

Name: KEY
Math 144: Business Calculus
Final Exam

Instructions: Complete all problems, showing all relevant work. Reduce or simplify answers as much as possible without using a calculator. Circle or clearly mark your final answers.

1. [5 points each] Calculate the limits.

a. $\lim_{x \rightarrow 6} \frac{x^2 - 36}{x^2 - 8x + 12} = \lim_{x \rightarrow 6} \frac{(x+6)(\cancel{x-6})}{(x-2)(\cancel{x-6})} = \frac{12}{4} = \boxed{3}$

b. $\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \frac{d}{dx}(x^3) = \boxed{3x^2}$

2. [10 points] Use the limit definition of derivative to find $\frac{d}{dx}[x^2]$. You must use the limit definition to receive any points.

$$\begin{aligned} \frac{d}{dx}(x^2) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h} = \boxed{2x} \end{aligned}$$

3. [5 points each] Find the derivatives of the functions.

a. $f(x) = \frac{x^2}{x+1}$ $f'(x) = \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \boxed{\frac{x^2 + 2x}{(x+1)^2}}$

b. $g(x) = x \ln(x) + e^{x^2+x}$ $g'(x) = 1 \cdot \ln(x) + x \frac{1}{x} + e^{x^2+x} (2x+1)$
 $= \boxed{\ln x + 1 + (2x+1)e^{x^2+x}}$

4. [10 points] Find dy/dx for the equation: $xy = x^2 + \ln(y^2)$

$$x \frac{dy}{dx} + y = 2x + \frac{2}{y} \frac{dy}{dx}$$

$$\left(x - \frac{2}{y}\right) \frac{dy}{dx} = 2x - y \quad \Rightarrow \quad \frac{dy}{dx} = \boxed{\frac{2xy - y^2}{xy - 2}}$$

5. [10 points] a. Find an equation for the tangent line to $y = \sqrt{x}$ at the point (25, 5).

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \text{ at } (25, 5): \frac{dy}{dx} = m = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

$$\text{line: } y - 5 = \frac{1}{10}(x - 25) \quad \Rightarrow \quad y = \frac{1}{10}x - \frac{5}{2} + 5 \quad \Rightarrow \quad \boxed{y = \frac{1}{10}x + \frac{5}{2}}$$

b. Use part a. to estimate $\sqrt{25.1}$.

$$\sqrt{25.1} \approx \frac{1}{10} \cdot (25.1) + \frac{5}{2} = \frac{1}{10} \cdot \frac{251}{10} + \frac{5}{2} = \frac{251}{100} + \frac{250}{100} = \frac{501}{100} = \boxed{5.01}$$

6. [10 points] A clown is blowing up a spherical balloon. When the volume of the balloon is $\frac{32}{3}\pi$ cm³, the surface area is changing at a rate of 1 cm²/sec. At what rate is the radius increasing at this moment? [volume: $V = \frac{4}{3}\pi r^3$, surface area: $A = 4\pi r^2$]

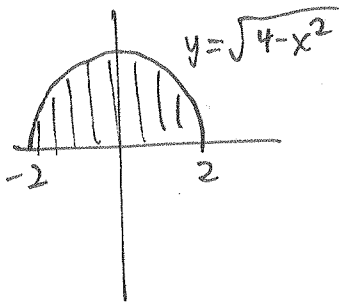
$$V = \frac{32}{3}\pi = \frac{4}{3}\pi r^3 \Rightarrow r^3 = 8 \Rightarrow r = 2$$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$1 = 8\pi(2) \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \boxed{\frac{1}{16\pi} \text{ cm/sec}}$$

7. [10 points] Calculate: $\int_{-2}^2 \sqrt{4-x^2} dx$



$$\text{Area} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \cdot 4 = \boxed{2\pi}$$

8. [10 points] Find the average value of the function $f(x) = x^2 + x$ between $x = -1$ and $x = 2$.

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2+1} \int_{-1}^2 x^2 + x dx = \frac{1}{3} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) \Big|_{-1}^2 \\ &= \frac{1}{3} \left(\frac{1}{3}(8) + 2 \right) - \frac{1}{3} \left(-\frac{1}{3} + \frac{1}{2} \right) \\ &= \frac{8}{9} + \frac{2}{3} + \frac{1}{9} - \frac{1}{6} = 1 + \frac{1}{2} = \boxed{\frac{3}{2}} \end{aligned}$$

9. [5 points each] Find the (definite and indefinite) integrals.

a. $\int x^4 + 5x - \frac{1}{x} + \frac{3}{x^2} dx = \boxed{\frac{1}{5}x^5 + \frac{5}{2}x^2 - \ln|x| - \frac{3}{x} + C}$

~~b. $\int_1^e 4t^3 \ln(t^4) dt = \int_1^e \ln u du$
 $u = t^4$ $u(e) = e^4$
 $du = 4t^3 dt$ $u(1) = 1$~~

$\int_1^e \frac{4t^3}{t^4} dt = 4 \int_1^e \frac{1}{t} dt$
 $= 4 \ln(t) \Big|_1^e$
 $= 4(\ln e - \ln 1)$
 $= \boxed{4}$

c. $\frac{1}{2} \int_0^4 \frac{2x}{\sqrt{x^2+9}} dx = \frac{1}{2} \int_9^{25} u^{-1/2} du = \frac{1}{2} (2u^{1/2}) \Big|_9^{25} = 5 - 3 = \boxed{2}$
 $u = x^2 + 9$ $u(4) = 16 + 9 = 25$
 $du = 2x dx$ $u(0) = 9$

d. $-\int \frac{1 - \ln(x)}{-x} dx = -\int u du = -\frac{1}{2} u^2 + C$
 $u = 1 - \ln(x)$
 $du = -\frac{1}{x} dx$
 $= \boxed{-\frac{1}{2} (1 - \ln(x))^2 + C}$