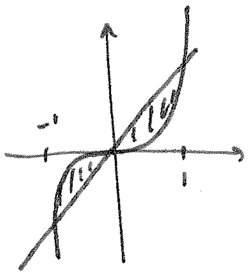
Math 243: Calculus II
Exam 3: Ch 7

Monday, 8 October 2012

Name: KEY

Instructions: Complete all problems, showing all work. Problems are graded not only on whether the answer is correct, but if the work leading up to the answer is correct. Simplify as necessary. Leave any answers involving π or irreducible square roots or logs in terms of such (no rounded off decimals).

1. Find the area of the region $\mathcal{R}$ bounded by the curves $y = x$ and $y = x^3$.



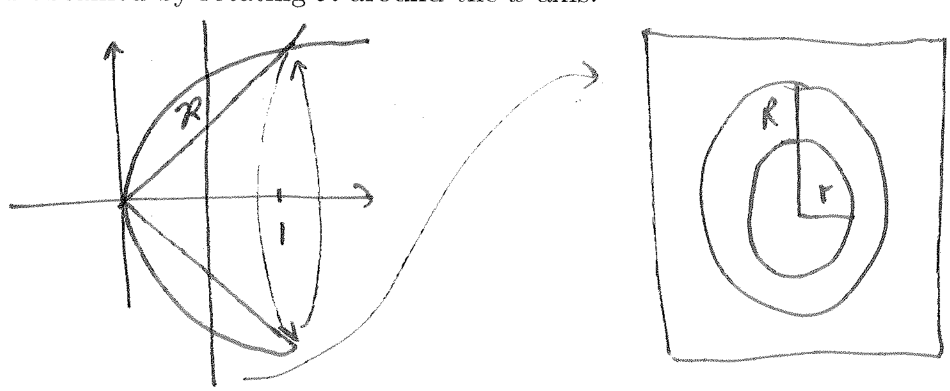
$$A = \int_{-1}^{0} x^3 - x \, dx + \int_0^1 x - x^3 \, dx$$

$$= 2\int_0^1 x - x^3 \, dx$$

$$= 2\left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right]_0^1$$

$$= 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right) = 2\left(\tfrac{1}{4}\right) = \boxed{\tfrac{1}{2}}$$

2. Let $\mathcal{R}$ be the region bounded by the curves $y = x$ and $y = \sqrt{x}$. Find the volume of the solid obtained by rotating $\mathcal{R}$ around the x-axis.



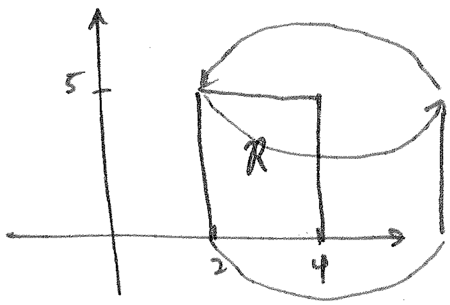
$$\left.\begin{array}{l} R = \sqrt{x} \\ r = x \end{array}\right\} \quad \begin{aligned} A &= \pi R^2 - \pi r^2 \\ &= \pi x - \pi x^2 \end{aligned}$$

$$V = \int A$$

$$V = \pi\int_0^1 x - x^2 \, dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \boxed{\tfrac{\pi}{6}}$$

3. The following integral represents the volume of a solid of revolution. Describe the solid, then find its volume (by any method).

$$\text{Volume} = 2\pi \int_2^4 5(4-x)\,dx$$



Cylinder centered at $x=4$
w/ radius 2 and height 5

$V = \pi r^2 h = \pi(2^2)(5) = \boxed{20\pi}$

4. You wish to use the trapezoidal rule with 10 "boxes" to estimate the area under the curve $y = e^{x^2}$ from $x = 0$ to $x = 1$. Find the maximum error $|E_{T_{10}}|$. (Simplify your answer as much as possible without using a calculator.)

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{6e(1)^3}{12(10^2)} = \frac{6e}{1200} = \boxed{\frac{e}{200}}$$

$a = 0$
$b = 1$
$n = 10$

K is upper bound on 2^{nd} deriv.

$y = e^{x^2}$
$y' = 2xe^{x^2}$
$y'' = 2e^{x^2} + 4x^2e^{x^2}$

so $K = y''(1) = 2e + 4e = 6e$

5. Write an integral to represent the arc length of the curve $y = \ln(x^2 - 1)$ from $x = \sqrt{2}$ to $x = e$. Do not evaluate it.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \boxed{\int_{\sqrt{2}}^{e} \sqrt{1 + \frac{4x^2}{(x^2-1)^2}}\, dx}$$

$a = \sqrt{2}$
$b = e$

or

$$\boxed{\int_{\sqrt{2}}^{e} \frac{x^2+1}{x^2-1}\, dx}$$

$$\frac{dy}{dx} = \frac{2x}{x^2-1}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{(x^2-1)^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{(x^2-1)^2 + 4x^2}{(x^2-1)^2} = \frac{x^4 - 2x^2 + 1 + 4x^2}{(x^2-1)^2} = \frac{x^4 + 2x^2 + 1}{(x^2-1)^2} = \frac{(x^2+1)^2}{(x^2-1)^2}$$

6. When a particle is located x meters from the origin, a force of $x^2 + 2x$ Newtons acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

$$W = \int_a^b f(x)\, dx = \int_1^3 x^2 + 2x\, dx = \left[\tfrac{1}{3}x^3 + x^2\right]_1^3$$

$a = 1$
$b = 3$
$f(x) = x^2 + 2x$

$$= 9 + 9 - \left(\tfrac{1}{3} + 1\right)$$

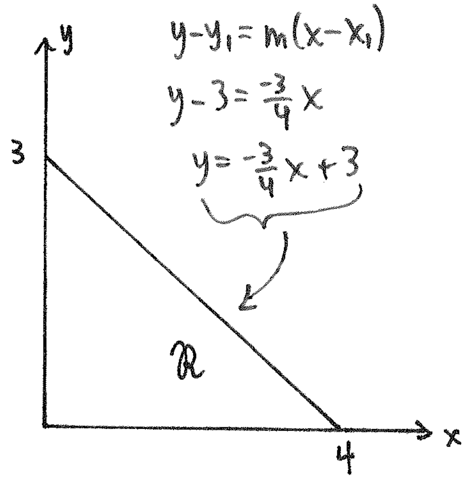
$$= 18 - \frac{4}{3} = \frac{54}{3} - \frac{4}{3} = \boxed{\frac{50}{3}}$$

7. Find the center of mass $C = (\overline{x}, \overline{y})$ of the region $\mathcal{R}$ in the figure.

$y - y_1 = m(x - x_1)$

$y - 3 = \frac{-3}{4}x$

$y = -\frac{3}{4}x + 3$



$$\overline{x} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad a=0,\ b=4,\ A=\tfrac{1}{2}bh=6$$

$$= \frac{1}{6}\int_0^4 -\frac{3}{4}x^2 + 3x\,dx = \frac{1}{6}\left[-\frac{1}{4}x^3 + \frac{3}{2}x^2\right]_0^4$$

$$= \frac{1}{6}\left[-16 + 24\right] = \frac{8}{6} = \frac{4}{3}$$

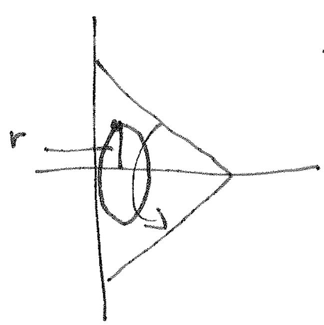
$$\overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx = \frac{1}{12}\int_0^4 \left(-\frac{3}{4}x + 3\right)^2 dx$$

$$= \frac{1}{12}\int_0^4 \frac{9}{16}x^2 - \frac{18}{4}x + 9\,dx$$

$$= \frac{1}{12}\left[\frac{3}{16}x^3 - \frac{9}{4}x^2 + 9x\right]_0^4 = \frac{1}{12}\left(12 - 36 + 36\right) = 1$$

So, $\boxed{C = (\overline{x}, \overline{y}) = \left(\frac{4}{3}, 1\right)}$

8. Use *Pappus's Theorem* to calculate the volume of the solid obtained by rotating the region $\mathcal{R}$ of the previous problem around the y-axis.

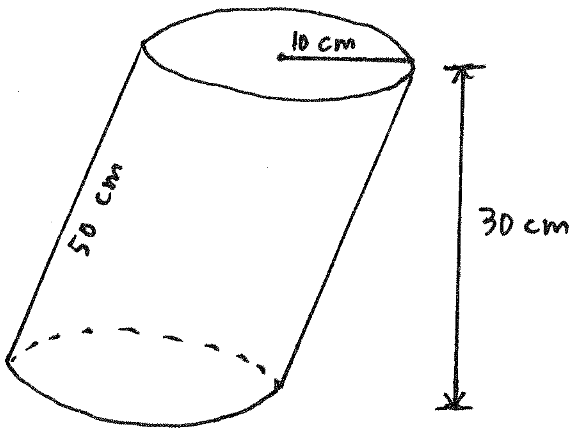


(Turn this picture around :)

Pappus: $V = A \cdot C \qquad C = 2\pi r,\ r = \overline{x},\ A = 6$

So $V = 2\pi(6)\left(\frac{4}{3}\right) = \boxed{16\pi}$

9. Use *Cavalieri's Principle* to calculate the volume of the solid.



Cavalieri: $V = A \cdot h$

$A = 2\pi r = 20\pi$

$h = 30$

So, $\boxed{V = 600\pi \text{ cm}^3}$

10. Solve the differential equation $(x^2+1)y' = xy$.

$$(x^2+1)\frac{dy}{dx} = xy$$

$$\int \frac{dy}{y} = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx \qquad u = x^2+1,\quad du = 2x\,dx$$

$$\ln y = \frac{1}{2}\ln(x^2+1) + C$$

$$\ln y = \ln\sqrt{x^2+1} + C$$

$$\boxed{y = C\sqrt{x^2+1}}$$

11. Find an equation of the curve that satisfies $\frac{dy}{dx} = 4x^3y$ and whose y-intercept is 7.

$$\frac{dy}{y} = 4x^3\,dx$$

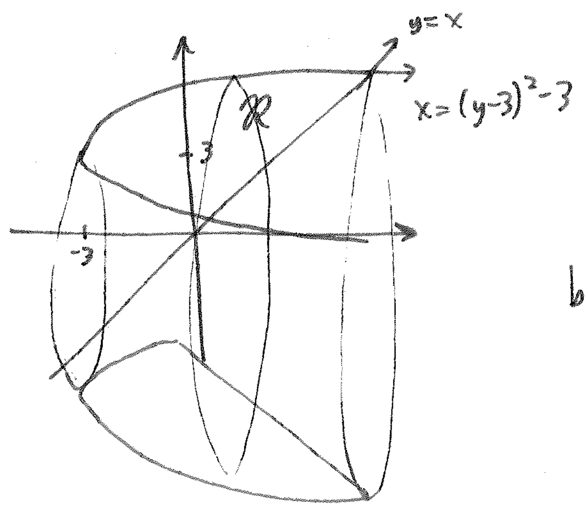
$$e^{\ln y} = e^{x^4 + C}$$

$$y = Ce^{x^4}$$

$$y(0) = 7 \Rightarrow C = 7$$

so, $\boxed{y = 7e^{x^4}}$

12. Let $\mathcal{R}$ be the region bounded by $x = (y-3)^2 - 3$ and $y = x$. Consider the solid obtained by rotating $\mathcal{R}$ about the x-axis. What method should you use to find the volume of the solid, and why? Write an integral that represents the volume, but don't evaluate it.



a.) Should use shells because a slice would be difficult to find the radii of.

b.) $y = (y-3)^2 - 3$

$$y + 3 = y^2 - 6y + 9$$

$$y^2 - 7y + 6 = 0$$

$$(y-6)(y-1) = 0$$

$$y = 6 \quad y = 1$$

so $\boxed{V = \int_1^6 2\pi y \left[y - (y-3)^2 + 3\right] dy}$