Beginning Algebra

Chapter 9 Notes

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1 Review: Quadratic Equations

The title of this chapter is *Quadratic Equations*. We have already studied these types of equations in some depth, so let's take a minute to review.

A *quadratic equation* is an equation which is equivalent to an equation of the type

$$ax^2 + bx + c = 0, \quad a > 0,$$

where a, b, and c are real constants, and x is the variable. We say that the form given above is the *standard form* of a quadratic equation.

We know how to solve equations of this type by factoring and using the *principle of zero products*. Let's take a moment to review this method.

Example 1. *Solve:* $4x^2 + 5x - 6 = 0$

Example 2. *Solve:* $(x - 1)(x + 1) = 5(x - 1)$

The number of diagonals d of a polygon with n sides is given by the formula

$$d = \frac{n^2 - 3n}{2}.$$

Example 3. *If a polygon has 27 diagonals, how many sides does it have?*

2 Solving Quadratic Equations by Completing the Square

Unfortunately, many quadratic equations cannot be factored, so our old method doesn't always work. In the next two sections we'll learn two new methods of solving quadratic equations.

The Principle of Square Roots Consider the equation $x^2 = d$.

a. The equation $x^2 = d$ has two real solutions when $d > 0$. The solutions are $\sqrt{d}$ and $-\sqrt{d}$; sometimes written together as $\pm\sqrt{d}$.

b. The equation $x^2 = d$ has no real-number solution when $d < 0$.

c. The equation $x^2 = d$ has one solution when $d = 0$. The solution is 0.

Example 4. *Solve.*

a. $x^2 = 10$

b. $6x^2 = 0$

c. $2x^2 - 3 = 0$

We can solve equations of the form $(x - c)^2 = d$ analogously.

Example 5. *Solve* $(x - 5)^2 = 9$.

$$\begin{aligned} (x-5)^2 &= 9 \\ x - 5 &= \pm\sqrt{9} \\ x - 5 &= \pm 3 \end{aligned}$$

Now break the equation into two separate equations; one for the principle square root, and one for the negative. We get,

$$\begin{aligned} x - 5 &= 3 \quad \text{or} \quad x - 5 = -3 \\ x &= 8 \quad \text{or} \quad x = 2 \end{aligned}$$

The solutions of the equation are $x = 8$ *and* $x = 2$.

Example 6. *Solve.*

a. $(x-3)^2 = 16$

b. $x^2 - 2x + 1 = 5$

This leads us into the main idea of this section – completing the square. Given a quadratic equation of the form $ax^2 + bx + c = 0$, we wish to manipulate it into the form $(x-c)^2 = d$. Here's how it goes.

Completing the Square Suppose we have a quadratic equation in standard form with $a = 1$,

$$x^2 + bx + c = 0.$$

We begin by moving the constant term to the opposite side of the equation

$$x^2 + bx = -c.$$

Now, add $(b/2)^2$ to each side of the equation

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2.$$

Next, factor the left hand side

$$\left(x + \frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2.$$

Now we can solve for x by taking the square root of both sides of the equation, as we did in the previous examples. With this method we can solve *any* quadratic equation that is thrown at us, even the ones that do not factor! Obviously, if we can factor, then we should. This method always works, but sometimes it takes more effort. And, as we know, we mathematicians are lazy.

Example 7. *Solve* $x^2 - 4x - 7 = 0$ *by completing the square.*

Begin by filling in the following table

$b =$	
$b/2 =$	
$(b/2)^2 =$	

Now, follow the process above to complete the square, then solve for x*.*

Example 8. *Solve by completing the square.*

a. $x^2 - 12x + 23 = 0$

b. $x^2 - 3x - 10 = 0$

c. $2x^2 + 3x - 3 = 0$

d. $2x^2 = 9x + 5$

3 The Quadratic Formula

It is tedious and time-consuming to always work through the process of completing the square. We can eliminate the bulk of the work in completing the square by solving the general equation $ax^2 + bx + c = 0$, $a \neq 0$. Then we'll just have to plug a, b, and c into the equation we find, and we'll have an answer. Having a formula that we can just plug into will make life easier, but there is no such thing as a free lunch. The process that we are about to carry out is messy, but at least we only have to do it once! Here goes nothing.

1. Start with the equation $ax^2 + bx + c = 0$, and divide through by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Move the constant term c/a to the right hand side of the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Half of b/a is $b/2a$, and $(b/2a)^2 = b^2/4a^2$. So, we add $b^2/4a^2$ to both sides of the equation.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

4. The left hand side of the equation is a perfect square.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

5. Add the fractions on the right hand side of the equation.

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} \cdot \frac{4a}{4a} + \frac{b^2}{4a^2} \\ &= \frac{-4ac + b^2}{4a^2} \\ &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

6. Take the square root of both sides.

$$\begin{aligned} \sqrt{\left(x + \frac{b}{2a}\right)^2} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

7. Solve for x.

$$\begin{aligned} x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

And so we have derived the *quadratic formula*!

The Quadratic Formula The solutions of any quadratic equation $ax^2 + bx + c = 0$ are given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Since this equation is derived from the completing the square method, it solves *every* quadratic equation. To use it, we need only get our quadratic equation in standard form, and then plug in for a, b, and c.

Example 9. *Solve using the quadratic formula.*

a. $2x^2 = 4 - 7x$

b. $x^2 - 3x - 10 = 0$

c. $x^2 + 4x = 7$

d. $x^2 = x - 1$

The Discriminant Test The last example illustrates the fact that some quadratic equations do not have any real-number solutions at all. In fact, quadratic equations can have either 0, 1, or 2 real solutions. A quick way to check and see how many solutions an equation has is the Discriminant Test. The *discriminant* of a quadratic equation is $b^2 - 4ac$; the radicand of the quadratic formula.

$b^2 - 4ac$	Number of real solutions
> 0	2
$= 0$	1
< 0	0

Example 10. *Determine the number of real solutions of the quadratic equation. Do not actually find them.*

a. $x^2 - 16 = 0$

b. $x^2 = 6x - 9$

c. $2t^2 + 6t + 5 = 0$

4 Formulas

We are going to skip this section. There are some homework problems online for you to work on. You shouldn't have any problems with them, but in case you do, I will answer any questions in class.

5 Applications and Problem Solving

In this section we apply our newfound ability to solve quadratic equations to real life problems. In short, this section is all about word problems!

Example 11. *The area of a rectangular framed painting is 52* ft^2*. The length is 5 ft longer than twice the width. Find the dimensions of the framed painting.*

Example 12. *The hypotenuse of a right triangular animal pen at the zoo is 7 yd long. One leg is 2 yd longer than the other. Find the lengths of the legs.*

Example 13. *The speed of a boat in still water is 12 km/hr. The boat travels upstream 45 km and downstream 45 km in a total time of 8 hr. What is the speed of the stream?*

6 Graphs of Quadratic Equations

Yeah, we really have to do this.

The graphs of quadratic equations take the shape of a *parabola*; a "cup" that opens either up or down. Every parabola has a turning point, called the *vertex*. Moreover, if you were to draw a vertical line through the vertex of a parabola, and fold your paper in half exactly on that line, then the two sides of the parabola would land on top of one another. This property is called *symmetry*. Therefore, if we find the vertex of a parabola, and one or two points on one side of the *axis of symmetry*, then we can easily sketch the graph of our parabola.

The Vertex Formula For a parabola given by the quadratic equation $y = ax^2 + bx + c$,

a. The x-coordinate of the vertex is given by $-\frac{b}{2a}$.

b. The *axis of symmetry* is the line $x = -\frac{b}{2a}$.

c. The y-coordinate of the vertex is found by substituting the x-coordinate into the equation, and solving for y.

Example 14. *Find the vertex and axis of symmetry for the parabola.*

a. $y = -2x^2 + 3$

b. $y = x^2 + 2x - 3$

x - intercepts A parabola can have either 0, 1, or 2 x-intercepts. To find the x-intercepts of a parabola, set $y = 0$ and solve for x.

Example 15. *Find the x-intercepts.*

a. $y = x^2 - 3$

b. $y = x^2 + 3$

As we mentioned above, to graph a parabola we need to know the vertex, and a few more points. If the parabola has x-intercept(s), then we should use those. If not, we have to plug in some values for x.

Example 16. *Find the vertex, fill in the table, and plot the points. Connect the points with a smooth parabola.*

$$y = x^2 - 2$$

x	$y = x^2 - 2$

Example 17. *Graph the parabola:* $y = x^2 + 4x + 4$

Example 18. *Graph the parabola:* $y = x^2 - 4x + 6$

7 Functions

The final section of this course is one of the most important. In future algebra courses you will spend much time and effort studying functions and their properties.

Definition of a Function A *function* is a correspondence between two sets such that for *every* member of the first set, there corresponds *exactly one* member of the second set. The first set is called the *domain*, and the second is called the *range*.

To have a function we must have *all three* pieces; the domain, the range, and the correspondence between them. If we think of the correspondence as a collection of arrows, we can illustrate this idea as follows.

Example 19. *Determine whether the correspondence is a function.*

Domain	*Range*	*Domain*	*Range*
1	*\$107.40*	*New York*	*Mets*
2	*\$34.10*		*Yankees*
3	*\$ 29.60*	*St. Louis*	*Cardinals*
4	*\$ 19.60*	*San Diego*	*Padres*
		Chicago	*Cubs*
			White Sox

Another way to think of a function is as a machine. You take an element of the domain, put it in the machine, it does some work, and gives you back exactly one element of the range.

We frequently denote functions by the letter f. In particular, f represents the correspondence, or the arrows, of the function. Members of the domain are represented by x, and members of the range are represented by y. We write

$$y = f(x).$$

This tells us that y is the member of the domain which corresponds to the member x of the domain under the function f. We say "y equals f of x".

Caution Here, $f(x)$ does not mean f times x.

Example 20. *Evaluate the function* $f(x) = 5x - 3$ *at the specified values.*

a. $f(-6)$

b. $f(0)$

c. $f(1)$

d. $f(20)$

e. $f(\frac{1}{5})$

In the future you will need to be able to evaluate functions when the entry inside the parentheses is not a number.

Example 21. *If* $f(x) = x^2$, *find* $f(x + h)$.

Example 22. *Let* $f(x) = 3 - |x|$. *Find* $f(-x)$.