Manifolds in Fluid Dynamics

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1 Preliminary Remarks

In studying fluid dynamics it is useful to employ two different perspectives of a fluid flowing through a domain $\mathcal{D}$. The *Eulerian* point of view is to consider a fixed point $\boldsymbol{x} \in \mathcal{D}$, and observe the fluid flowing past. The *Lagrangian* point of view is to consider a fixed but arbitrary volume of fluid, called a *parcel of fluid*, and follow the parcel as it travels with the flow. One would think that over time one of these two descriptions would win out over the other, but that has not been the case. The reason for this is that while the Lagrangian description holds some mathematical and conceptual advantages, it is difficult to apply to physical flows – measurements are often taken at fixed points in the real world, matching the Eulerian description [3].

Regardless of our choice of perspective, we must make some preliminary assumptions. We assume that the domain $\mathcal{D}$ is a *continuum*; a nonempty, compact, connected metric space. This assumption can be thought of as ignoring the molecular structure that is incumbent to any physical fluid (liquid or gas). Due to the nature of molecules, if we tried to follow the flow of a single molecule of fluid it would act very chaotically, and would not be representative of the fluid as a whole. To avoid complications of this sort, we restrict ourselves to dealing with small volumes of fluid that are not "too small"; hence the need to follow a parcel through the flow, rather than just a point. In [3], Stephen Childress gives reasonable lower bounds for the diameter of a parcel: 10^{-7} cm for liquids, 10^{-3} cm for gases.

The following definitions will prove useful. A *Riemannian metric* (or metric tensor) is a positive definite symmetric bilinear form g on the tangent bundle of a manifold. A *Riemannian manifold* is a manifold $\mathcal{M}$ together with a Riemannian metric, usually written as a pair $(\mathcal{M}, g)$. Given a Riemannian manifold, the tangent and cotangent bundles are naturally isomorphic, $T\mathcal{M} \cong T^*\mathcal{M} : \boldsymbol{v} \mapsto g(\ ,\boldsymbol{v})$.

For our purposes we will assume that $\mathcal{D}$ is a smooth[1] n-manifold, $n = 2, 3$. This is not a far cry from the assumption that $\mathcal{D}$ be a continuum, but it requires mention. In fact, $\mathcal{D}$ being a Riemannian manifold implies that $\mathcal{D}$ is a continuum, but not conversely. Furthermore, every parcel is a smooth manifold; a submanifold of $\mathcal{D}$.

2 The time-T Map, ϕ_T

Let us begin by thinking in the Lagrangian point of view, with a fixed but arbitrary parcel of fluid $\mathcal{M}$. A *fluid flow* (or *flow field*) is a vector field on $\mathcal{D}$, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$, where the vectors attached at each point represent the velocity of the fluid at that point. In the language of manifolds and bundles, $\boldsymbol{u}(\ , t)$ is a section of the tangent bundle $T\mathcal{D}$ for each time t. That is, $\boldsymbol{u}$ is a *time dependent* vector field. Over a time interval, a fluid flow will distort any embedded manifold; in particular, the parcel $\mathcal{M}$.

Suppose we turn on the flow at some time t_0, and let the flow carry $\mathcal{M}$ for a time T. The flow will move and distort the parcel to some new form, which we will call $\mathcal{M}_T$. We denote the map which carries $\mathcal{M}$ to $\mathcal{M}_T$ by $\phi_T : \mathcal{D} \to \mathcal{D}$, and call it the *time-$T$ map* for the flow $\boldsymbol{u}$. The time-T map is then a diffeomorphism between manifolds which carries every parcel to its image under the flow after time T. Note that in [9] we denote the flow of a vector field by $\mathbf{c}$. We may use this convention for flows of arbitrary vector fields, but will reserve ϕ for the flow map of the fluid flow.

Example 2.1 (Childress [4]) A steady flow is one in which $\partial_t \boldsymbol{u} = 0$. That is, the flow does not change in time, and the velocity vectors are constant at each point. Suppose we let $\mathcal{D}$ be a sphere, and $\boldsymbol{u}$ be a steady flow with horizontal velocity vectors at each point. Since we require that ϕ_T be a diffeomorphism, then the Brouwer Fixed Point Theorem implies that (at least) one point on the sphere must be fixed; $\phi_T(p) = p$. This implies that the associated velocity vector at p is the zero vector.

Physically, this means that the horizontal wind velocity on the surface of the Earth must vanish at (at least) one point. Unfortunately, this point is almost never in Kansas.

Now, suppose we have a vector field $\boldsymbol{v} \in \mathfrak{X}$, and any diffeomorphism $f : \mathcal{D} \to \mathcal{D}$. The *transport* of $\boldsymbol{v}$ by f is given by $f_*(\boldsymbol{v})$, the pushforth[2] of $\boldsymbol{v}$.

[1] C^∞, as usual

[2] terminology of [8]

In particular, if f is the time-T map ϕ_T, then the transport of $\boldsymbol{v}$ by the flow $\boldsymbol{u}$ is given by $\phi_{T*}\boldsymbol{v}$.

Example 2.2 Suppose $\mathcal{D}$ is the surface of a lake, or a portion of the surface of a river, and consider a very tiny boat inside a (very large, in comparison to the boat) parcel of fluid. The boat's motor propels it along the lake and determines a vector field $\boldsymbol{v}$ along the path of the boat. The flow of the water in the lake is given by $\boldsymbol{u}$. Thus, the motion of the boat as it is being propelled by its motor, and simultaneously carried in the flow of the water, is given by $\phi_{t*}\boldsymbol{v}(\boldsymbol{x}, t)$.

3 Covariant and Material Derivatives

In his notes on Mathematical Relativity [9], Parker writes

> The notion of connection is *the* fundamental concept in global differential geometry. Intuitively, it is supposed to connect the fibers of the tangent bundle by parallel translations.

I leave any general and historical discussion of connections to Freeman in her thesis [6]. Instead, we will focus on the specific example of the covariant derivative.

Since our immediate focus is not geometry, we will focus solely on the essentials here. A covariant derivative is a generalization of the directional derivative of calculus. It can be thought of as a way to slide vectors along a curve in the manifold without changing their "direction" or magnitude; *i.e.*, a means of parallel transport.

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vector fields, and f, g be smooth functions. The *covariant derivative* in the direction of $\boldsymbol{v}$, written $\nabla_{\boldsymbol{v}}$, satisfies the following conditions

$$\begin{aligned} \nabla_{f\boldsymbol{v}+g\boldsymbol{w}}\boldsymbol{u} &= f\nabla_{\boldsymbol{v}}\boldsymbol{u} + g\nabla_{\boldsymbol{w}}\boldsymbol{u}, \\ \nabla_{\boldsymbol{v}}(\boldsymbol{u}+\boldsymbol{w}) &= \nabla_{\boldsymbol{v}}\boldsymbol{u} + \nabla_{\boldsymbol{v}}\boldsymbol{w}, \\ \nabla_{\boldsymbol{v}}f\boldsymbol{u} &= f\nabla_{\boldsymbol{v}}\boldsymbol{u} + \boldsymbol{u}\nabla_{\boldsymbol{v}}f\,. \end{aligned}$$

Our interest in the covariant derivative is in using it to define the material derivative of a vector field, as follows.

The *material derivative* operator is defined to be

$$\frac{D}{Dt} = \partial_t + (\boldsymbol{u} \cdot \nabla_{\boldsymbol{u}}).$$

This operator can be applied to scalar fields (C^∞ functions) or vector fields on $\mathcal{D}$, just as a covariant derivative can. In fact, the material derivative can be thought of as a covariant derivative on the manifold $\mathcal{D} \times \mathbb{R}$, where $\mathbb{R}$ represents the time dimension. A vector field $\boldsymbol{v} \in \mathfrak{X}$ is said to be *material* if its material derivative is equal to zero,

$$\frac{D\boldsymbol{v}}{Dt} = \partial_t \boldsymbol{v}|_{\boldsymbol{x}} + \boldsymbol{u} \cdot \nabla_{\boldsymbol{u}} \boldsymbol{v} = 0.$$

Notice that in this formula we are working in Eulerian coordinates; made evident by the restriction to a fixed $\boldsymbol{x}$. Every material vector field $\boldsymbol{v}$ has the property that $\boldsymbol{v}(\phi_T \boldsymbol{x}) = \phi_{T*} \boldsymbol{v}(\boldsymbol{x})$ [4].

Example 3.1 For a 3-dimensional flow, define a vector field $\omega := \nabla \times \boldsymbol{u}$, the curl of the flow field. We call ω the vorticity field of the flow. Vorticity measures the rotation of the fluid flow in $\mathcal{D}$.

Example 3.2 To match our physical intuition, we wish to define the acceleration of a fluid flow to be the "derivative" of velocity. It turns out that the material derivative is the proper derivative for this purpose. Thus, we define the acceleration of a fluid parcel to be the material derivative of $\boldsymbol{u}$, given by $\boldsymbol{a} := \frac{D\boldsymbol{u}}{Dt} = \partial_t \boldsymbol{u}|_{\boldsymbol{x}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}$.

In [9], vorticity and acceleration are defined in more generality, and the two concepts are related via the Lie derivative. To do so, however, requires some more geometry. There is a unique connection ∇ on the tangent bundle of a Riemannian manifold that satisfies the following properties, for any vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\nabla g = 0, \text{ and} \tag{i}$$

$$\nabla_{\boldsymbol{u}} \boldsymbol{v} - \nabla_{\boldsymbol{v}} \boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}] \tag{ii}$$

where g is the Riemannian metric defined in section 1. Condition i) means that ∇ preserves the metric, while condition ii) means that ∇ is *torsion-free*, or symmetric. The connection ∇ is called the *Levi-Civita connection*; named after the Italian geometer Tullio Levi-Civita. This notation gets confusing (∇ means three different things), but is justified as all three objects are essentially covariant derivatives. Henceforth, we will assume ∇ denotes the Levi-Civita connection.

We can now define the acceleration of a fluid flow by $\boldsymbol{a} := \nabla_{\boldsymbol{u}} \boldsymbol{u}$. Next, we define the vorticity of the flow field $\boldsymbol{u}$ to be $\omega := d\boldsymbol{u}^\flat$, where $\boldsymbol{u}^\flat := g^\flat(\boldsymbol{u}) = g(\ ,\boldsymbol{u})$. Here the musical flat symbol denotes the map between $\boldsymbol{u}$

and the unique one-form associated to $\boldsymbol{u}$ via the metric. One often refers to this relationship as a "lowering of indices" as elements of the tangent bundle are written with superscript indices, and those of the cotangent bundle with subscripts. We will use sharp and flat notation hereafter to denote the lowering ($\flat$) and raising ($\sharp$) of indices.

We now have the following proposition from [9]:

Proposition 3.3 *The velocity change along integral curves is related to the vorticity of acceleration:* $\pounds_{\boldsymbol{u}}\omega = d\boldsymbol{a}^\flat$.

For a 3-dimensional flow, we define the *field lines* of a flow to be the integral curves of

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}.$$

This definition can be easily generalized to higher (or lower) dimensions. In manifold and bundle terminology, this is a *foliation* of the flow field $\boldsymbol{u}$. In particular, it is called a one-dimensional foliation. We may think of these field lines as being frozen in the fluid. That is, the flow follows the field lines without distorting them.

4 Euler's Equation

In this section we closely follow portions of [5], as on pages 110–113 of [9].

Let us begin by defining an operator $\mathcal{A}_{\boldsymbol{u}} := \pounds_{\boldsymbol{u}} - \nabla_{\boldsymbol{u}}$ on a Riemannian manifold $(\mathcal{M}, g)$. Since ∇ is torsion-free, $\mathcal{A}_{\boldsymbol{u}}\boldsymbol{v} = -\nabla_{\boldsymbol{v}}\boldsymbol{u}$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathfrak{X}$. Furthermore, for each $\boldsymbol{u} \in \mathfrak{X}(\mathcal{M})$, $\mathcal{A}_{\boldsymbol{u}}$ is a derivation of $\mathfrak{T}$ which vanishes on $C^\infty(\mathcal{M})$. Thus, it follows that if $\alpha \in \Omega^1$, then $\langle \mathcal{A}_{\boldsymbol{u}}\alpha, \boldsymbol{v}\rangle = \langle \alpha, \nabla_{\boldsymbol{v}}\boldsymbol{u}\rangle$.

Now observe that for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathfrak{X}$, $\langle \mathcal{A}_{\boldsymbol{u}}\boldsymbol{v}^\flat, \boldsymbol{w}\rangle = g^\flat(\boldsymbol{v}, \nabla_{\boldsymbol{w}}\boldsymbol{u})$ whence

$$\begin{aligned}
\langle dg^\flat(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w}\rangle &= \nabla_{\boldsymbol{w}} g^\flat(\boldsymbol{u}, \boldsymbol{v}) \\
&= g^\flat(\nabla_{\boldsymbol{w}}\boldsymbol{u}, \boldsymbol{v}) + g^\flat(\boldsymbol{u}, \nabla_{\boldsymbol{w}}\boldsymbol{v}) \\
&= \langle \mathcal{A}_{\boldsymbol{u}}\boldsymbol{v}^\flat, \boldsymbol{w}\rangle + \langle \mathcal{A}_{\boldsymbol{v}}\boldsymbol{u}^\flat, \boldsymbol{w}\rangle,
\end{aligned}$$

so that $dg^\flat(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{A}_u\boldsymbol{v}^\flat + \mathcal{A}_v\boldsymbol{u}^\flat$. Note here that $\nabla_{\boldsymbol{u}}\boldsymbol{v}^\flat = (\nabla_{\boldsymbol{u}}\boldsymbol{v})^\flat$. We now have the following.

Lemma 4.1 *For any* $\boldsymbol{u} \in \mathfrak{X}$, $\pounds_{\boldsymbol{u}}\boldsymbol{u}^\flat = \nabla_{\boldsymbol{u}}\boldsymbol{u}^\flat + \frac{1}{2}dg^\flat(\boldsymbol{u}, \boldsymbol{u})$.

As a proof, take $\boldsymbol{u} = \boldsymbol{v}$ in the preceding computation and apply the definition of $\pounds_{\boldsymbol{u}}$. Moreover, we can also now prove Proposition 3.1. To do so, apply this lemma and use the fundamental properties of $\pounds$ and d.

Suppose $\mathcal{S}$ is a Riemannian manifold, and $\mathcal{M} = \mathcal{S}\times\mathbb{R}$. In this case $\mathcal{M}$ is called a *spacetime*. We have already seen an example of this when we noted that the material derivative on a manifold is actually a covariant derivative on an associated spacetime. Let $\boldsymbol{u}(\ ,t)$ be the flow field on $\mathcal{S}$ and consider $\boldsymbol{u} \in \mathfrak{X}(\mathcal{M})$. Furthermore, let $p, \rho \in C^\infty(\mathcal{M})$ represent pressure and density, respectively. Then *Euler's equation* is

$$\nabla_{\partial_t+\boldsymbol{u}}\boldsymbol{u} = -\frac{1}{\rho}\text{grad } p, \quad \text{or, dually,}$$

$$\nabla_{\partial_t+\boldsymbol{u}}\boldsymbol{u}^\flat = -\frac{1}{\rho}dp.$$

A *barotropic* flow is one in which $p = f(\rho)$. Since $\nabla_{\partial_t+\boldsymbol{u}}dt = 0$, $\nabla_{\partial_t+\boldsymbol{u}}\boldsymbol{u}^\flat = \nabla_{\partial_t+\boldsymbol{u}}(\partial_t+\boldsymbol{u})^\flat$, and in case $\boldsymbol{u}$ is barotropic, the Euler equation becomes

$$\nabla_{\partial_t+\boldsymbol{u}}(\partial_t+\boldsymbol{u})^\flat \quad \text{is exact.}$$

In this case it follows from Proposition 3.1 that $\pounds_{\partial_t+\boldsymbol{u}}d(\partial_t+\boldsymbol{u})^\flat = 0$.

Theorem 4.2 (Cauchy-Lagrange formula) *Let $\boldsymbol{u} \in \mathfrak{X}(\mathcal{S})$ be a time-dependent vector field satisfying the Euler equation on $\mathcal{S}\times\mathbb{R}$ with a barotropic flow. If ϕ_t is the flow of $\boldsymbol{u}$, then $\phi_t^*(d\boldsymbol{u}_t^\flat) = d\boldsymbol{u}_0^\flat$.*

Proof: Define $\mu_t : \mathcal{S} \to \mathcal{S}\times\mathbb{R} : \boldsymbol{x} \mapsto (\boldsymbol{x}, t)$ and note that $d\mu_t^*(\partial_t+\boldsymbol{u})^\flat = d\boldsymbol{u}_t^\flat$. Define a flow by $\psi_t(\boldsymbol{x}, s) := (\phi_{s+t}\phi_s^{-1}(\boldsymbol{x}), s+t)$ and observe that this is the flow of $\partial_t+\boldsymbol{u}$. Also it is clear that $\mu_s \circ \phi_s = \psi_s \circ \mu_0$. Now compute

$$\begin{aligned}
\phi_t^*(d\boldsymbol{u}_t^\flat) &= \phi_t^* d(\mu_t^*(\partial_t+\boldsymbol{u})^\flat) = \phi_t^*\mu_t^* d(\partial_t+\boldsymbol{u})^\flat \\
&= \mu_0^*\psi_t^* d(\partial_t+\boldsymbol{u})^\flat = \mu_0^* d(\partial_t+\boldsymbol{u})^\flat \\
&= d\boldsymbol{u}_0^\flat.
\end{aligned}$$

□

A *relativistic fluid flow* is a vector field $\boldsymbol{u} \in \mathfrak{X}$ on a Riemannian manifold $(\mathcal{M}, g)$ such that $g(\boldsymbol{u}, \boldsymbol{u}) = 1$. A vector field $\boldsymbol{v} \in \mathfrak{X}$ on $(\mathcal{M}, g)$ is called a *Killing field* if $\pounds_{\boldsymbol{v}} g = 0$. A relativistic fluid flow is called *isometric* if there is a nonzero Killing field parallel to the velocity; *i.e.*, $\nabla_{\boldsymbol{u}}\boldsymbol{v} = 0$ and $\pounds_{\boldsymbol{v}} g = 0$. It is a result in [5] that vorticity is conserved only when the flow is (at least) locally isometric.

Remark 4.3 An *incompressible* flow preserves the volume of parcels. An isometric flow preserves not only the volume of parcels, but also lengths and angles, and anything measured by the metric g.

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