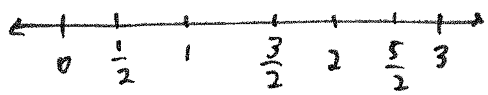
REs from last class.

$f(x) = x + x^2 \qquad a = 0\ ,\ b = 3$

1. Find R_6.

Step 1. Partition the interval $[0, 3]$ into 6 pieces



$$\left.\begin{array}{l} x_0 = 0 \\ x_1 = \frac{1}{2} \\ x_2 = 1 \\ x_3 = 3/2 \\ x_4 = 2 \\ x_5 = 5/2 \\ x_6 = 3 \end{array}\right] \text{Right end points}$$

$$\Delta x = \frac{a+b}{n} = \frac{0+3}{6} = \frac{1}{2}$$

Step 2. $R_6 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$

Find $f(x_n)$'s:

$$f(x_1) = f\left(\tfrac{1}{2}\right) = \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}$$
$$f(x_2) = f(1) = 1 + 1^2 = 2$$
$$f(x_3) = f\left(\tfrac{3}{2}\right) = \tfrac{3}{2} + \left(\tfrac{3}{2}\right)^2 = \tfrac{3}{2} + \tfrac{9}{4} = \tfrac{15}{4}$$
$$f(x_4) = f(2) = 2 + 2^2 = 6$$
$$f(x_5) = f\left(\tfrac{5}{2}\right) = \tfrac{5}{2} + \left(\tfrac{5}{2}\right)^2 = \tfrac{5}{2} + \tfrac{25}{4} = \tfrac{35}{4}$$
$$f(x_6) = f(3) = 3 + 3^2 = 12$$

Plug in:

$$R_6 = \frac{1}{2}\left(\frac{3}{4} + 2 + \frac{15}{4} + 6 + \frac{35}{4} + 12\right) = \frac{1}{2}\left(\frac{53}{4} + 20\right) = \frac{1}{2}\left(\frac{133}{4}\right) = \boxed{\frac{133}{8}}$$

phew!

2. Find $\int_0^3 x+x^2\,dx = \lim_{N\to\infty} \sum_{n=1}^{N} f(x_n)\cdot\Delta x$.

1. $\underline{\Delta x} = \frac{a+b}{N} = \frac{0+3}{N} = \frac{3}{N}$

2. $\underline{x_n} = a + \Delta x\cdot n = 0 + \frac{3}{N}\cdot n = \frac{3n}{N}$

3. $\underline{f(x_n)} = f\left(\frac{3n}{N}\right) = \frac{3n}{N} + \left(\frac{3n}{N}\right)^2 = \frac{3n}{N} + \frac{9n^2}{N^2}$

4. $\underline{f(x_n)\Delta x} = \left(\frac{3n}{N} + \frac{9n^2}{N^2}\right)\cdot\frac{3}{N} = \frac{9n}{N^2} + \frac{27n^2}{N^3}$

5. $\underline{\sum_{n=1}^{N} f(x_n)\Delta x} = \sum_{n=1}^{N}\left(\frac{9n}{N^2} + \frac{27n^2}{N^3}\right)$

$$= \sum_{n=1}^{N}\frac{9n}{N^2} + \sum_{n=1}^{N}\frac{27n^2}{N^3}$$

$$= \frac{9}{N^2}\underbrace{\sum_{n=1}^{N} n}_{} + \frac{27}{N^3}\underbrace{\sum_{n=1}^{N} n^2}_{}$$ use formulas for these.

$$= \frac{9}{N^2}\cdot\frac{N(N+1)}{2} + \frac{27}{N^3}\cdot\frac{N(N+1)(2N+1)}{6}$$

$$= \frac{9N^2+9N}{2N^2} + \frac{27\left(2N^3+3N^2+N\right)}{6}$$

$$= \frac{9N^2+9N}{2N^2} + \frac{54N^3+81N^2+27N}{6}$$

6. $\underline{\lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)\Delta x} = \lim_{N\to\infty}\left(\frac{9N^2+9N}{2N^2} + \frac{54N^3+81N^2+27N}{6}\right) = \frac{9}{2} + \frac{54}{6}$

$$= 9 + \frac{9}{2} = \boxed{\frac{27}{2}}$$

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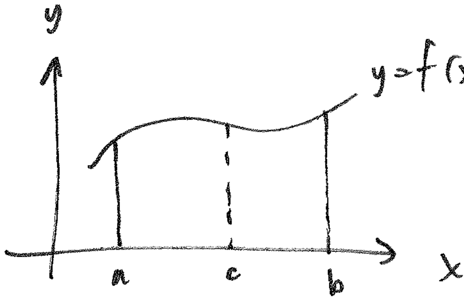
3. Some Properties of Definite Integrals



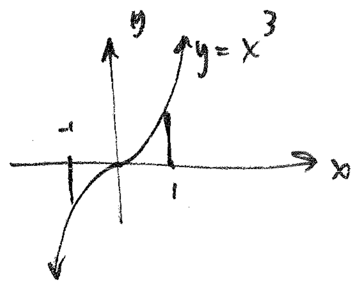
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

for any $c \in (a,b)$.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

If graph of $y=f(x)$ lies below the x-axis, then the area is negative. Justification: the "heights" of the boxes are negative.

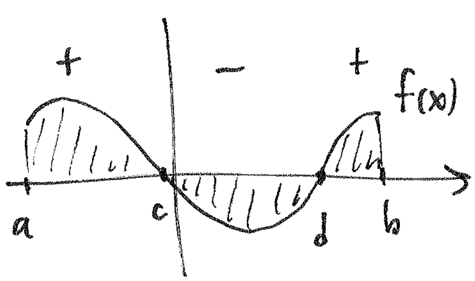
Ex.



$$\int_{-1}^{1} x^3\,dx = \int_{-1}^{0} x^3\,dx + \int_0^1 x^3\,dx$$

$= 0$ bc they cancel each other.

Ex.



$$\int_a^b f(x)\,dx =$$

$$\underset{+}{\int_a^c f(x)\,dx} + \underset{-}{\int_c^d f(x)\,dx} + \underset{+}{\int_d^b f(x)\,dx}$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx, \quad \forall k \in \mathbb{R}$$

$$\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

BUT

$$\left[\ \int_a^b f(x)\,g(x)\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx \ !\right.$$

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4. The Fundamental Theorem of Calculus

Part I. If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} = F(x)\Big|_a^b$$

Ex. $\int_0^2 x^2\,dx = \frac{1}{3}x^3\Big|_0^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(0)^3 = \frac{8}{3} - 0 = \boxed{\frac{8}{3}}$

Ex. $\int_0^3 x + x^2\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^3\Big|_0^3 = \frac{1}{2}\cdot 3^2 + \frac{1}{3}\cdot 3^3 - \cancel{\left(\frac{1}{2}\cdot 0^2 + \frac{1}{3}\cdot 0^3\right)}^{0}$

$= \frac{9}{2} + 9 = \boxed{\frac{27}{2}}$

Ex. $\int_{-1}^1 x^3\,dx = \frac{1}{4}x^4\Big|_{-1}^1 = \frac{1}{4}(1)^4 - \frac{1}{4}(-1)^4 = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

Ex. $\int_1^3 \left(4x - 2e^x + \frac{5}{x}\right) dx = 2x^2 - 2e^x + 5\ln x \Big|_1^3$

$= 2(3^2) - 2e^3 + 5\ln(3) - \left[2(1^2) - 2e^1 + \cancel{5\ln(1)}\right]$

$= 18 - 2e^3 + 5\ln(3) - 2 + 2e$

$= 16 - 2e^3 + 2e + 5\ln(3)$

$\approx \boxed{-13.24}$

Can you imagine doing a Riemann Sum for that!?

Ex. $\int_0^5 \frac{x}{x^2+10}\, dx$ $\quad u = x^2 + 10 \quad u(0) = 10$

$\quad du = 2x\, dx \quad u(5) = 35$

$\frac{1}{2}\int_0^5 \frac{2x}{x^2+10}\, dx = \frac{1}{2}\int_{10}^{35} \frac{1}{u}\, du = \frac{1}{2}\ln|u| \Big|_{10}^{35}$

$= \frac{1}{2}\ln(35) - \frac{1}{2}\ln(10)$

$= \frac{1}{2}\ln\left(\frac{35}{10}\right) = \boxed{\frac{1}{2}\ln\left(\frac{7}{2}\right)}$

Ex. $-1\int_{-4}^{1} \sqrt{5-t}\,(-1\,dt)$ $\quad u = 5 - t \quad u(-4) = 9$

$\quad du = -1\, dt \quad u(1) = 4$

$= -\int_9^4 \sqrt{u}\, du = \int_4^9 \sqrt{u}\, du = \frac{2}{3}u^{3/2}\Big|_4^9 = \frac{2}{3}9^{3/2} - \frac{2}{3}4^{3/2}$

$= \frac{2}{3}(27 - 8) = \boxed{\frac{38}{3}}$

Ex. A company makes x HDTVs per month. The monthly marginal profit is given by $P'(x) = 165 - 0.1x$, $0 \le x \le 4{,}000$.

The company currently manufactures 1500 HDTVs per month, but is planning to increase production. Find the change in monthly profit if they increase to 1600 HDTVs per mo.

i.e. $P(1600) - P(1500) = ?$

$$P(1600) - P(1500) = \int_{1500}^{1600} P'(x)\,dx$$

$$= \int_{1500}^{1600} 165 - 0.1x \, dx$$

$$= 165x - 0.05x^2 \Big|_{1500}^{1600}$$

$$= 165(1600) - 0.05(1600)^2 - \left(165(1500) - 0.05(1500)^2\right)$$

$$= \boxed{\$1000}$$

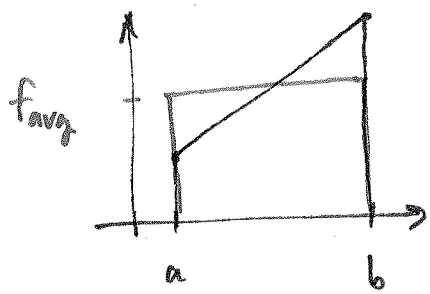
Average Values: $\quad \text{Avg}_f(a,b) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Ex. $f(x) = x - 3x^2$, on $[-1, 2]$

$$\text{Avg} = \frac{1}{2+1}\int_{-1}^{2} x - 3x^2\,dx = \frac{1}{3}\int_{-1}^{2} x - 3x^2\,dx = \frac{1}{3}\left(\frac{1}{2}x^2 - x^3\right)_{-1}^{2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\cdot 4 - 8 - \frac{1}{2} - 1\right] = \frac{1}{3}\left(-\frac{15}{2}\right) = \boxed{\frac{-15}{6}} = \boxed{\frac{-5}{2}}$$

Interpretation of avg. value:



$$\int_a^b f(x)\,dx = (b-a)\cdot f_{avg}.$$

Exs. $\displaystyle\int_6^7 \frac{\ln(t-5)}{t-5}\,dt$

$\displaystyle\int_0^1 \frac{x-1}{x^2-2x+3}\,dx$

$\displaystyle\int_0^1 xe^{x^2}\,dx$

~~Monday~~: Indefinite Integrals

Part II of FTC: $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Pf. $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}\left[F(x)-F(a)\right]$

$= F'(x) - 0 = F'(x) = f(x).$ $\square$

For antiderivatives we write.

$$\boxed{\int f(x)\,dx = F(x) + C}$$

and call this an indefinite integral.