A complex differentiable map \( f: \mathbb{C} \to \mathbb{C} \) is also called a \textit{holomorphic} map, or an \textit{analytic} map. If \( f \) is holomorphic at a point \( a \), and if \( f'(a) = 0 \), show that \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are both zero. (Recall: We related \( f'(a) \) to partial derivatives of \( f \) with respect to \( x \) and \( y \).) Conclude that if \( f: \mathbb{C} \to \mathbb{C} \) is holomorphic and \( f'(a) = 0 \) for all \( a \) then \( f \) is constant.

Comparing \( \mathbb{R} \)-linear maps of \( \mathbb{C} \) to \( \mathbb{C} \)-linear maps of \( \mathbb{C} \).

1. Let \( L: \mathbb{C} \to \mathbb{C} \) be an arbitrary \( \mathbb{R} \)-linear map. Let \( A: \mathbb{C} \to \mathbb{C} \) and \( B: \mathbb{C} \to \mathbb{C} \) be the linear maps \( A(z) \equiv \frac{1}{2} (L(z) - iL(iz)) \) and \( B(z) \equiv \frac{1}{2} (L(z) + iL(iz)) \). Confirm that \( A(\alpha z) = \alpha A(z) \) and \( B(\alpha z) = \overline{\alpha}B(z) \) for any \( \alpha \in \mathbb{C} \). Conclude that there exist \( a, b \in \mathbb{C} \) such that \( L(z) = az + b\overline{z} \). Thus an arbitrary \( \mathbb{R} \) linear map from \( \mathbb{C} \) to \( \mathbb{C} \) is of this form. Check that the map \( L \) is \( \mathbb{C} \)-linear iff \( b = 0 \).

In general, to say a function is differentiable at a point \( a \) means that if we look close enough around the point \( a \) we can make the function \( f \) look arbitrarily close to being linear.

For a function \( f: \mathbb{C} \to \mathbb{C} \) to be differentiable at \( a \in \mathbb{C} \) in the usual multivariable calculus sense (i.e. as if \( f \) was just a map from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)) just means that near \( a \), \( f \) looks like an \( \mathbb{R} \)-linear map of \( \mathbb{C} \). For \( f \) to be complex differentiable at \( a \), \( f \) must look like a \( \mathbb{C} \)-linear map close enough to the point \( a \). Hence it is important to have a very precise understanding of \( \mathbb{C} \)-linear maps of \( \mathbb{C} \). Of course, all such maps are just of the form \( z \mapsto bz \), since \( \mathbb{C} \) is one dimensional as a \( \mathbb{C} \) vector space. For an arbitrary value of \( b \in \mathbb{C} \), answer the following questions about the action of the linear map \( L_b(z) = bz \).

2. Show that the circle \( S_r \) of radius \( r \) centered about the origin, is mapped to the circle \( S_{r|b|} \) of radius \( r \cdot |b| \).
3. Show that the entire plane is rotated by an amount \( \arg b \).

4. If we think of \( \mathbb{C} \) as being \( \mathbb{R}^2 \), what is the value of the Jacobian of the linear map \( L_b \)?

Thus the effect of a \( \mathbb{C} \)-linear map \( z \mapsto bz \) on \( \mathbb{C} \) is to rotate the plane by an amount \( \arg b \) and to stretch it uniformly by an amount \( |b| \). Of course, when \( b = 0 \), then \( L_b \) just maps all of \( \mathbb{C} \) to a point.

Based on this, give an argument (not a proof) answering the following question.

5. If a holomorphic map \( f: \mathbb{C} \to \mathbb{C} \) maps all of \( \mathbb{C} \) onto a single smooth curve in the complex plane, must it actually map all of \( \mathbb{C} \) to a single point.

Getting used to \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \bar{z}} \).

6. If \( f \) is holomorphic, what is \( \frac{\partial f}{\partial \bar{z}} \)? What is the relationship between \( f'(z) \) and \( \frac{\partial f}{\partial \bar{z}} \)?

7. Check by explicit calculation that for any differentiable \( f: \mathbb{C} \to \mathbb{C} \) one has \( df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \). (Hint: Convert it into a statement in terms of \( x \) and \( y \).)

8. Simplify the above expression for \( df \) under the assumption that \( f \) is holomorphic. Use \( f'(z) \) in your answer.

9. Check that \( \frac{\partial f}{\partial \bar{z}} = \frac{\bar{f}}{\partial z} \). (Hint: Convert it into a statement in terms of \( x \) and \( y \) and write \( f = u + iv \).)