Calculus I: Chapter 3 Notes

Fall 2011

These are blank lecture notes for the week of 10 – 14 Oct 2011. They cover sections 3.3 through 3.6 of our textbook. These notes are intended to help us cover the material more quickly. You are still expected to pay attention, participate, and fill in any missing information in these notes.

There are 8 exercises scattered throughout these notes. You are expected to complete these on your own, outside of class. If you have questions, and you can prove that you've put forth *some* effort of your own, then I will offer some help. You will want to complete these exercises, as some of them will certainly be on your next exam.

Section 3.3: Derivatives of Logarithmic and Exponential Functions

We began our study of this section last Friday. Let's recall what we learned:

Theorem 1. *The function* $f(x) = \log_a x$ *is differentiable, and*

$$f'(x) = \frac{1}{x}\log_a e.$$

Proof.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h} \\
&= \lim_{h\to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\log_a\left(1+\frac{h}{x}\right) \\
&= \lim_{h\to 0} \frac{1}{x}\frac{x}{h}\log_a\left(1+\frac{h}{x}\right) \\
&= \frac{1}{x}\lim_{h\to 0}\log_a\left(1+\frac{h}{x}\right)^{x/h} \\
&= \frac{1}{x}\log_a\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/(h/x)}\right) \\
&= \frac{1}{x}\log_a\left(\lim_{u\to 0}(1+u)^{1/u}\right) \\
&= \frac{1}{x}\log_a e
\end{aligned}$$

□

J. Ryan: 8 Oct 2011

Remark 1. If $a = e$ we can use this theorem to obtain the formula for $(\ln x)'$. Since $\log_e e = \ln e = 1$, we have

$$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}}$$

Remark 2. We can rewrite the result of theorem 1 by utilizing the change of base formula.

$$\log_a e = \frac{\ln e}{\ln a} = \frac{1}{\ln a}, \quad \text{so}$$

$$\boxed{\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}}$$

Example 1. Find $\frac{dy}{dx}$ for $y = \ln\left(x^3 + 1\right)$.

Example 2. In general, if $u = u(x)$ and $y = \ln u$, then $\frac{dy}{dx}$ can be found by using the chain rule. We have:

$$\begin{aligned} f &= \ln u & \qquad f' &= \frac{1}{u} \\ u &= u & \qquad u' &= u' \end{aligned}$$

so $\frac{dy}{dx} = f'u' = \frac{1}{u}u' = \frac{u'}{u}$.

$$\boxed{\frac{d}{dx}[\ln u] = \frac{u'}{u}}$$

Example 3. Find $f'(x)$ if $f(x) = \ln|x|$.

Example 4. Find $\frac{dy}{dx}$ for $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$.

Exercise 1. When we introduced the power rule for derivatives, we only proved it for positive integers. Use the method of example 4 to prove the power rule for all $n \in \mathbb{R}$. That is, if $f(x) = x^n$, show that $f'(x) = nx^{n-1}$.

Next, we consider exponential functions $f(x) = a^x$ for $a > 0$, $a \neq 1$, and try to find their derivatives. We'll use the same method as example 4.

Example 5. Use the method of example 4 to calculate $\frac{d}{dx}[a^x]$.

Remark 3. Letting $a = e$ in the formula obtained in example 5, we have $(e^x)' = e^x \ln e = e^x$. We sketched a proof of this fact in section 3.1, so this should not be a shock to anyone. We also mentioned that $f(x) = e^x$ is the only function other than $f(x) = 0$ whose derivative is itself. Awesome!

Example 6. Find $f'(x)$ for $f(x) = e^{3x^2}$.

Remark 4. In general, if $u = u(x)$ and $f(x) = e^u$, then we can use the chain rule to find $f'(x)$. It is fairly easy to show that $f'(x) = u'e^u$. Do it.

$$\boxed{\frac{d}{dx}[e^u] = u'e^u}$$

Exercise 2. Find a similar formula for $\frac{d}{dx}[a^u]$ where $u = u(x)$.

Example 7. Find $f'(x)$ for $f(x) = x^{\sqrt{x}}$.

Example 8. Evaluate the limit: $\lim_{x\to\pi} \dfrac{e^{\sin x} - 1}{x - \pi}$.

Example 9. Show that $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

Example 10. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' + 5y' - 6y = 0$?

__**Exercise 3.**__ If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

Example 11. Find $\frac{dy}{dx}$ if $x^y = y^x$.

Section 3.4: Exponential Growth and Decay

This section is an application of the calculus of exponential functions. In many real-life situations, quantities grow or decay at a rate proportional to their size. For example, if $y = f(t)$ represents the number of individuals in a population at time t, then it makes sense to assume that the population is growing at a rate proportional to $f(t)$; that is $f'(t) = kf(t)$. It turns out that, in most applications, this equation fairly accurately predicts what really happens in nature.

In general, if $y(t)$ represents a quantity at time t, and if the rate of change of y with respect to t is proportional to y for all times, then

$$\boxed{\tfrac{dy}{dt} = ky}$$

This equation is called the *law of natural growth* if $k > 0$, or the *law of natural decay* if $k < 0$.

Theorem 2. *The only solutions of the differential equation* $\frac{dy}{dt} = ky$ *are the exponential functions*

$$y(t) = y(0)e^{kt} \qquad (*)$$

Let's investigate.

$$\frac{dy}{dt} = ky \qquad \text{means} \qquad \frac{1}{y}\frac{dy}{dt} = \frac{y'}{y} = k,$$

but we learned in the last section that $\frac{y'}{y} = (\ln y)'$. So our equation becomes

$$(\ln y)' = k.$$

Now, we work backwards. We want to take away the derivative on the left hand side, which means that we need to replace the right hand side with a function whose derivative is k (remember, k is a constant). Since our independent variable in this setting is t, we know that $(kt + C_1)' = k$, for any constant C_1. Therefore we have

$$\ln y = kt + C_1.$$

To solve for y, we take the exponential of both sides. We have

$$e^{\ln y} = e^{kt+C_1} = e^{C_1}e^{kt}, \quad \text{or}$$

$$y = C_2 e^{kt},$$

where $C_2 = e^{C_1}$ is another constant.

Finally, we want $y(0)$ to equal the initial value of our quantity (population, mass, temperature, whatever), but $y(0) = C_2 e^{k\cdot 0} = C_2 e^0 = C_2 \cdot 1 = C_2$. This tells us that we should let the constant C_2 be equal to the initial value of the function y; $C_2 = y(0)$. Making this substitution, we obtain equation $(*)$ of the theorem:

$$\boxed{y(t) = y(0)e^{kt}}$$

Example 12. *Population Growth.* Let P be the population at time t, and P_0 be the initial population. Then equation $(*)$ becomes

$$P(t) = P_0 e^{kt}$$

where $k > 0$ represents the growth rate of the population.

If the world population was 2560 million in 1950 and 3040 million in 1960, find an equation to model the population growth in the second half of the 20^{th} century. Use the equation you find to estimate the population in 1993, and in 2020.

Example 13. *Radioactive Decay.* When $k < 0$, equation $(*)$ models a quantity that is decaying in time. If we let m be the mass of a radioactive substance at time t, with initial mass m_0, then equation $(*)$ becomes

$$m(t) = m_0 e^{kt}.$$

Here $k < 0$ is the rate of decay of the substance.

The half-life of radium-226 is 1590 years. A sample of radium-226 has a mass of 100 mg. Find a formula for the mass after t years. Use the formula to find the mass after 1000 years, correct to the nearest milligram.

Example 14. *Newton's Law of Cooling.* Let T represent the temperature of an object at time t, and T_s be the ambient temperature of the surroundings. Newton's Law of Cooling says

$$\frac{dT}{dt} = k(T - T_s).$$

Making the substitution $y(t) = T(t) - T_s$, the equation becomes the familiar

$$\frac{dy}{dt} = ky.$$

A bottle of beer at room temperature (72° F) is placed in a refrigerator where the temperature is 44° F. After half an hour the beer has cooled to 61° F. What is the temperature of the beer after another half hour? How long does it take for the beer to cool to 50° F?

Remark 5. *Compound Interest.* Let A be the amount of money in a bank account after time t, P be the principal, or initial amount invested, k be the interest rate, and n be the number of times the interest is compounded per year. Then,

$$A = P\left(1 + \frac{k}{n}\right)^{nt}.$$

Exercise 4. Find a formula for continuously compounded interest; that is, calculate

$$A = \lim_{n\to\infty} P\left(1 + \frac{k}{n}\right)^{nt}.$$

Hint: make a change of variables: $u = \frac{k}{n}$.

Example 15. If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded (a) semianually, (b) monthly, and (c) continuously.

Section 3.5: Inverse Trigonometric Functions

Recall from section 3.1 that a function has an inverse function if and only if it is one-to-one. Trigonometric functions are *not* one-to-one. We want to define inverse functions for sine, cosine, and tangent, so first we must restrict them to intervals on which they *are* one-to-one. We'll develop everything for sine, and then state the results for cosine and tangent.

We see that $f(x) = \sin x$ is one-to-one on the right hand side of the unit circle, or for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and it also takes on every one of its possible y-values on this interval. Therefore, we can define the inverse sine function, also called *arcsine*, as follows

$$\boxed{\sin^{-1} y = x \quad \Longleftrightarrow \quad \sin x = y, \quad -\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2}}$$

Sometimes we write $f(x) = \arcsin x$ for $f(x) = \sin^{-1} x$. Actually, I usually just write $f(x) = \text{asin}x$ for arcsine, because I am a lazy mathematician (in a good way). So, you can choose any of these three notations.

Since sine and arcsine are inverse functions of one another we have the following identities

$$\boxed{\begin{aligned} \sin(\sin^{-1} x) &= x, \quad &-1 \leq x \leq 1, \\ \sin^{-1}(\sin x) &= x, \quad &-\tfrac{\pi}{2} \leq x \leq \tfrac{\pi}{2} \end{aligned}}$$

Example 16. If $f(x) = \sin^{-1} x$, find $f'(x)$. What is the domain of f'?

Example 17. Let $f(x) = \sin^{-1}(x^2 - 1)$. Find (a) domain($f$), (b) $f'(x)$, and (c) domain(f').

Now, we want to state similar results for the cosine and tangent functions. We'll start with tangent. First, notice that tangent is also one-to-one on the right half of the unit circle, except that tangent is not defined at $x = \pm\frac{\pi}{2}$. Therefore, we define arctangent by

$$\boxed{\tan^{-1} y = x \quad \Longleftrightarrow \quad \tan x = y, \quad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}}$$

Arctangent has the inverse (or cancellation) properties

$$\begin{array}{|c|}\hline \tan(\tan^{-1} x) = x, \quad -\infty < x < \infty \\ \tan^{-1}(\tan x) = x, \quad -\tfrac{\pi}{2} < x < \tfrac{\pi}{2} \\ \hline\end{array}$$

To develop the same theory for arccosine, we need to make a small change. Cosine is *not* one-to-one on the right half of the unit circle, but it is one-to-one on the *top* half. So we can define arccosine by

$$\boxed{\cos^{-1} y = x \quad \Longleftrightarrow \quad \cos x = y, \quad 0 \le x \le \pi}$$

Arccosine satisfies the following inverse (cancellation) properties

$$\begin{array}{|c|}\hline \cos(\cos^{-1} x) = x, \quad -1 \le x \le 1 \\ \cos^{-1}(\cos x) = x, \quad 0 \le x \le \pi \\ \hline\end{array}$$

Example 18. Let $f(x) = \tan^{-1} x$. Find $f'(x)$.

<u>**Exercise 5.**</u> Show that $(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$

Example 19. Define arc-functions for secant, cosecant, and cotangent. Do we really *need* these?

Example 20. Find the exact values of the expressions

(a) $\sin^{-1}(\sqrt{3}/2)$

(b) $\sec(\tan^{-1} 1)$

(c) $\csc^{-1}(2)$

(d) $\sin(\tan^{-1} \frac{x}{2})$

Example 21. Let $f(x) = \sin^{-1}(e^x)$. Find (a) domain(f), (b) $f'(x)$, and (c) domain(f').

Exercise 6. Use the chain rule to find formulas for the derivatives of $\sin^{-1} u$, $\cos^{-1} u$, and $\tan^{-1} u$, where $u = u(x)$.

Section 3.6: Hyperbolic Functions

Our book says: Certain combinations of exponential functions arise so frequently in mathematics and physics that they deserve to be given their own names. In many ways they are analogous to trig functions, so they are given similar names. They have the same relationship to the hyperbola that the trig functions have to the circle. Therefore they are called hyperbolic functions, or *hyperbolic trig functions.*

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{csch } x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{sech } x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Example 22. Find the domain and range of sinh, cosh, and tanh. Sketch their graphs.

Hyperbolic Identities

$\sinh(-x) = -\sinh x$	$\cosh(-x) = \cosh x$
$\cosh^2 x - \sinh^2 x = 1$	$1 - \tanh^2 x = \text{sech}^2 x$
$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Exercise 7. Verify these.

Derivatives of Hyperbolic Functions

$\frac{d}{dx}[\sinh x] = \cosh x$	$\frac{d}{dx}[\text{csch}x] = -\text{csch}x \coth x$
$\frac{d}{dx}[\cosh x] = \sinh x$	$\frac{d}{dx}[\text{sech}x] = -\text{sech}x \tanh x$
$\frac{d}{dx}[\tanh x] = \text{sech}^2 x$	$\frac{d}{dx}[\coth x] = -\text{csch}^2 x$

Inverse Hyperbolic Functions

$$\sinh^{-1} y = x \quad \Leftrightarrow \quad \sinh x = y$$

$$\cosh^{-1} y = x \quad \Leftrightarrow \quad \cosh x = y, \quad \text{and} \quad y \geq 0$$

$$\tanh^{-1} y = x \quad \Leftrightarrow \quad \tanh x = y$$

Example 23. Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Formulas for Inverse Hyperbolic Functions

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), \quad x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1$$

Example 24. Find $\frac{d}{dx}[\sinh^{-1} x]$.

Derivatives of Inverse Hyperbolic Functions

$\frac{d}{dx}[\sinh^{-1} x] = \dfrac{1}{\sqrt{1+x^2}}$	$\frac{d}{dx}[\text{csch}^{-1} x] = -\dfrac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\frac{d}{dx}[\cosh^{-1} x] = \dfrac{1}{\sqrt{x^2-1}}$	$\frac{d}{dx}[\text{sech}^{-1} x] = -\dfrac{1}{x\sqrt{1-x^2}}$
$\frac{d}{dx}[\tanh^{-1} x] = \dfrac{1}{1-x^2}$	$\frac{d}{dx}[\coth^{-1} x] = \dfrac{1}{1-x^2}$

Example 25. Find $\frac{d}{dx}[\tanh^{-1}(\sin x)]$.

Example 26. Show that $\cosh x \pm \sinh x = e^{\pm x}$

Example 27. Show that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for all $n \in \mathbb{R}$.

Exercise 8. Evaluate $\lim\limits_{x\to\infty} \dfrac{\sinh x}{e^x}$.