Math 243: Calculus II
Exam 4: Ch 8

Monday, 5 November 2012

Name: KEY

Instructions: Complete all problems, showing all work. Simplify as necessary. Leave any answers involving π or irreducible square roots in terms of such (no rounded off decimals).

You may select two problems to be considered as "Extra Credit." These problems will be worth 5 points each, while the rest of the problems are worth 10 points each. Circle or clearly mark the two problems that you choose.

1. Show that the series is divergent.

$$\sum_{n=0}^{\infty}\left(\frac{5n^2-2n+3}{(2n+1)^2}\right)$$

By test for divergence $\lim \frac{5n^2-2n+3}{(2n+1)^2} = \lim \frac{5n^2-2n+3}{4n^2+4n+1} = \frac{5}{4} \neq 0$

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \arctan(2n)$$

$\lim \arctan(2n) = \arctan(\lim 2n) = \lim_{x\to\infty} \arctan x = \boxed{\frac{\pi}{2}}$

$\boxed{\text{Convergent}}$

3. Determine whether the series is convergent or divergent. Justify your answer.

$$1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

$\boxed{\text{Converges by p-test}}$

4. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$$

$\left|\cos\left(\frac{n\pi}{3}\right)\right| \leq 1$

so $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n!}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n!}$ which is convergent $(= e-1)$

So $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{3}\right)}{n!}$ is $\boxed{\text{absolutely conv}}$

5. Is the 50^{th} partial sum S_{50} of the alternating series $\sum_{n=1}^{\infty}(-1)^n/n$ an over- or underestimate of the total sum? Explain.

$S_1 = -1$ is under-, so all odd will be under- and all even will be over-

Therefore S_{50} is an $\boxed{\text{over-estimate}}$

6. Prove that $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ whenever $|r| < 1$.

$$S_n = a + ar + ar^2 + \cdots + ar^n$$

$$rS_n = \quad ar + ar^2 + \cdots + ar^n + ar^{n+1}$$

$$S_n - rS_n = a - ar^{n+1}$$

$$\Rightarrow S_n(1-r) = a(1-r^{n+1})$$

$$\Rightarrow S_n = \frac{a(1-r^{n+1})}{1-r}$$

Then $S = \lim S_n = \lim \dfrac{a(1-r^{n+1})}{1-r}$

$= \dfrac{a}{1-r}$ when $|r| < 1$ $\quad$ //

7. Find a power series representation for the function and determine the radius of convergence.

$$f(x) = \ln(1 - 4x^2)$$

$f(x) = \int \frac{-8x}{1-4x^2}\,dx$

$\frac{-8x}{1-4x^2} = \cancel{-8x \text{ the}}$ Sum of geom. series w/ $a = (-8x)$, $r = 4x^2$

$= \sum_{n=0}^{\infty} (-8x)(4x^2)^n = \sum_{n=0}^{\infty} (-2)\, 4^{n+1}\, x^{2n+1}$

Then $f(x) = \ln(1-4x^2) = \int \sum_{n=0}^{\infty} (-2)\, 4^{n+1} x^{2n+1} = -2 \sum_{n=0}^{\infty} 4^{n+1} \frac{x^{2n+2}}{2n+2} + C$

but when $x = 0$ we see $C = 0$, and

$f(x) = \ln(1-4x^2) = \boxed{-2 \sum_{n=0}^{\infty} 4^{n+1} \frac{x^{2n+2}}{2n+2}}$

Since it came from geom. series. conv. when $|4x^2| < 1$ or $|x|^2 < \frac{1}{4}$ or $|x| < \frac{1}{2}$

so $\boxed{R = \frac{1}{2}}$

8. Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{t}{1 - t^8}\,dt$$

$a = t$ $\quad r = t^8$ $\quad ar^n = t(t^{8n}) = t^{8n+1}$

so $\int \frac{t}{1-t^8}\,dt = \sum_{n=0}^{\infty} \int t^{8n+1}\,dt = \boxed{\sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2} + C}$

$\boxed{R = 1}$ since it's geometric.

9. Find the Taylor series representation for $f(x) = \sin x$ centered at $a = \pi/2$.

$n=0 \quad f^{(0)}(x) = \sin x \qquad f^{(0)}(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$n=1 \quad f'(x) = \cos x \qquad f'(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$

$\vdots \quad -1$

0

so $\sin(x) = 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - \frac{1}{6!}(x-\frac{\pi}{2})^6 + \frac{1}{8!}(x-\pi/2)^8 - + \cdots$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}}$$

10. Find the radius of convergence and interval of convergence for the power series.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

$\left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right| = |x-3| \frac{n}{n+1} \longrightarrow |x-3|$ as $n \to \infty$

so $|x-3| < 1$ means $\boxed{R=1}$

when $x=2$, get $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST

when $x=4$, get $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test

So I.C. is $\boxed{[2,4)}$

11. If $f^{(n)}(0) = (n+1)!$ for all $n = 0, 1, 2, \ldots$, find (a simplified formula for) the MacLaurin series for f, and find its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} x^n = \sum_{n=0}^{\infty} \frac{(n+1)\cancel{n!}}{\cancel{n!}} x^n = \boxed{\sum_{n=0}^{\infty} (n+1)x^n}$$

$\boxed{R=1}$ since ~~[illegible]~~.

$$\left| \frac{n+2}{n+1} \frac{x^{n+1}}{x^n} \right| = |x| \frac{n+2}{n+1} \to |x| < 1$$

12. Find the 3rd Taylor polynomial $T_3(x)$ for $f(x) = \cos(x)$ centered at $a = \pi/4$.

$f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin x$, $f'(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos x$, $f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin x$, $f'''(\pi/4) = \frac{\sqrt{2}}{2}$

$$\boxed{T_3(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\tfrac{\pi}{4}) + \frac{\sqrt{2}}{4}(x-\tfrac{\pi}{4})^2 - \frac{\sqrt{2}}{12}(x-\tfrac{\pi}{4})^3}$$

b/c

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$$

Write the formula for the Taylor remainder $R_3(x)$ (for the same function), and find the maximum error for $T_3(x)$. ← if $x \in (0, \pi/2)$

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x-a)^4$$

$f^{(4)}(x) = \cos x$ so $f^{(4)}(z) = \cos z$

$$\boxed{R_3(x) = \frac{\cos(z)}{24}(x-\tfrac{\pi}{4})^4}$$

$\cos(z) \leq 1$ for all $z \in (0, \frac{\pi}{2})$

$|(x-\frac{\pi}{4})| \leq \frac{\pi}{4}$ for all $z \in (0, \pi/2)$

so $R_3(x) \leq \frac{1}{24} \cdot \left(\frac{\pi}{4}\right)^4 = \boxed{\frac{\pi^4}{6144}}$

≈ 0.0159