due to O'Neill [1].

1. The speed of light c plays no role in Newtonian physics. In particular, it is theoretically possible to build a spaceship that could travel from one point in the universe to any other point in an arbitrarily small amount of time; *e.g.,* from Wichita to the sun in 5 seconds. We know that it takes light from the sun approximately 8 minutes to reach the earth, and we have yet to observe anything moving at a speed faster than the speed of light, so this should be impossible.

2. Newtonian physics also fails to model another property of light: that its speed is constant to all observers, no matter how fast one is moving toward or away from it.

   For example, suppose a spaceship is moving toward you, a fixed observer, at a constant speed of $c/10$. Both you and the spaceship shine lights at one another. According to Newtonian vector addition, the light from the space ship should be traveling at a speed of $c + c/10$ according to the observer. Similarly, the light from the observer should be traveling at a speed of $c - c/10$ according to the spaceship. However, observations of the light from stars in our night sky indicate that the speed of light should always be constant c.

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3. In Newtonian physics, a particle is either at rest or it is not. However, if a spaceship is far off in space, away from any outside influences, and has zero acceleration, then it is impossible to tell if the spaceship is actually at rest or not. One can only determine whether or not the spaceship is at rest *relative* to other objects (like the earth, sun, moon, stars, *etc.*).

We can sum up all of these problems by saying that Newtonian physics treats light *relatively* when it should be treated *absolutely*, and treats motion *absolutely* when it should be treated *relatively*.

2   Minkowski Spacetime

To remedy problem 1, consider a fixed event $(t_0, \alpha(t_0))$ of any particle α in $\mathcal{N}$. The tangent line to the world line W of α makes an angle θ with the time axis $\mathcal{T}$. The speed of α can be related to θ.

In particular, any particle moving at the speed of light forms an angle of $\pi/4$ with the time axis. The tangent directions of light then form a cone about $\mathcal{T}$ in $\mathcal{T} \oplus \mathcal{S}$. To ensure that no particle travels faster than the speed of light, we thus require that the worldline of every material particle (see *infra*) stays within this *light cone*.

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The light cone arises naturally by making a simple change to our Newtonian space time.

Definition 11 *Minkowski spacetime* is the space $\mathcal{M} = \mathcal{T} \oplus \mathcal{S}$ together with a Minkowski inner product (*metric*) defined by

$$\langle \vec{u}, \vec{u} \rangle = -t^2 + x_1^2 + x_2^2 + x_3^2,$$

where $\vec{u} = \langle t, x_1, x_2, x_3 \rangle$.

Just like Newtonian spacetime, Minkowski spacetime looks like four-dimensional space $\mathbb{R}^4 = \mathbb{V}^4$ as a vector space. However, this simple change in the inner product has a profound effect on the geometry of the space.

Remark 12 The Minkowski inner product is traditionally written as

$$\langle \vec{u}, \vec{u} \rangle = -c^2\, t^2 + x_1^2 + x_2^2 + x_3^2,$$

where $\vec{u} = \langle t, x_1, x_2, x_3 \rangle$ and c is the speed of light. We will always use geometric units (see the project), so we set $c = 1$. Thus our definition agrees with this traditional one.

Example 13 Consider the equation $-t^2 + x_1^2 + x_2^2 + x_3^2 = 1$. Solutions $\vec{u} = (t, x_1, x_2, x_3)$ of this equation are the *unit vectors* of $\mathcal{M}$. Find the traces of this equation and determine that the "unit sphere" of $\mathcal{M}$ is actually two hyperboloids (one of one sheet, and one of two sheets) that are asymptotic to the light cone. Sketch a graph.

It turns out that a reasonable attempt to recover Newtonian mechanics in this space also solves the other two problems outlined at the end of the previous section, thus producing special relativity.

There is a light cone at every point in $\mathcal{M}$. The nappe in the positive time direction is called the *future*, while the negative time-nappe is called the *past*. The particles of interest to us will be those whose worldlines have tangent vectors inside of the future time cone at every point. We call these particles *timelike future-pointing*.

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Unlike on $\mathcal{N}$, there is no canonical time function on $\mathcal{M}$. However, there are many times.

Definition 14 A *material particle* in $\mathcal{M}$ is a timelike future-pointing parametrized curve $\alpha : I \to \mathcal{M}$ such that $\|\dot{\alpha}(\tau)\| = 1$ for all $\tau \in I$. The parameter τ is called the *proper time* of the particle.

This means that every particle in $\mathcal{M}$ measures time differently. The proper times of two different particles can be compared *via* the Lorentz transformation, as we will see later.

Definition 15 A *lightlike particle* is a freely falling future-pointing curve $\alpha : I \to \mathcal{M}$ with $\|\dot{\alpha}(t)\| = 0$ for all $t \in I$. Recall that a particle is in free fall if and only if $\ddot{a}(t) = 0$ for all $t \in I$.

In contemporary physics there are three types of lightlike particles: photons (light itself), neutrinos (not perfectly lightlike), and "confidently conjectured" gravitons, which are outside of the special relativity framework [1].

As in the Newtonian case, we can assign mass to particles. The mass of a material particle is positive, and the mass of a lightlike particle is necessarily zero.

The *Fundamental Hypothesis* of special relativity is that light moves *geodesically*; *i.e.,* light is in free fall. Since $\langle \dot{\alpha}, \dot{\alpha} \rangle = 0$ for a lightlike particle, then parametrization by proper time is out of the question (this is the reason for using t rather than τ in definition 15). Being massless, light cannot carry a clock!

Definition 16 Let p and q be points in $\mathcal{M}$. The *separation* of p and q is $\mathscr{S}(p, q) := |\langle \vec{pq}, \vec{pq} \rangle| \geq 0$.

Example 17 Write a formula for $\mathscr{S}(p, q)$ is coordinates.

The separation of two points p and q in $\mathcal{M}$ has physical significance.

1. If $\vec{pq}$ is timelike and future-pointing, then there is a unique freely falling, timelike future-pointing curve σ connecting them. Then $\mathscr{S}(p, q) = L(\sigma)$ is the ellapsed proper time of the freely falling material particle σ from the event p to the event q.

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2. There is a lightlike particle through p and q if and only if $\mathscr{S}(p, q) = 0$ if and only if $\vec{pq}$ is lightlike.

3. If $\vec{pq}$ is *spacelike* (*i.e.,* outside the light cone), then $\mathscr{S}(p, q) \geq 0$ is the distance measured by any freely falling observer orthogonal to $\vec{pq}$.

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Lemma 18 *Let o, p, q be points in $\mathcal{M}$. If $\vec{op}$ is spacelike and $\vec{oq}$ is timelike, then any two of the following imply the third:*

1. *$\vec{pq}$ is lightlike;*

2. *$\vec{op}$ is orthogonal to $\vec{oq}$;*

3. *$\mathscr{S}(\vec{op}, \vec{op}) = \mathscr{S}(\vec{oq}, \vec{oq}) = 0$.*

<center>*Put a diagram here*</center>

One of Einstein's most important contributions to special relativity was his insistence that any physical equation must be *invariant of coordinates*. If two people go into two different rooms and calculate the same physical quantity, each choosing their own coordinates, then they should get the same answer regardless of their choice of coordinates.

<center>*Need some discussion and hypotheses here...*</center>

This leads us to the *Lorentz transformation*. It says that, if we choose two different reference frames E and E' with coordinates (t, x, y, z) and $(t', x', y'z')$, respectively, then these coordinates are related by the following equations

$$
\begin{aligned}
t' &= (t - vx)(1 - v^2)^{-1/2} \\
x' &= (x - vt)(1 - v^2)^{-1/2} \\
y' &= y \\
z' &= z.
\end{aligned}
$$

Remark 19 The factor

$$
\gamma := \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
$$

is called the *Lorentz factor*. We've left out the c^2 in our formulas since we are using geometric units. (Yet another reason why geometric units are awesome: they simplify equations.)

<center>*Final version will be on my web page, eventually...*</center>

References

[1] O'Neill, Barrett, *Semi-Riemannian Geometry: With Applications to Relativity*, New York: Academic Press, 1983.

[2] Stewart, James, *Essential Calculus: Early Transcendentals*, first edition, Belmont: Brooks/Cole, 2007.