Math 243: Calculus II
Exam 5: Ch 9

Mon, 19 Nov 2012

Name: KEY

Instructions: Complete all problems, showing all work. Simplify as necessary. Leave any answers involving π or irreducible square roots in terms of such (no rounded off decimals).

Do all 10 problems (no "omits" on this test).

1. Consider a circle centered at the Cartesian point $(4, 3)$ with radius $r = 5$.

7 **a.** Write parametric equations for the circle.

$$\begin{cases} x = 5\cos\theta + 4 \\ y = 5\sin\theta + 3 \end{cases}$$

3 **b.** Write a single polar equation for the circle.

$r = 8\cdot\cos\theta + 6\sin\theta$ since $\frac{\sqrt{8^2+6^2}}{2} = \frac{\sqrt{100}}{2} = \frac{10}{2} = 5$

2. Consider the parametric equations:

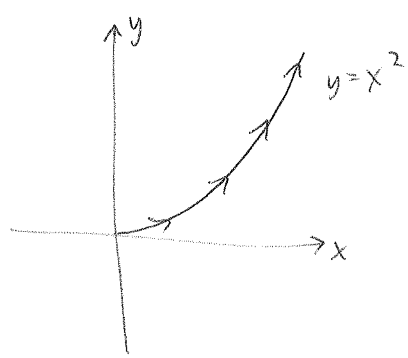
$$x = t^2,$$
$$y = t^4.$$

5 3

Eliminate the parameter to find a Cartesian equation of the curve, $y = f(x)$. What is the domain of f? Sketch the graph, with arrows to indicate the direction that the curve is traced as t increases. 2

$y = (t^2)^2 = x^2$

domain $= \{x \geq 0\}$



3. Eliminate the parameter to find a Cartesian equation of the curve:

5

$$x = \sin t,$$
$$y = \csc t,$$
$$0 < t < \pi/2.$$

3

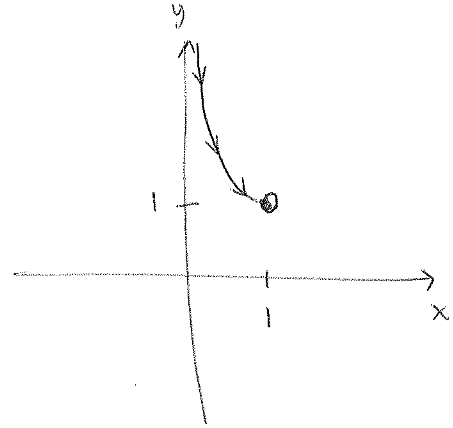
What is the domain of the curve? Sketch its graph, including arrows to indicate the direction the curve traces as t increases.

2

$y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$

$\sin(0) = 0$
$\sin(\pi/2) = 1$

domain $= x \in (0,1)$ or $\{0 < x < 1\}$



4. Find an equation of the tangent line to the parametric curve at the indicated value.

$$x = t^4 + 1, \quad y = t^3 + t, \quad t = -1$$

$m = \dfrac{dy}{dx} = \dfrac{\dot{y}(t)}{\dot{x}(t)} = \dfrac{3t^2+1}{4t^3} = \dfrac{3+1}{-4} = -1$ 5

$x(-1) = 2$
$y(-1) = -2$ 2

so line: $y = -1(x-2) - 2$ 3

$y = -x + 2 - 2$

$\boxed{y = -x}$

10 **5.** Find the Cartesian points on the curve where the tangent line is horizontal.

$$x = \cos 3\theta, \quad y = 2\sin\theta$$

$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2\cos\theta}{-\frac{1}{3}\sin 3\theta}$

So pts are:

$\boxed{(0,2) \text{ and } (0,-2)}$

horizontal when $\dot{y} = 0$, but $\dot{x} \neq 0$.

$2\cos\theta = 0$ at $\frac{\pi}{2}, \frac{3\pi}{2}$

$x(\pi/2) = \cos(3\pi/2) = 0$

$y(\pi/2) = 2(\sin(\pi/2)) = 2(1) = 2$

$x(\frac{3\pi}{2}) = \cos(\frac{9\pi}{2}) = 0$

$y(\frac{3\pi}{2}) = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2$

10 **6.** Find the length of the curve:

$$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \pi$$

$Lr(0,\pi) = \int_0^{\pi} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt = \int_0^{\pi} \sqrt{2e^{2t}}\, dt = \sqrt{2}\int_0^{\pi} e^t\, dt = \boxed{\sqrt{2}e^{\pi} - \sqrt{2}}$

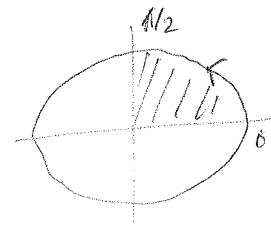
$\dot{x} = e^t\cos t - e^t \sin t$, $(\dot{x})^2 = (e^t)^2(\cos t - \sin t)^2 = e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t)$

$\dot{y} = e^t\cos t + e^t\sin t$, $(\dot{y})^2 = e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t)$

$(\dot{x})^2 + (\dot{y})^2 = e^{2t}(2(\cos^2 t + \sin^2 t)) = 2e^2 t$

10 **7.** Use the parametric equations of the ellipse to find the area that it encloses:

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$$



$$A = -4\int_0^{\pi/2} y\,\dot{x}\,d\theta = +4ab\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= +2ab\int_0^{\pi/2} 1-\cos 2\theta\,d\theta$$

$$= +2ab\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\tfrac{\pi}{2} - 0 - 0 + 0$$

$$= \boxed{\pi ab}$$

8. Identify the curves by finding Cartesian equations:

5 **a.** $r = 3\sin\theta$

$r^2 = 3r\sin\theta$

$x^2 + y^2 = 3y$

$x^2 + y^2 - 3y + \frac{9}{4} = 0 + \frac{9}{4}$

$\boxed{x^2 + (y - 3/2)^2 = 9/4}$

circle w/ radius $r = 3/2$, centered at $(0, 3/2)$.

5 **b.** $r = \csc\theta$

$r = \frac{1}{\sin\theta}$

$r\sin\theta = 1$

$\boxed{y = 1}$

horizontal line passing thru $y = 1$.

9. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

$\cancel{a}\sin\theta = \cancel{a}\cos\theta$

$\cancel{\frac{\sin\theta}{\cos\theta} = \frac{a}{a}}$

$\cancel{\tan\theta = 1}$

$\cancel{\theta = \pi/4}$

$\theta = \arcsin\left(\frac{r}{a}\right)$

$\theta = \arccos\left(\frac{r}{a}\right)$

$\arcsin\left(\frac{r}{a}\right) = \arccos\left(\frac{r}{a}\right)$

only if $\frac{r}{a} = \frac{\sqrt{2}}{2}$

10 Area of 1 leaf of 4-leaved rose:

$A = 2\int_0^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\, d\theta = \cdots = \pi/8$ Done a few times in notes.

10 **10.** Find the exact length of the curve $r = \theta^2$, $0 \le \theta \le 2\pi$.

$$L_r(0,2\pi) = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 2\theta\sqrt{\theta^2+4}\, d\theta$$

$u = \theta^2 + 4$ $\quad u(0) = 4$

$du = 2\theta\, d\theta$ $\quad u(2\pi) = 4(\pi^2+1)$

$$= \frac{1}{2}\int_4^{4(\pi^2+1)} \sqrt{u}\, du = \frac{1}{\cancel{2}}\frac{\cancel{2}}{3} u^{3/2}\Big|_4^{4(\pi^2+1)}$$

$$= \boxed{\frac{1}{3}\left[(4\pi^2+4)^{3/2} - 4^{3/2}\right]}$$