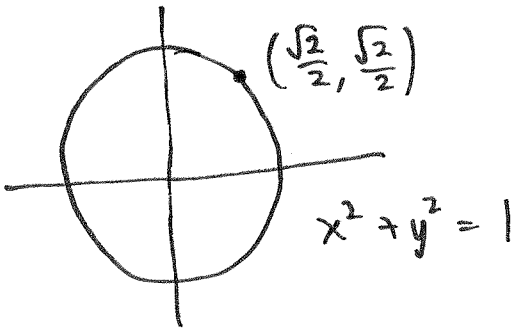
§4.5. Implicit Differentiation

Ex. tangent lines on a circle.



Find the eqn of the tan line at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$:

1. solve for y:

$$y = \sqrt{1 - x^2} = (1-x^2)^{1/2}$$

2. Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}}$$

3. Plug in $\frac{\sqrt{2}}{2}$:

slope at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $\frac{-\sqrt{2}/2}{\sqrt{1-1/2}} = \frac{-\sqrt{2}/2}{\sqrt{1/2}} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$

4. write the eqn:

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

$$\boxed{y = -x + \sqrt{2}}$$

A different way to find $\frac{dy}{dx}$:

leave it as $x^2+y^2=1$ and use the chain rule,

$$\frac{d}{dx}\left[x^2+y^2=1\right]$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Is this the same!?

Ex. $y - xy^2 + x^2 + 1 = 0$

Find $\frac{dy}{dx}$.

$$\frac{dy}{dx} - y^2 - x\,2y\frac{dy}{dx} + 2x = 0$$

$$\frac{dy}{dx}\left(1 - x2y\right) = y^2 - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{y^2-2x}{1-2xy}}$$

Ex let $x = x(t)$. Find $\frac{dx}{dt}$ for:

$$t\ln x = xe^t - 1$$

$$\ln x - \frac{t}{x}\frac{dx}{dt} = e^t\frac{dx}{dt} + xe^t$$

$$\frac{dx}{dt}\left(1+\frac{t}{x}\right) = \ln x - xe^t$$

$$\frac{dx}{dt} = \frac{\ln x - xe^t}{1+t/x}$$

Ex. Find $\frac{dy}{dx}$ at $(x,y) = (1,1)$

$$x^3 - y = \ln y$$

$$3x^2 - \frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(1 + \frac{1}{y}\right) = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{1 + 1/y}$$

at $(1,1)$:

$$\frac{dy}{dx} = \boxed{\frac{3}{2}}$$

Ex. $e^{xy} - 2x = y + 1$ at $(0,0)$

$\frac{d}{dx}$:

$$e^{xy}\left(y + x\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(1 - xe^{xy}\right) = ye^{xy}$$

$$\frac{dy}{dx} = \frac{ye^{xy}}{1 - xe^{xy}}$$

at $(0,0)$:

$$\frac{dy}{dx} = \frac{0}{1} = \boxed{0}$$

Ex. If $xe^y = 1$ Find $\frac{dy}{dx}$ in two ways

$$xe^y\frac{dy}{dx} + e^y = 0$$

$$\frac{dy}{dx} = \frac{-1}{x}$$

or solve for y:

$$e^y = \frac{1}{x}$$

$$y = \ln\left(\frac{1}{x}\right)$$

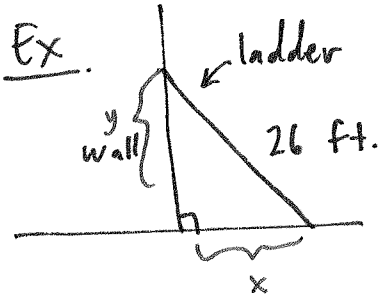
$$y = -\ln(x)$$

$$\frac{dy}{dx} = -\frac{1}{x}$$

§4.6. Related Rates:

If two quantities are related, then their derivatives are also related. (derivatives are rates)

Ex.



slides down wall at 2 ft/sec
how fast does it move away from base?
when it is 10 ft away?

$$\frac{d}{dt}\left[x^2 + y^2 = 26^2\right]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$10\left(\frac{dx}{dt}\right) + 24(-2) = 0$$

$$\frac{dx}{dt} = 4.8 \text{ ft/sec}$$

$10^2 + y^2 = 26^2$

$y = \sqrt{676 - 100}$

$y = \sqrt{576}$

$y = 24$

Ex. A pt moves along the graph of $x^2 + y^2 = 25$. when the pt is at $(-3,4)$ its x-coord is increasing at 0.4 units/sec. How fast is the y-coord changing?

$$\boxed{\frac{dy}{dt} = 0.3 \text{ units/sec.}}$$

Ex. Flash Drive Company.

$C(x) = 5000 + 2x$

$R(x) = 10x - 0.001x^2$

$P(x) = R(x) - C(x)$

production for 1 week is x f.d.s

$\frac{dx}{dt} = 500$ f.d/wk when $x = 2000$ f.d.

Find rate of change of C, R, P.

$$\frac{dC}{dt} = 2\frac{dx}{dt} = 2(500) = \$1000/\text{wk}$$

$$\frac{dR}{dt} = 10\frac{dx}{dt} - 0.002x\frac{dx}{dt}$$

$$= 10(500) - 0.002(2000)(500)$$

$$= 5000 - 2000 = \$3000/\text{wk}.$$

$$\frac{dP}{dt} = \frac{dR}{dt} - \frac{dC}{dt} = 3000 - 1000 = \$2000/\text{wk}.$$

Ex. 15. Boyle's Law : $\frac{P}{T} = k$

P = pressure
T = temperature
k = constant

$\frac{dT}{dt} = 3\text{k/hr}$ $\frac{dP}{dt} = ?$ when $T = 250$ k and $P = 500$ psi

$$\frac{dP}{dt} = k\frac{dT}{dt} = \frac{P}{T}\frac{dT}{dt} = \frac{500}{250}\frac{\text{psi}}{\text{k}}(3)\,\text{k/hr} = \boxed{6\ \text{psi/hr}}$$

Ex. 29. Price-demand.

The price p in dollars and demand x for a product are related:

$$2x^2 + 5xp + 50p^2 = 80000$$

a) If price increases at \$2/mo when $p = \$30$ Find the rate of change for demand.

$$2x^2 + 5x(30) + 50(30)^2 = 80000$$

$$2x^2 + 150x + 45000 = 80000$$

$$x^2 + 75x - 17500 = 0$$

$$(x-100)(x+175) = 0$$

$$\boxed{x = 100} \qquad x = -175$$

$\frac{d}{dt}$:

$$4x\frac{dx}{dt} + 5x\frac{dp}{dt} + 5p\frac{dx}{dt} + 100p\frac{dp}{dt} = 0$$

$$4(100)\frac{dx}{dt} + 5(100)(2) + 5(30)\frac{dx}{dt} + 100(30)(2) = 0$$

$$550\frac{dx}{dt} = -7000$$

$$\frac{dx}{dt} = \frac{-7000}{550} = \boxed{\frac{-140}{11}} \text{ units/mo.}$$