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Lectures on Differential Manifolds, Bundles, and Groups

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Preface

In 1972 Serge Lang wrote, "The three great modern 'differential' theories [are] differential topology, differential geometry, and differential equations." Now we should add differential dynamics to this list.

In these lecture notes I focus on the common aspect of all these differential theories: smooth *manifolds*, *bundles*, *groups*, and structures *inherent* on them rather than *imposed* on them.

For some of this I consulted Bröcker & Jänich [8] and Lang [44], but followed [17] for the best ideas about bundles. I also used Poor [66, Ch. 1] and Warner [82, Ch. 1–3] for some things.

One unique feature is that I introduce all the different types of bundles in the same way, *via* transition cocycles, so that they do not appear so different from one another. Another is coverage of the natural symplectic structure on the cotangent bundle, always omitted from introductory texts. Yet another is the treatment of smooth groups, usually called Lie groups, largely based on Tondeur [77]. I admired this as a graduate student, and have been continually bemused that no one else ever used it, or even seemed to appreciate it much.

It will be apparent that this book has a somewhat more "algebraic" flavor than some others on the same general subject. In particular, concepts and constructions frequently are expressed *via* categories and functors. The book of Nestruev [61] is relatively close in spirit for the more classical algebraic aspects, eliding the category theory and functional analysis for example. There is always a choice between coverage and prerequisites: between completeness and accessibility.

These lectures were prepared for a group of mostly second year graduate students who had already taken at least one semester each of real analysis, algebra, and topology before we began. I have taken advantage of this by approaching some things in a more advanced way than is customary in the usual advanced-undergraduate/beginning-graduate level texts, and by including many things that are never seen there. I hope that even those already familiar with manifolds, bundles, and Lie groups will find some new ideas and new insights into old ideas.

In truth, I wrote the book as much for myself as for my students. There are quite a few things that seem to be taken for granted in many journal articles that I could not find in any book about smooth manifolds. I hope this book will spare others, especially newcomers, most of the time and effort I spent extracting those things from other journal articles over the past 25 years.

Finally, I wish to thank R. Balakrishnan and J. Ryan for transcribing my lectures into LaTeX. I also wish to thank J. Ryan and S. Sahraei for help with indexing, and all three plus K. Freeman and J. Shamas for comments, suggestions, and clarifications. I thank G. Greenfield for advice on algebra in general, and M. Patel for help with commutative algebras and some scheme theory.

Foreword

Most exercises are more conceptual than computational: determining how to think about the situation is often harder than doing the work. Some may appear to be nothing but tedious slogging through brutal calculations: that is precisely how one learns what it is like to *use* that system for detailed calculating, and why one might prefer one system over another for some computations.

As always, the only way to understand mathematics is to do it and use it. To encourage this, **Ex** denotes an exercise; but the choice is usually up to you, the reader, on the amount of work you wish to do. Some are explicitly stated directions to write something out: here the claim is that doing so is truly necessary for a *reasonable* understanding of the conceptual and computational machinery. Some are explicitly stated as (unanswered) questions. In such cases, the implicit claim is that you will greatly benefit from at least *thinking* about how to answer them. Those appearing as bald statements of fact may be safely taken as such, but I urge you to play with at least a few of them.

Contents

Standard Symbols

Number sets and friend

$\mathbb{N}$ natural

$\mathbb{Z}$ integer

$\mathbb{Q}$ rational

$\mathbb{R}$ real

$\mathbb{C}$ complex

$\mathbb{T}$ 1-torus S^1

Categories

Aff affine spaces (usually real and finite-dimensional)

Alg algebras

Bdl bundles

cgH k-spaces

Grp groups

Hsdf Hausdorff spaces

LAlg Lie algebras

LGrp Lie groups (sometimes their germs)

Mfld manifolds

Mod modules

Rng rings

Set sets

Top topological spaces

Vec vector spaces (usually real and finite-dimensional)

1 Preliminaries

Here we review some relevant concepts and results from elementary [4] and functional [79] analysis, algebra [51, 77], and topology [72, 74].

1.1 Analysis

Let $U \subseteq \mathbb{R}^m$ be open and $f : U \to V \subseteq \mathbb{R}^n$ be *smooth*, where V is open. Then $Df : U \times \mathbb{R}^m \to V \times \mathbb{R}^n$ is given by $(x, v) \mapsto (f(x), Df_x(v))$ where Df_x is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, for each $x \in U$. With respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$, Df_x is the *Jacobian matrix* of f at x and can be written as

$$\left[(Df_x(v))^i_j\right] = \begin{bmatrix} \dfrac{\partial f^1}{\partial x^1} & \dfrac{\partial f^1}{\partial x^2} & \cdots & \dfrac{\partial f^1}{\partial x^m} \\ \dfrac{\partial f^2}{\partial x^1} & \dfrac{\partial f^2}{\partial x^2} & \cdots & \dfrac{\partial f^2}{\partial x^m} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial f^n}{\partial x^1} & \dfrac{\partial f^n}{\partial x^2} & \cdots & \dfrac{\partial f^n}{\partial x^m} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^m \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f^1}{\partial x^j} \cdot v^j \\ \dfrac{\partial f^2}{\partial x^j} \cdot v^j \\ \vdots \\ \dfrac{\partial f^n}{\partial x^j} \cdot v^j \end{bmatrix}$$

where $x = \left(x^1, x^2, \ldots, x^m\right)^T$, $f(x) = \left(f^1(x), f^2(x), \ldots, f^n(x)\right)^T$, and $v = \left(v^1, v^2, \ldots, v^m\right)^T$ with ^{T} denoting the transpose. Note we denote a matrix A by $A = [a^i_j]$, where a^i_j is the element in the i-th row and j-th column. We shall also employ Einstein's summation convention, by which the preceding matrix equation becomes

$$(Df_x(v))^i = \frac{\partial f^i}{\partial x^j} \cdot v^j = \sum_{j=1}^{m} \frac{\partial f^i}{\partial x^j} \cdot v^j. \tag{1.1.1}$$

Observe that summation is assumed on any index appearing as a superscript and as a subscript in the same term whenever the term looks like a product.

Furthermore, the elements of a vector space, also known as *contravariant* vectors, will be denoted with superscripts, as in $x = \left(x^1, x^2, \ldots, x^m\right)^T$. Similarly, the elements of the dual vector space, *covariant* vectors, will be denoted with subscripts, as in $y = (y_1, y_2, \ldots, y_n)$.

Since x acts as a parameter in Df_x we can think of it as $Df : U \to L(\mathbb{R}^m, \mathbb{R}^n)$, the space of all linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$. This {Df} operator D has the following properties:

1. $D(\alpha) = 0$, where α is a constant map;
2. $D(L) = L$, where L is a linear transformation;
3. $(Dg \circ Df)(v) = Dg \circ Df_p(v) = Dg_{f(p)}(Df_p(v)) = D(g \circ f)_p(v)$, which implies that $D(g \circ f) = Dg \circ Df$, the chain rule;
4. $D(1_U) = 1_{\mathbb{R}^n}$;
5. This diagram commutes:

$$\begin{array}{ccc} U \times \mathbb{R}^m & \xrightarrow{f \times Df} & V \times \mathbb{R}^n \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U & \xrightarrow[f]{} & V \end{array}$$

{Dfct} The last three mean D is a functor. (Between which categories?)

Here is a version of the Inverse Function Theorem, which Bartle calls the Inversion Theorem [4, 41.8].

{bift} **Theorem 1.1.1** *Let $\Omega \subseteq \mathbb{R}^n$ be open and suppose that $f : \Omega \to \mathbb{R}^m$ is C^1. If $c \in \Omega$ is such that $Df(c)$ is a bijection, then there exists an open neighborhood U of c such that $V = f(U)$ is an open neighborhood of $f(c)$ and $f|_U : U \to V$ is a bijection with continuous inverse g. Moreover, g is C^1 on V and $Dg(y) = Df(g(y))^{-1}$ for all $y \in V$.*

Next, the Implicit Function Theorem [4, 41.9].

{bmft} **Theorem 1.1.2** *Let $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ be open. If $F : \Omega \to \mathbb{R}^m$ is C^1 with $F(0) = 0_m$ and $\ker(DF_0) = \mathbb{R}^n \times \{0_m\}$, then there exists W an open neighborhood of 0_n, a unique $f : W \to \mathbb{R}^m$ with $f(0_n) = 0_m$, and an open neighborhood U of 0_{n+m} such that for all $(x, y) \in U$, $F(x, y) = 0_m$ if and only if $y = f(x)$.*

In other words, $F^{-1}(0_m) \cap U = \Gamma(f) \cap U$ where $\Gamma(f)$ denotes the graph of f; also, we may as well assume $W = \mathrm{pr}_1(U)$.

The usual induction argument extends both to smooth functions. One may wish to review the entire Section 41 in [4], *Mapping Theorems and Implicit Functions.*

Now a somewhat less elementary result that is also convenient.

Theorem 1.1.3 (Sard) *If f is any smooth function $\mathbb{R}^n \to \mathbb{R}^m$, then the set of critical points of f is of (Lebesgue) measure zero.* {sard}

We continue with some functional analysis [79]. Recall that a *topological vector space* (TVS) is a vector space E together with a translation-invariant topology in which vector addition and scalar multiplication are continuous. It is *locally convex* (LC) if and only if there is a convex basis of neighborhoods at 0. It is *metrizable* if the topology can be induced by a metric.

A *seminorm* p on a topological vector space E is a continuous real-valued function on E such that:

(i) $p(x) \geq 0$ for all $x \in E$;

(ii) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$; and

(iii) $p(\lambda x) = |\lambda| p(x)$ for all scalars λ and all $x \in E$.

A family of seminorms $\{p_i\}$ determines a topology *via* the sets $\{p_i(x) \leq r\}$ for $r \geq 0$ and i in some index set. It is locally convex, and is Hausdorff if and only if $\bigcap_i \{p_i(x) = 0\} = \{0\}$. A family of seminorms $\{p_i\}$ is a *basis* if and only if for every seminorm q there exist a seminorm p_i and a positive constant C such that $q \leq C p_i$ on E. Furthermore, E is metrizable if and only if there exists a countable basis of seminorms.

Ex 1.1.4 Let E be a metrizable LCTVS, $(p_j)_{j \geq 1}$ a nondecreasing countable basis of seminorms, and $(a_j)_{j \geq 1}$ a sequence with $a_j > 0$ and $\sum_j a_j$ converges. Define {tim}

$$d(x,y) := \sum_{j=1}^{\infty} a_j \frac{p_j(x-y)}{1 + p_j(x-y)} \,.$$

Show that d is a translation-invariant metric for E [79, p. 71].

A *Fréchet space* is a metrizable TVS with a translation-invariant metric given in which it is complete. Clearly, Fréchet spaces are Hausdorff.

Example 1.1.5 We will show that if $\Omega \subseteq \mathbb{R}^n$ is open, then $C^\infty(\Omega)$ is a Fréchet space. {fsx1}

Observe that we can write $\Omega = \bigcup_{j=1}^{\infty} K_j$ where each K_j is compact and $K_j \subseteq \mathring{K}_{j+1}$. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ with each $\alpha_i \in \mathbb{N}$ and define $|\alpha| := \sum_{i=1}^{n} \alpha_i$. We define a countable family of seminorms as follows:

$$|f|_{m,K_j} = \sup_{|\alpha|<m} \sup_{K_j} |D^\alpha f(x)|$$

where x varies in K_j and

$$D^\alpha = \Big(\frac{\partial}{\partial x^1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x^n}\Big)^{\alpha_n}.$$

Thus $C^\infty(\Omega)$ is locally convex and metrizable. We may use the metric from Ex 1.1.4. To show that it is complete, it suffices to consider Cauchy sequences [79, p. 89]. This topology is called the *Schwartz* or *weak C^∞ topology;* it is the topology of uniform convergence of functions and all their derivatives on compact sets.

In general, Hausdorff LCTVSs have two distinct completed tensor products: one with the projective (initial, induced) topology (π) and one with the injective (final, coinduced) topology (ε). When they coincide, it is worthy of special note.

{ntvs} **Definition 1.1.6** For Hausdorff LCTVSs E and F, E is *nuclear* if and only if $E \mathbin{\widehat{\otimes}_\pi} F \cong E \mathbin{\widehat{\otimes}_\varepsilon} F$. Moreover, the isomorphism is natural.

Almost all function spaces relevant to manifolds are nuclear, and many are also Fréchet. Here is a short list of examples [79, p. 530]. Let Ω be open and K be compact in $\mathbb{R}^n$.

- $C^\infty(\Omega)$ is both.
- $C_c^\infty(\Omega)$ is nuclear.
- $C_c^\infty(K)$ is both.
- $\mathscr{S}(\mathbb{R}^n)$ is nuclear.

Recall that the last one is the space of rapidly decreasing (roughly, faster than 1 over any polynomial) smooth functions. Rapid decrease can be slower than exponential. It is square-integrable globally.

Here are some "stability" results for nuclear spaces [79, p. 514]. The following operations on nuclear spaces result in a nuclear space: strong dual, completion, subspace, quotient by a closed subspace, product, Hausdorff inverse (projective) limit, countable direct (inductive, injective) limit, and finite completed tensor product.

For nuclear spaces E and F, $L(E,F) \cong E'\widehat{\otimes}F$ and $L(E',F) \cong E\widehat{\otimes}F$ where E' is the strong dual of E and $L(\,,\,)$ has the topology of uniform convergence on bounded sets [79, p. 525].

What really sets complete nuclear spaces apart from Banach and Hilbert spaces is that the Heine-Borel (Montel) property holds: *closed and bounded sets are compact.* If the space is not complete, closed and bounded sets are *precompact* [79, p. 519].

1.2 Algebra

The elementary category theory needed may be found in Chapts. IV and XV of MacLane and Birkhoff [51] or in Dodson [16], among many other sources. For modules see Chapt. V and the latter half of Chapt. IX in [51]; for lattices see Chapt. IV.

MacLane and Birkhoff [51, III.3] define the localization K_P of an integral domain K at a prime ideal P. We shall need the general version for K a commutative ring with 1. The construction almost mimics that of the field of quotients in [51, III.5] but for two significant differences, both due to the possibility of having zero-divisors in K. Let $S := K \neg P$, and note that S is *multiplicatively closed*: if $a, b \in S$, then $ab \in S$. This S is the set of possible denominators. We also modify the equivalence relation on $K \times S$ to

$$(a,b) \sim (c,d) \text{ if and only if } s(ad - bc) = 0$$

for some $s \in S$. Elements in K_P are equivalence classes of this relation, and are denoted by either $[a,b]$ or a/b, as usual.

Ex 1.2.1 The standard remark here is that the latter modification is necessary for $\sim$ to be transitive. Can you give an explicit example illustrating the need for this? {trnsty}

Ex 1.2.2 The quotient space K_P is in fact a *local ring*: it has the unique maximal ideal P_P. More generally, all ideals of K_P are localizations I_P of ideals in K. {locrng}

The rest of this section is a review of group actions. I have followed Chapt. 1 of Tondeur [77] rather closely. This exposition is more thorough than most of the other review topics as this material is not typically included in standard courses in much detail.

1.2.1 actions

Let C denote a category (usually concrete) in which we shall work, let X be an object in C, and let G be a group. Recall that $\mathrm{Aut}(X) = \{f \mid f : X \cong X\}$ is a group (is in Grp).

{act} **Definition 1.2.1.3** An *action* of G on X is a morphism $\tau : G \to \mathrm{Aut}(X)$. We call X a *G-object* (with respect to τ).

Thus an action of G on X is a *representation* of G on X. Simply put, a representation is just a morphism from a group to some other group. Recall that a representation is *faithful* when the morphism is monic and *full* when it is epic. When X is a module the representation is called *linear*.

Example 1.2.1.4 A G-object in Set is a set X with a morphism $\tau : G \to S(X)$, where $S(X)$ is the group of permutations of X.

Equivalently, we may regard $\tau : G \times X \to X$ by $(g, x) \mapsto \tau_g(x)$ such that $\tau_{g_1 g_2} = \tau_{g_1}\tau_{g_2}$ and $\tau_1 = 1_X$. More precisely, this is a *left* action of G on X. A morphism $G^{\mathrm{op}} \to \mathrm{Aut}(X)$ is then a *right* action of G on X. Default actions are left unless otherwise specified.

{lt} **Example 1.2.1.5** Let G be a group; denote by L_g the *left* translation by $g \in G$; *i.e.,* $L_g x = gx$. Then $g \mapsto L_g$ is a morphism from G to $S(X)$. This action is called the *left regular representation* of G on itself; see [51].

Example 1.2.1.6 Let $\rho : G \to H$ be a morphism. We get an action τ of G on H *via* $\tau_g = L_{\rho(g)} \in S(H)$.

{sbg} **Example 1.2.1.7** Let H be a subgroup of G and consider the map $G \times H \to G$ obtained by restricting the binary operation $G \times G \to G$. This defines a right action of H on G by translation.

{cj} **Example 1.2.1.8** Let G be a group, $g \in G$ and κ_g be *conjugation* by g; that is, for any $x \in G$ we have $\kappa_g(x) = gxg^{-1}$. Thus $\kappa_g \in \mathrm{Inn}(G)$, the group of inner automorphisms. We call κ the action of G on itself *by conjugation*; see also [51]. Note that $\kappa : G \to \mathrm{Inn}\, G \leq \mathrm{Aut}\, G \leq S(G)$.

{lat} **Ex 1.2.1.9** Consider the lattice of subgroups $\Lambda(G)$. If $H \leq G$, then each $\kappa_g \in \mathrm{Inn}(G)$ acts *via* $H \mapsto gHg^{-1}$. Note that each conjugate of a subgroup is a subgroup. Determine how this action on $\Lambda(G)$ behaves with respect to the lattice structure on $\Lambda(G)$.

Example 1.2.1.10 Let H act on N by means of $\theta : H \to \mathrm{Aut}\, N$. On the set $N \times H$ define a binary operation by {sdp}

$$(n_1, h_1)(n_2, h_2) := (n_1\theta_{h_1}(n_2), h_1h_2)\,.$$

The set $N\times H$ with this binary operation makes it a group called the *semidirect product* and denoted as $N \ltimes_\theta H$.

Ex 1.2.1.11 Prove that $N \ltimes_\theta H$ is in fact a group.

We have these group morphisms.

$i : N \to N \ltimes_\theta H : n \mapsto (n, 1)$

$j : H \to N \ltimes_\theta H : h \mapsto (1, h)$

$p : N \ltimes_\theta H \to H$ as projection on the second factor

Ex 1.2.1.12 The following is a short exact sequence and j splits it; *i.e.*, $pj = 1_H$.

$$1 \longrightarrow N \xrightarrow{\;i\;} N \ltimes_\theta H \xrightarrow{\;p\;} H \longrightarrow 1$$

Conversely, any splitting $s : H \to G$ of the short exact sequence $1 \to N \to G \to H \to 1$ induces an action of H on N (*i.e.*, a $\theta : H \to \mathrm{Aut}\, N$) such that $G \cong N \ltimes_\theta H$. [Hint: Consider $\kappa_{s(h)} \in \mathrm{Inn}\, G$.] Prove that $N \trianglelefteq G$. Conclude that H-groups are bijective with such split short exact sequences.

Example 1.2.1.13 Let V be a finite-dimensional vector space over $\mathbb{R}$ and $GL(V)$ the automorphism group. Then $GL(V)$ acts by definition on V. For appropriate θ, define $GA(V) := V \ltimes_\theta GL(V)$, the group of *affine* motions of V. {affgrp}

One may wish to think of the *semidirect product* as an algebraic analogue of a nontrivial fiber bundle.

Next we examine the effect of a functor on a group action.

Proposition 1.2.1.14 *Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor. An action of G on $X \in \mathcal{C}_1$ induces a well-defined action of G on $\mathcal{F}(X) \in \mathcal{C}_2$. If $\mathcal{F}$ is a cofunctor, then an action of G on X induces a well-defined action of G^{op} on $\mathcal{F}(X)$.* {abf}

Ex 1.2.1.15 Prove this result. [Hint: $\mathcal{F}$ defines a morphism $\operatorname{Aut} X \to \operatorname{Aut} \mathcal{F}(X)$. Now compose $G \to \operatorname{Aut} X \to \operatorname{Aut} \mathcal{F}(X)$. The proof then follows.] Is this induced action natural (in any sense)?

{ps} **Example 1.2.1.16** Let $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ be the *power set* functor defined as follows: for each set X there is a corresponding $\mathcal{P}X$ and to each map $X \to X'$ there is a corresponding induced map $\mathcal{P}X \to \mathcal{P}X'$ of subsets. Let X be a G-set. This implies that $\mathcal{P}X$ is a G-set by Proposition 1.2.1.14. The functor $\mathcal{P}^{-1} : \mathsf{Set} \to \mathsf{Set}$ has the same effect as $\mathcal{P}$ on objects of Set but for a map $f : X \to X'$ it assigns $\mathcal{P}^{-1}(f) = f^{-1}$. Thus $\mathcal{P}^{-1}X$ is a G^{op}-set.

{chf} **Example 1.2.1.17** Let C be a fixed object in C. The hom-cofunctor (the contravariant hom-functor) $h^C = \hom(\ ,C)$ maps arrows by $h^C(\alpha)f = f\alpha$ for any $f \in \hom(X', C)$ and $\alpha : X \to X'$. Any left action of G on X induces a right action of G on $\hom(X, C)$. If $\tau : G \to \operatorname{Aut} X$ is a given morphism, we write τ^* for the induced morphism of G^{op} into the group of automorphisms of $\hom(X, C)$.

{rmod} **Example 1.2.1.18** Let R be a ring and $R\mathsf{Mod}$ be the category of all left R-modules. A *G-module* A is defined by a left action of G on A *via* R-linear morphisms; *i.e.,* a linear representation of G in $\operatorname{Aut}_R A$. By previous results, this induces an action of G on the lattice of submodules of A.

Ex 1.2.1.19 Verify the action is by lattice morphisms.

{inv} **Definition 1.2.1.20** An element x_0 (subobject Y) of the G-object X is called *(G-)invariant* if and only if $\tau_g(x_0) = x_0$ ($\tau_g(Y) \subseteq Y$) for all $g \in G$. Invariant elements are also called *fixed points.*

{hom} **Ex 1.2.1.21** Let X, X' be G-objects in C with respect to morphisms $\tau : G \to \operatorname{Aut} X$ and $\tau' : G \to \operatorname{Aut} X'$. Then $\sigma_g(f) = \tau'_g f \tau_{g^{-1}}$ for any $f \in \hom(X, X')$, defines an action of G on the arrows from X to X'. The previous example is the special case for the trivial action $\tau'_g = 1_{X'}$. Find the functor inducing this action according to Proposition 1.2.1.14.

Let G and G' be groups and C a category. Suppose X is a G-object in C with $\tau : G \to \operatorname{Aut} X$ and X' is a G'-object in C with $\tau' : G' \to \operatorname{Aut} X'$. Let $\rho : G \to G'$ be a morphism of groups.

{eqv} **Definition 1.2.1.22** An arrow $\alpha : X \to X'$ is called *ρ-equivariant* if and only if for all $g \in G$ this diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ {\scriptstyle\tau_g}\downarrow & & \downarrow{\scriptstyle\tau'_{\rho(g)}} \\ X & \xrightarrow[\alpha]{} & X' \end{array}$$

We denote the collection of all such ρ-equivariant arrows by $\hom^{\rho}_{\mathsf{C}}(X, X')$. Intuitively, equivariant arrows commute with group actions. If $G = G'$ and $\rho = 1_G$, then α is just called *equivariant* and the notation becomes $\hom^{\mathrm{eqv}}_{\mathsf{C}}(X, X')$.

Remark 1.2.1.23 The group G may be regarded as a category G with one object $*$ and all arrows invertible. The elements of G are the arrows $* \to *$. Now a functor $\tau : \mathsf{G} \to \mathsf{C}$ is a G-action on $\tau(*) \in \mathsf{C}$. A ρ-equivariant arrow is then a natural transformation of functors $\tau \mapsto \tau'$. When τ, τ' are linear actions (linear representations), equivariant is called *intertwining*. {cat}

Example 1.2.1.24 Let X be a G-set and X' a G'-set. Let $f : X \to X'$ and $\rho \in \hom(G, G')$. Then f is ρ-equivariant if and only if the following diagram commutes.

$$\begin{array}{ccc} G \times X & \xrightarrow{\tau} & X \\ {\scriptstyle\rho\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X' & \xrightarrow[\tau']{} & X' \end{array}$$

Example 1.2.1.25 Let $\rho : G \to G'$ be a morphism of groups. Let G, G' act on themselves by left translation (also known as the left regular representations). Then $\varphi : G \to G'$ is ρ-equivariant if and only if $\varphi(g_1 g_2) = \rho(g_1)\varphi(g_2)$. Thus ρ is ρ-equivariant with respect to left translations (and dually for right). {lrceqv}

Now let G, G' act on themselves by conjugation. Then the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & G' \\ {\scriptstyle\kappa_g}\downarrow & & \downarrow{\scriptstyle\kappa_{\rho(g)}} \\ G & \xrightarrow[\rho]{} & G' \end{array}$$

Ex 1.2.1.26 So what?

Example 1.2.1.27 Let H be a subgroup of G and H act on G as in Example {sbgeqv}
1.2.1.7. Let $\rho : G \to G$ be a morphism; this maps $H \to H$. Then ρ is $\rho|_H$-equivariant.

{assoc} **Example 1.2.1.28** Any right translation (of the group G) is equivariant for the action of left translation of G on itself (and dually). This is just the associative property of the binary operation on G.

{cjeqv} **Ex 1.2.1.29** Let $\tau : G \to \operatorname{Aut} X$ be an action (recall, default left). Then for any $g \in G$, $\tau_g : X \to X$ is κ_g-equivariant.

Ex 1.2.1.30 Let X be a G-set and $x_0 \in X$ be fixed. Then $p(g) = \tau_g(x_0)$ defines $p : G \to X$. This p is left-translation-equivariant.

{eqvcat} **Ex 1.2.1.31** For G fixed, G-objects in a given category C and equivariant arrows comprise a new category C^G. If one regards G as a category as in Remark 1.2.1.23, this is the functor category C^G.

We continue along the trail from Proposition 1.2.1.14.

{feqv} **Theorem 1.2.1.32** *Let $\mathcal{F} : \mathsf{C}_1 \to \mathsf{C}_2$ be a (covariant) functor and let X, X' be G, G'-objects, respectively. Consider $\mathcal{F}X, \mathcal{F}X'$ as induced G, G' objects, respectively. Let $\rho : G \to G'$ be a group morphism and $\varphi : X \to X'$ be ρ-equivariant.*

Then $\mathcal{F}(\varphi) : \mathcal{F}X \to \mathcal{F}X'$ is ρ-equivariant with respect to these induced actions. Moreover, for a fixed G we obtain an extension of Prop. 1.2.1.14 to a functor $\mathcal{F}^G : \mathsf{C}_1^G \to \mathsf{C}_2^G$.

Proof: This diagram commutes by definition

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle \tau_g}\downarrow & & \downarrow{\scriptstyle \tau'\rho g} \\ X & \xrightarrow[\varphi]{} & X' \end{array}$$

whence, applying $\mathcal{F}$, so does this one.

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}X' \\ {\scriptstyle \mathcal{F}(\tau_g)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\tau'_{\rho g})} \\ \mathcal{F}X & \xrightarrow[\mathcal{F}(\varphi)]{} & \mathcal{F}X' \end{array}$$

The rest of the proof now follows easily. □

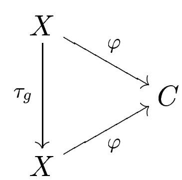
Ex 1.2.1.33 $\mathcal{F}^G$ is "canonical"—in what sense?

Ex 1.2.1.34 If G and G' remain distinct, can you make an analogous extension of Prop. 1.2.1.14? In other words, is there an $\mathcal{F}^\rho : \mathsf{C}_1^G \to \mathsf{C}_2^{G'}$?

Ex 1.2.1.35 The inverse of an equivariant morphism (if it exists) is also equivariant, and hence an isomorphism in C^G.

Define an inclusion functor $\mathcal{J} : \mathsf{C} \to \mathsf{C}^G$ *via* the trivial action $\tau : G \to \operatorname{Aut} X : g \mapsto 1_X$ for all $g \in G$. Then there exist equivariant morphisms $\varphi : X \to C \in \mathsf{C}$ for all $X \in \mathsf{C}^G$.

Definition 1.2.1.36 We say $\varphi : X \to C$ is an *invariant* arrow if and only if for all $g \in G$ the diagram commutes. {invar}



The collection of all such invariant arrows is denoted by $\hom_{\mathsf{C}}^{\text{inv}}(X, C)$.

Proposition 1.2.1.37 *Let X, X' be G, G'-sets (G, G'-objects), respectively, and $\varphi : X \to X'$ be ρ-equivariant with respect to $\rho : G \to G'$ a group morphism. If $x \in X$ ($Y \leq X$) is G-invariant, then $\varphi(x)$ ($\varphi(Y)$) is $\rho(G)$-invariant.* {equiv}

Proof: Note that $\tau_g(x) = x$. This implies that $\tau'_{\rho(g)}(\varphi(x)) = \varphi(\tau_g x) = \varphi(x)$. □

Ex 1.2.1.38 Let X, X' be G-objects in C, so $\hom(X, X')$ is a G-object by Ex 1.2.1.21. The invariant elements are equivariant morphisms $X \to X'$. As a special case, the invariant morphisms from X to C are the invariant elements for the action of Example 1.2.1.17.

1.2.2 orbits

Definition 1.2.2.1 Let X be a G-object and $\tau : G \to \operatorname{Aut} X$. We define the ($G$-)*orbit* (or τ-orbit) of an element $x \in X$ with respect to the given action as {orb}

$$\mathcal{O}(x) = [x] = Gx := \{\tau_g(x) \mid g \in G\}.$$

Ex 1.2.2.2 Extend the definition to subobjects.

Lemma 1.2.2.3 *For a G-set X, the orbits partition X.* {par}

Proof: Being in the same orbit is an equivalence relation: it is reflexive since $1 \in G$, symmetric since $\tau_g x = y$ if and only if $\tau_{g^{-1}} y = x$, and transitive since an action is a group morphism $G \to \operatorname{Aut} X$; see [51] p. 70. □

Therefore an orbit is an equivalence class for the *orbit equivalence relation.* We denote the equivalence quotient by X/G.

{uniq} **Lemma 1.2.2.4** *Let X be a G-object, $\pi : X \to X/G$ the natural projection, and C a fixed but arbitrary object in C. For any invariant $\varphi : X \to C$, there exists a unique $\psi : X/G \to C$ such that $\varphi = \psi\pi$.*

Proof: By universality of quotients; *e.g.,* see [51, Ch. I, Thm. 19]. □

Thus the orbit object, being universal, is well defined in any category. Its *existence* in a particular category is another matter.

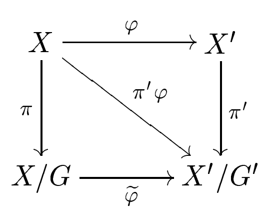
On the other hand, $\psi : X/G \to C$ composed with $\pi : X \to X/G$ gives an invariant morphism $\varphi = \psi\pi$. This proves

{unipi} **Theorem 1.2.2.5** *Let X be a G-object, $\pi : X \to X/G$ the natural projection, and $C \in \mathsf{C}$. The correspondence $\psi \mapsto \psi\pi$ is an isomorphism*

$$\hom_{\mathsf{C}}(X/G, C) \cong \hom_{\mathsf{C}}^{\mathrm{inv}}(X, C)$$

in C. □

{unio} **Corollary 1.2.2.6** *Let X be a G-object, X' a G'-object, $\rho : G \to G'$ a group morphism, and $\varphi : X \to X'$ be ρ-equivariant. Then there exists a unique $\widetilde{\varphi} : X/G \to X'/G'$ such that the diagram commutes.*



Proof: Note that

$$\begin{aligned}(\pi'\varphi)\tau_g &= \pi'(\varphi\tau_g) = \pi'(\tau'_{\rho g}\varphi) \\ &= (\pi'\tau'_{\rho g})\varphi = \pi'\varphi\,.\end{aligned}$$

□

Ex 1.2.2.7 Write out $\widetilde{\varphi}$ explicitly.

We follow [51] in defining a *right* coset of H in G to be gH, for $g \in G$.

Example 1.2.2.8 Let G be a group and let $H \leq G$ act on G by right translation as in Example 1.2.1.7. Then G/H is the orbit set and the elements are the right cosets of H (in G). Let G' and H' be defined similarly. Let $\varphi : G \to G'$ be defined by $gh \mapsto \varphi(g)\varphi(h)$, for $g \in G$ and $h \in H$. Thus φ is equivariant for the action of H on the right. Note that φ need *not* be a morphism of groups. Then $\varphi|_H : H \to H'$ is a group morphism and φ is $\varphi|_H$-equivariant. Corollary 1.2.2.6 implies that there exists a unique $\widetilde{\varphi} : G/H \to G'/H'$ such that the diagram commutes. {quouni}

$$\begin{array}{ccc} G & \xrightarrow{\ \varphi\ } & G' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ G/H & \xrightarrow[\widetilde{\varphi}]{} & G'/H' \end{array}$$

If H and H' are normal, this is the usual induced morphism of quotient groups.

Fix G and assume that X is a G-object. Then X/G is the orbit object. Corollary 1.2.2.6 implies that for any equivariant $\varphi : X \to X'$ there exists a unique $\widetilde{\varphi} : X/G \to X'/G$ whence it defines a functor

$$\mathcal{B} : \mathsf{C}^G \to \mathsf{C} : X \mapsto X/G : \varphi \mapsto \widetilde{\varphi}.$$

Therefore an isomorphism $\varphi : X \to X'$ in C^G induces an isomorphism $\widetilde{\varphi} : X/G \to X'/G$ in C.

Ex 1.2.2.9 Let $\mathcal{U} : \mathsf{C}^G \to \mathsf{C}$ be the forgetful functor. Prove that $\pi : \mathcal{U} \to \mathcal{B}$ may be regarded as a natural transformation.

Remark 1.2.2.10 Note that $\mathcal{O} : X \to \mathcal{P}X$ is equal to $(\pi^{-1} \circ \pi) : X \to \mathcal{P}X$. {oext} The latter extends naturally to $\mathcal{P}X \to \mathcal{P}X$ so $\mathcal{O}$ also can be so extended. We continue to denote this extension by $\mathcal{O}$.

If Y is a subobject of X, then $\mathcal{O}(Y)$ is the orbit of Y in $\mathcal{P}X$ *via* the induced G-action of Proposition 1.2.1.14 on $\mathcal{P}X$. That is,

$$\mathcal{O}(Y) = \{\tau_g(y) \mid g \in G, y \in Y\} = \bigcup_{x\in X} \{\mathcal{O}(x) \mid \mathcal{O}(x) \cap Y \neq \varnothing\}.$$

One calls $\mathcal{O}(Y)$ the *saturation* of Y with respect to the orbit equivalence relation.

The invariance of $Y \subseteq X$ can now be expressed by $\mathcal{O}(Y) = Y$. For any Y, the set $\mathcal{O}(Y)$ is the intersection of all invariant subsets of X containing Y. The orbits are the minimal invariant sets in X.

{itpy} **Definition 1.2.2.11** Let X be a G-object and $x \in X$. The *isotropy subgroup of* x (or *at* x) is defined as $G_x = \{g \in G \mid \tau_g(x) = x\}$.

{cjitpy} **Proposition 1.2.2.12** *Let X be a G-object, $x \in X$, and consider G_x. For any $g \in G$ we have $G_{\tau_g x} = gG_xg^{-1}$.*

Proof: For convenience, we write $\tau_g(x) = gx$. If $h \in G_x$ then $\left(ghg^{-1}\right) gx = g(hx) = gx$, so $gG_xg^{-1} \subseteq G_{gx}$.

Conversely, if $h \in G_{gx}$ then $\left(g^{-1}hg\right) x = \left(g^{-1}h\right) gx = x$. This gives $g^{-1}G_{gx}g \subseteq G_x$ or $G_{gx} \subseteq gG_xg^{-1}$. □

Alternatively, consider $\varphi : X \to \Lambda(G)$ by $x \mapsto G_x$. From Ex 1.2.1.9, a G-orbit in $\Lambda(G)$ is a conjugacy class of subgroups. By Proposition 1.2.2.12, this diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \Lambda(G) \\ {\scriptstyle \tau_g}\big\downarrow & & \big\downarrow{\scriptstyle \kappa_g} \\ X & \xrightarrow[\varphi]{} & \Lambda(G) \end{array}$$

so φ is equivariant. From Corollary 1.2.2.6, φ induces $\widetilde{\varphi} : X/G \to \Lambda(G)/G$.

{ot} **Definition 1.2.2.13** This $\widetilde{\varphi}$ maps each orbit to the *orbit type* conjugacy class.

Particular types are $\{1\}$ and G.

Let $x_0 \in \mathcal{O}(x_0)$ of orbit type $\{1\}$. Then for any $x \in \mathcal{O}(x_0)$ there exits a unique $g \in G$ such that $x = gx_0$. Indeed, $g_1x_0 = g_2x_0$ implies that $g_2^{-1}g_1x_0 = x_0$ which implies that $g_2^{-1}g_1 = 1$ and thus $g_1 = g_2$. This proves that g is unique.

On the other hand, orbit type G means that x_0 is invariant, or $\mathcal{O}(x_0) = \{x_0\}$. Thus fixed points are exactly the orbits of orbit type G.

{otx1} **Example 1.2.2.14** Consider the general linear group GL_n acting on $\mathbb{R}^n$ in the usual way. The origin 0 and its complement $\mathbb{R}^n - \{0\}$ are the orbits. The orbit-type of 0 is GL_n.

Example 1.2.2.15 Consider $\mathbb{R}^n$ with the standard Euclidean metric and the corresponding orthogonal group O_n. The orbits of this action are spheres with the origin as center (including 0 as the degenerate sphere of radius 0). The isotropy group of a point different from the origin is isomorphic to the orthogonal group O_{n-1}. {otx2}

Example 1.2.2.16 Let $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ denote the complex projective line (or Riemann sphere). Let G denote the group of all (fractional linear or Möbius) transformations {otx3}

$$z \longmapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We find that $\mathcal{O}(z) = \mathbb{C}P^1$ for any $z \in \mathbb{C}P^1$; the action is *transitive*.

Example 1.2.2.17 Let G be a group acting on itself by conjugation. The set of fixed points is the center of the group $Z(G)$. Referring to Example 1.2.1.9, we find that the orbit of a subgroup is its conjugacy class; *i.e.*, $\mathcal{O}(H)$ is the conjugacy class of the subgroup H of G. Therefore, the normal subgroups are the fixed points under this action. (This explains the older terminology of *invariant* subgroups.) Also, the conjugacy classes partition $\Lambda(G)$.

Here is the effect of an equivariant map on the isotropy groups.

Proposition 1.2.2.18 *Let X be a G-object, X' a G'-object, $\rho : G \to G'$ a morphism of groups, and $\varphi : X \to X'$ a ρ-equivariant map. Then $\rho(G_x) \subseteq G'_{\varphi(x)}$.* {eqvitpy}

Proof: Let $g \in G_x$. Then $\rho(g)\varphi(x) = \varphi(\tau_g x) = \varphi(x)$ so $\rho(g) \in G'_{\varphi(x)}$. □

Ex 1.2.2.19 Suppose X is a G-set, X' is a G'-set, $\rho : G \to G'$ is a group morphism and $\widetilde{\varphi} : X/G \to X'/G'$ is a map. Determine (or characterize) when $\widetilde{\varphi}$ is induced by a ρ-equivariant map $\varphi : X \to X'$ as in Corollary 1.2.2.6. {eqvitpyx}

Ex 1.2.2.20 Let X be a G-set, $\mathcal{O} : \mathcal{P}X \to \mathcal{P}X$ the extension of Remark 1.2.2.10. Show that $\mathcal{O}$ has these properties: {ostop}

1. $\mathcal{O}(\varnothing) = \varnothing$;

2. $Y \subseteq \mathcal{O}(Y)$ for $Y \subseteq X$;

3. $\mathcal{O}(\mathcal{O}(Y)) = \mathcal{O}(Y)$;

4. $\mathcal{O}(Y_1 \cup Y_2) = \mathcal{O}(Y_1) \cup \mathcal{O}(Y_2)$.

Then $\mathcal{O}$ is a *Kuratowski closure operator* [72, p. 139] and we may topologize X *via* $Y \subseteq X$ is closed if and only if $\mathcal{O}(Y) = Y$.

This remains true for any equivalence relation R on X and the extended saturation map $\mathcal{S} = \pi^{-1} \circ \pi : \mathcal{P}X \to \mathcal{P}X$ where $\pi : X \to X/R$ is the natural projection. Even more generally, for any relation R on X the saturation operator $\mathcal{S}(Y) = \{x \in X \mid yRx \text{ for some } y \in Y\}$ has properties 1–4 if and only if R is reflexive and transitive.

{ostopx1} **Ex 1.2.2.21** Consider the topology of Ex 1.2.2.20 on X with an equivalence relation R. Show that:

1. $Y \subseteq X$ is closed if and only if Y is a union of equivalence classes;

2. $Y \subseteq X$ is closed if and only if Y is open.

By looking at X/R determine when X is: second countable; compact; connected.

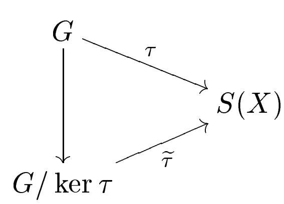
{ostopx2} **Ex 1.2.2.22** Let X be a G-set, Y a set, and $\varphi : X \to Y$. Let X be topologized as in Ex 1.2.2.20 and let Y have the discrete topology. Then φ is invariant if and only if it is continuous.

1.2.3 types

Let X be a G-set with $\tau : G \to S(X)$ the action.

{eff} **Definition 1.2.3.1** An action is *effective* if and only if τ is a monomorphism.

Note that $\ker \tau = \bigcap_{x \in X} G_x$. If τ is not effective, there is a factorization



with $\widetilde{\tau}$ effective.

{effx} **Example 1.2.3.2** The action of G on itself by conjugation κ has $\ker\kappa = Z(G)$.

Definition 1.2.3.3 An action is *free* (of fixed points) if and only if $\tau_g(x) = x$ for some $x \in X$ implies $g = 1$. {free}

Each isotropy group is trivial. We also say that X is a *principal* G-set. Note that free implies effective.

Ex 1.2.3.4 Any equivariant map of principal G-sets is an isomorphism. {mpgs}

Example 1.2.3.5 The action of G on itself by left (or right) translations is free. The action of $H \leq G$ by right (or left) translations (Example 1.2.1.7) is also free. {freex}

Definition 1.2.3.6 An action is *transitive* if and only if for $x, x' \in X$, there exists a $g \in G$, such that $\tau_g(x) = x'$. If g is unique, τ is *simply transitive*. If we replace x, x' by finite sets of k points each, we define a *k-transitive* action. {trans}

Simply transitive implies free, and a free action is simply transitive on each orbit. Indeed, if $x = gx_0 = g'x_0$, then $x_0 = g^{-1}g'x_0$ whence $g^{-1}g' \in G_{x_0} = \{1\}$, so $g = g'$.

Alternatively, an action is transitive if and only if there exists $x_0 \in X$ with $\mathcal{O}(x_0) = X$. Since X is the only orbit, X/G then is a point.

Definition 1.2.3.7 A G-set X is *homogeneous* if and only if the action is transitive, and a *G-torsor* if and only if it is simply transitive. {homog}

Remark 1.2.3.8 Let X be a G-torsor. Then we may define a G-valued binary operation on X as follows. If $y = gx$, define $y/x := g$. This is clearly well defined as the action of G is simply transitive; indeed, for each $(x, y) \in X \times X$ it is immediate that g is unique. (For a right action, we write $x\backslash y$ instead; when G is abelian we write $y - x$ as usual.) Now, just as a G-torsor may be regarded as a (bijective) copy of G in which one "forgets" where the identity element is, we may consider $(X, /)$ by "forgetting" about G. {heap}

One may use this nonstandard binary operation to define a genuine ternary operation on X *via* $(X \times X) \times X \to G \times X \to X : (x, y, z) \mapsto (x/y)z$ $[x(y\backslash z)$ for a right action]. With this more algebraic/categorical view, X is called a *heap*.

Example 1.2.3.9 The orthogonal group O_n acts transitively on $S^{n-1} \subseteq \mathbb{R}^n$. More generally, any action is transitive on each orbit. {on}

Example 1.2.3.10 Let V be a vector space over a field $\Bbbk$. If V acts simply transitively on a set $\mathbb{A}$, then $\mathbb{A}$ is an *affine space* (over $\Bbbk$) with *translations* V, or a V-torsor. This is equivalent to more traditional definitions of an affine space; see [51, pp. 564ff]. {affsp}

More generally, for a module M over a commutative ring K we can consider an M-torsor as an extension of *affine over a field* to *affine over a (commutative) ring.*

{hol} **Ex 1.2.3.11** The group of holomorphisms of the unit disk in $\mathbb{C}$ acts transitively. [What kind of unit disk: with or without boundary?]

The fundamental example of a homogeneous G-set is the following. Let H be a subgroup of G acting on G by right translations (see Example 1.2.1.7). We define an action σ of G on the orbit set G/H. Left translations $L_g : G \to G$ satisfy $L_g(g'H) = gg'H$ whence $\sigma_g(g'H) = (gg')H$ defines the *standard* action making G/H a G-set. Note that it is homogeneous and that the isotropy group at $H \in G/H$ is H itself. We shall show that this is the only kind of homogeneous G-set.

First, let X be an arbitrary G-set and $x_0 \in X$. Set $H = G_{x_0}$ and define $\varphi : G/H \to \mathcal{O}(x_0)$ by $gH \mapsto \tau_g x_0$.

{homogl} **Lemma 1.2.3.12** *φ is equivariant and injective.*

Proof: For $g \in G$, we have

$$(\tau_g \circ \varphi)(g'H) = \tau_g(\tau_g' x_0) = \tau_{gg'} x_0$$

and

$$(\varphi \circ \sigma_g)(g'H) = \varphi(gg'H) = \tau(gg')x_0\,,$$

whence $\tau_g \circ \varphi = \varphi \circ \sigma_g$. Thus φ is equivariant with respect to τ and σ.

To show φ is injective, consider $g_1, g_2 \in G$ such that $\varphi(g_1H) = \varphi(g_2H)$. Then $\tau_{g_1}x_0 = \tau_{g_2}x_0$ so that $\tau_{g_2^{-1}g_1}x_0 = x_0$ and $g_2^{-1}g_1 \in H$ whence $g_1 \in g_2H = g_1H$. □

If X is homogeneous, then $\mathcal{O}(x_0) = X$ and φ is an equivariant isomorphism.

{homog1} **Theorem 1.2.3.13 (Fundamental of Homogeneous Sets)** *If X is a homogeneous G-set, $x_0 \in X$, $H = G_{x_0}$ and $\sigma : G \to S(G/H)$ the standard action, then $\varphi : G/H \to X : gH \to \tau_g x_0$ is an isomorphism of G-sets.* □

The subgroup H of G depends on x_0 but its conjugacy class does not (by transitivity).

We conclude with some remarks on effective and transitive actions.

In view of the preceding theorem, we may as well consider homogeneous sets G/H. The kernel K of the standard action $\sigma : G \to S(G/H)$ is

$$K = \bigcap_{g \in G} gHg^{-1}.$$

Observe that K is normal in G and K is a (normal, by restriction) subgroup of H.

Conversely, if $L \trianglelefteq G$ with L a subgroup of H, then L is a subgroup of K. Indeed, $lgH = gl'H$ for some $l' \in L$ because $Lg = gL$ and $lgH = gH$, which means L is a subgroup of K. This proves

Theorem 1.2.3.14 *If G is a group, H a subgroup of G, and G acts on G/H via the standard action σ, then* $\ker \sigma$ *is the largest normal subgroup of G contained in H, given by* {ktr}

$$\ker \sigma = \bigcap_{g \in G} gHg^{-1}.$$

□

Corollary 1.2.3.15 *G acts effectively on G/H if and only if H contains no proper normal subgroups of G.* {etr} □

Note that H need *not* be a simple group.

Ex 1.2.3.16 Study the effect of the choice of x_0 in Theorem 1.2.3.13.

1.3 Topology

We shall work in the category *cgH* of compactly generated Hausdorff spaces [72, 74]. It will turn out that all smooth manifolds lie in *cgH*.

The goal is to find a *convenient category* of Hausdorff spaces which is:

1. large enough to contain all particular spaces commonly used;

2. closed under all standard operations; and

3. certain reasonable identities should hold (for instance, the commutativity of certain operations).

In particular, we want these to hold:

1. $C(X, Y \times Z)$ is isomorphic to $C(X, Y) \times C(X, Z)$;
2. $C(X \times Y, Z)$ is isomorphic to $C(X, C(Y, Z))$;
3. a product of identification maps/spaces is an identification map/space of the product;
4. a union of products is a product of unions; and
5. an identification space of a union is the union of the identification spaces.

Steenrod achieved this goal in his classic article [74]. We follow it to give a summary of relevant results.

{cg} **Definition 1.3.1** A Hausdorff space is *compactly generated* if and only if it has the weak topology relative to its compact sets [72]. These spaces are called *k-spaces* after their discoverer, John L. Kelley [39].

{cgx1} **Example 1.3.2** All locally compact or first countable Hausdorff spaces are k-spaces; in particular, all locally Euclidean Hausdorff spaces and all metric spaces are k-spaces.

{cgx2} **Example 1.3.3** For X compactly generated, any closed subset is also compactly generated. If $U \subseteq X$ is open and each $x \in U$ has a neighborhood V with $\overline{V} \subseteq U$, then U is compactly generated.

{idcg} **Proposition 1.3.4** *If $f : X \to Y$ is an identification map, X is a k-space, and Y is Hausdorff, then Y is a k-space.*

{cnt} **Proposition 1.3.5** *If X is compactly generated, Y is Hausdorff, and $f : X \to Y$ is continuous on compact sets in X, then f is continuous in the usual sense.*

{cgh} **Definition 1.3.6** Let Hsdf be the category of Hausdorff spaces and cgH the category of compactly generated Hausdorff spaces. Define a functor $k : \mathsf{Hsdf} \to \mathsf{cgH}$ by $k(X)$ is X made compactly generated and $k(f) = f$ for each f.

Remark 1.3.7 The application of k is called *Kelleyfication*. It makes each Hausdorff space into a k-space.

{corfl} **Theorem 1.3.8** *The following are true.*

1. *The identity function $1_X : k(X) \to X$ is continuous.*

2. *$k(X)$ is a Hausdorff space.*

3. *$k(X)$ and X have the same compact sets.*

4. *$k(X)$ is compactly generated.*

5. *If X is compactly generated, then $k(X) = X$ ($k^2 = 1$).*

6. *If $f : X \to Y$ is continuous on compact sets, then $k(f)$ is continuous.*

7. *$1_X : k(X) \to X$ induces isomorphisms of standard functors of algebraic topology.*

Alternatively, k is a *coreflection* of *Hsdf* in *cgH*: it is a right adjoint of the inclusion functor *cgH* $\hookrightarrow$ *Hsdf*. Thus *cgH* is a *coreflective* subcategory of *Hsdf*; it is also full, (small) complete and cocomplete, and Cartesian closed [50, VII.8].

A subcategory is *full* when the inclusion functor is epi on hom-sets. Thus k may create some new continuous functions but no old ones are destroyed. It preserves products because it is a right adjoint [50, V.5].

For clarity, denote the usual topological product in *Hsdf* by $X \times_t Y$.

Definition 1.3.9 Let $X, Y \in$ *cgH*. Then their product in *cgH* is defined by

$$X \times Y := k(X \times_t Y)\,.$$

Theorem 1.3.10 *The product $X \times Y$ for all $X, Y \in$ cgH is categorical.*

Proof: Any right adjoint functor preserves products and k is one such. □

Theorem 1.3.11 *If X is locally compact and $Y \in$ cgH, then $X \times_t Y \in$ cgH.*

Theorem 1.3.12 *If $f : X \to X'$ and $g : Y \to Y'$ are identification maps in cgH, then $f \times g : X \times Y \to X' \times Y'$ is also an identification map in cgH.* {prodid}

Lemma 1.3.13 *Let X and Y be two Hausdorff spaces. Then $k(X) \times k(Y)$ is homeomorphic to $k(X \times_t Y)$.* {prodl}

Definition 1.3.14 Suppose X is a subset of $Y \in$ *cgH* and let X_r be X with the relative topology inherited from Y. The *subspace* X of Y in *cgH* is given by $k(X_r)$. {sbsp}

Theorem 1.3.15 *If $f : X \hookrightarrow X'$ and $g : Y \hookrightarrow Y'$ are inclusions in cgH, then $f \times g : X \times Y \hookrightarrow X' \times Y'$ is also an inclusion in cgH.* {prodincl}

Definition 1.3.16 Let X and Y be Hausdorff spaces. We define {fcnsp}

$$Y^X := k(C(X,Y))$$

with the usual compact-open topology on $C(X,Y)$.

{eval} **Lemma 1.3.17** *If $X, Y \in$ cgH and the* evaluation *map*

$$e : C(X,Y) \times_t X \to Y : (f,x) \mapsto f(x)$$

is continuous on compact sets, then $e : Y^X \times X \to Y$ is also continuous.

{fsl} **Lemma 1.3.18** *If $X \in$ cgH and Y is a Hausdorff space, then $C(X, k(Y)) = C(X,Y)$ as sets and both topologies have the same compact sets. Thus*

$$k\,C(X,kY) \cong k\,C(X,Y)$$

in cgH.

{fsdist} **Theorem 1.3.19** *If $X, Y, Z \in$ cgH, then $(Y \times Z)^X \cong Y^X \times Z^X$.*

{fsit} **Theorem 1.3.20** *If $X, Y, Z \in$ cgH, then $Z^{X \times Y} \cong \left(Z^Y\right)^X$.*

{fscomp} **Theorem 1.3.21** *If $X, Y, Z \in$ cgH, then the composition $X \to Y \to Z$ is continuous as a map $Z^Y \times Y^X \to Z^X$.*

1.3.1 continuous actions

From now on, we assume all spaces to be k-spaces unless specified otherwise.

{tgrp} **Definition 1.3.1.1** A *topological group* G is a group with a topology such that $G \times G \to G$ given by $(g,h) \mapsto gh$ and $G \to G$ given by $g \mapsto g^{-1}$ are continuous.

Ex 1.3.1.2 Equivalently, $(g,h) \mapsto gh^{-1}$ is continuous.

{ttg} **Definition 1.3.1.3** A *topological G-space* X is a k-space and a G-set for a continuous action $G \to \operatorname{Aut} X$. The pair (G, X) is called a *topological transformation group* (or continuous transformation group).

Ex 1.3.1.4 Aut X is closed in X^X.

From Theorem 1.3.20, it follows immediately that continuity of $G \times X \to X$ is equivalent to continuity of $G \to \operatorname{Aut} X$ in cgH.

Definition 1.3.1.5 A *morphism* of topological groups is a continuous morphism of groups.

Let X and X' be G and G'-spaces, respectively, and $\rho : G \to G'$ a morphism of topological groups.

Definition 1.3.1.6 A ρ-equivariant *morphism* $\varphi : X \to X'$ is a continuous ρ-equivariant map of G-sets. {teqv}

Example 1.3.1.7 Let G be a topological group. Then left/right translation makes G a G-space, as does conjugation. {tltcj}

Remark 1.3.1.8 For X *any* topological space, setting $G = \operatorname{Aut} X$ with the discrete topology makes X a G-space.

Definition 1.3.1.9 Let X be a G-space, X/G the *orbit space*, and $\pi : X \to X/G$ the standard projection. So X/G has the quotient (or identification) topology, the strongest topology that makes π continuous (by definition), and the unique one making π continuous and open. {tos}

Ex 1.3.1.10 Verify the last claim. {tosx}

Now X/G is a k-space if X/G is Hausdorff by Proposition 1.3.4, and X/G is Hausdorff if the orbit equivalence relation (Lemma 1.2.2.3 and following) is closed by a standard result from general topology [72] (since π is open).

Example 1.3.1.11 Let G be a topological group and $H \leq G$ a subgroup. The action of H on G by right translations makes G an H-space with $\pi : G \to G/H$ continuous and open. If H is closed, G/H is a k-space. {tquo}

Theorem 1.3.1.12 *Let G be a topological group, X a G-space, $\pi : X \to X/G$ the projection, and Y an arbitrary space. Then* $\hom_{\mathsf{cgH}}(X/G, Y) \cong \hom^{\mathrm{inv}}_{\mathsf{cgH}}(X, Y)$ via $\psi \mapsto \psi \circ \pi$. □ {tunipi}

Corollary 1.3.1.13 *Let G, G' be topological groups, $\rho : G \to G'$ a morphism, and X, X' be G, G'-spaces, respectively. For each ρ-equivariant $\varphi : X \to X'$ there exists a unique $\widetilde{\varphi} : X/G \to X'/G'$ such that the following diagram commutes in cgH.* {tunio}

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ X/G & \xrightarrow[\widetilde{\varphi}]{} & X'/G' \end{array}$$

□

{tuniox1} **Example 1.3.1.14** Let X be a G-space with transitive action, $x_0 \in X$, and $H = G_{x_0}$ as in Theorem 1.2.3.13. Then $\varphi : G/H \to X$ is an isomorphism of G-sets and continuous but *not* necessarily a homeomorphism.

Indeed, let $\mathbb{R}$ act on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ by $\tau_\lambda(x_1, x_2) = (x_1 + \alpha(\lambda), x_2 + \alpha(\theta\lambda))$ where $\lambda \in \mathbb{R}$, $\alpha : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the projection, and θ is irrational. Fix a point $(x_1, x_2) \in \mathbb{T}^2$ and define $\varphi : \mathbb{R} \to \mathbb{T}^2$ as in 1.2.3.13; *i.e.*, $\varphi : \lambda \mapsto \tau_\lambda(x_1, x_2)$.

Now φ is continuous by construction and injective by Lemma 1.2.3.12. Consider $\mathcal{O}(x_1, x_2)$ with the relative topology. Then $\varphi : \mathbb{R} \to \mathcal{O}(x_1, x_2)$ is continuous and bijective, but *not* a homeomorphism: $\mathcal{O}(x_1, x_2)$ is dense in $\mathbb{T}^2$ and cannot be homeomorphic to the complete $\mathbb{R}$.

Ex 1.3.1.15 Do we really *need* completeness of $\mathbb{R}$? Find a better proof. How about using φ is not closed?

{afftvs} **Example 1.3.1.16** On the other hand, a simply transitive action is quite well behaved. In Example 1.2.3.10, if V is a TVS then the associated affine space $\mathbb{A}$ is homeomorphic to V. We leave it as an exercise for the reader to show that this is a general feature of simply transitive actions.

{osbgc} **Ex 1.3.1.17** Let G be a topological group and H be an open subgroup of G. Show that H is closed. [Hint: partition by cosets gH.]

{tgen} **Ex 1.3.1.18** Let G be a connected topological group and U be a neighborhood of 1. Set $V = U \cap U^{-1}$. Show that $V \subseteq U$ and $V^{-1} = V$. Now form the union $V^\infty = \bigcup_{i=1}^\infty V^i$ and note that $V^\infty \leq G$ is the subgroup generated by V. Since $1 \in V \subseteq V^\infty$, then 1 is in the interior. By translation, so is every other point of V^∞. Hence V^∞ is open, therefore closed by 1.3.1.17, and thus $V^\infty = G$ since G is connected. Provide any missing details.

{cc1cn} **Ex 1.3.1.19** Let G be a topological group with G_o the connected component of the identity. Show that G_o is a closed, normal subgroup of G. Note that G/G_o is a group with the discrete topology. What else can you prove about (or with) G/G_o?

We will also need the following more advanced result. Recall that a topological space is called *1-connected* if it is both connected and simply connected. A *local* morphism $G \to G'$ is a map $\varphi : U \to G'$, where U is an open neighborhood of $1 \in G$, such that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ whenever both sides are well defined.

Theorem 1.3.1.20 *If G is 1-connected, then any local morphism $\varphi : U \to G'$ of k-groups extends to a global morphism $\widetilde{\varphi} : G \to G'$.* {loc2glob}

1.3.2 covering spaces and sheaves

Continuous surjections that are local homeomorphisms are an important class of maps in topology. Here we consider two of the most notable.

Let $\widetilde{X}$ and X be connected, locally path connected spaces. (In general they need not be in *cgH* but we might as well assume they are.) Let $p : \widetilde{X} \twoheadrightarrow X$ be a continuous surjection (a projection) that is also a local homeomorphism. An open set U in X is said to be *evenly covered* if and only if the inverse image $p^{-1}U$ is a disjoint union (coproduct) of open sets in $\widetilde{X}$, each of which is homeomorphic to U.

Definition 1.3.2.1 The space $\widetilde{X}$ is called a *covering space* of X if and only if X has an open cover comprised of evenly covered open sets. If $\widetilde{X}$ is also simply connected, then it is called the *universal covering space* of X. {cvsp}

It is a standard fact [72] that the universal covering space is a covering space of any other covering space of X, whence the terminology. The main theorem on the existence of universal covering spaces [72, Thm. 14.5.3] requires a little homotopy theory. For our purposes, the following consequence of it will suffice.

Theorem 1.3.2.2 *Any connected locally Euclidean space has a universal covering space.* {leuc}

Note that each fiber $p^{-1}\{x\}$ is a discrete subspace of $\widetilde{X}$. It is simple to show that all fibers have the same cardinality n. One says that p is an *n-fold covering map* or that $\widetilde{X}$ is an *n-sheeted covering*.

Ex 1.3.2.3 Show that S^n is a double covering of real projective n-space $\mathbb{P}^n$. Moreover, it is the universal covering for $n \geq 2$.

Ex 1.3.2.4 If $\widetilde{G}$ is a covering space of a topological group G, then there {cstg}
exists a unique topological group structure on $\widetilde{G}$ such that the projection is a continuous group morphism.

{cstorus} **Ex 1.3.2.5** Show that $\mathbb{R}$ is the universal covering group of S^1. More generally, $\mathbb{R}^n$ is the universal covering group of $\mathbb{T}^n$

With some additional homotopy theory (PLP + HLP), one can show that there is a transitive action of the fundamental group $\pi_1(X)$ on each fiber of $\widetilde{X} \twoheadrightarrow X$ which is in general *not* effective.[72, 14.2.6]

It is commonly said that sheaves enable one to do algebra vertically and topology horizontally. We shall see that sheaves provide a means of encoding higher-order (such as differential) information topologically. It is also true that sheaves provide local-global relations of some other properties. See Warner [82, Ch. 5] for more on sheaves, or any of the several standard textbooks on sheaf theory.

Let X be a topological space. We may as well assume X is a k-space, since in our applications it will be a manifold. On the other hand, we let $\mathscr{S}$ be a completely arbitrary topological space; in fact it will usually *not* be Hausdorff.

{sheaf} **Definition 1.3.2.6** A *sheaf* on X is a topological space $\mathscr{S}$, called the *sheaf space* (French: *espace étalé*), together with a continuous, surjective, local homeomorphism $\pi : \mathscr{S} \twoheadrightarrow X$ such that

1. each *stalk* (fiber) $\mathscr{S}_x := \pi^{-1}\{x\}$ is an algebraic object (such as a K-module) with the discrete topology, and

2. all binary (algebraic) operations are continuous maps $\mathscr{S} \times_\pi \mathscr{S} \to \mathscr{S}$, and similarly for n-ary operations.

Recall that $\mathscr{S} \times_\pi \mathscr{S} := \{(s_1, s_2) \in \mathscr{S} \times \mathscr{S} \mid \pi(s_1) = \pi(s_2)\}$ is the *fibered product* [51, p. 153].

Example 1.3.2.7 If $\mathscr{S}$ is a sheaf of K-modules, then module addition/subtraction are continuous binary operations as indicated and scalar multiplication is continuous $K \times \mathscr{S} \to \mathscr{S}$ where K is given the discrete topology.

{shfm} **Definition 1.3.2.8** Given two sheaves $\mathscr{S}$ over X and $\mathscr{T}$ over Y, a sheaf *morphism* is a commutative square of continuous maps

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{u} & \mathscr{T} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X & \xrightarrow[f]{} & Y \end{array}$$

such that u preserves stalks and is a morphism of the algebraic structure on the stalks (such as K-linear on K-modules). We shall almost always take $X = Y$.

As usual, any continuous right inverse σ of π is called a *section* (of π or of $\mathscr{S}$); thus $\pi \circ \sigma = 1_X$. We denote the space of all sections of $\mathscr{S}$ over X by $\Gamma(\mathscr{S}, X)$, or just $\Gamma(\mathscr{S})$ when X is clear from context. It follows readily that section spaces are the same type of algebraic object as the stalks (for example, K-modules), that sections are open maps, and that two sections that agree at a point x must also agree on a neighborhood of x.

For the next example, we need some preparation. Let $C(X)$ denote the $\mathbb{R}$-algebra of continuous functions on X. Two such functions f, g are said to be *(germ-) equivalent at* $x \in X$ if and only if there exists an open neighborhood U of x such that $f|_U = g|_U$.

Ex 1.3.2.9 Verify this is an equivalence relation on $C(X)$ for each $x \in X$.

Definition 1.3.2.10 An equivalence class of continuous functions at x is called a *germ* of a continuous function at x. If f is a member of a germ at x, we denote the equivalence class containing it by $[f]_x$. {germ}

Example 1.3.2.11 Let $\mathscr{C}_x$ denote the union of all germs at x and consider the set {shcntf}

$$\mathscr{C}(X) := \bigcup_{x \in X} \mathscr{C}_x \,.$$

Define a projection $\pi : \mathscr{C}(X) \twoheadrightarrow X$ in the obvious way: $[f]_x \mapsto x$. We topologize $\mathscr{C}(X)$ as follows. Let U be open in X and let $f \in C(U)$. Form the sets

$$\bigcup_{x \in U} [f]_x \subseteq \mathscr{C}(X)$$

for each such U and f and then take the topology on $\mathscr{C}(X)$ that has them as a basis.

Ex 1.3.2.12 Verify that this collection is in fact eligible to be a basis.

Ex 1.3.2.13 Verify that the topology so defined on $\mathscr{C}(X)$ makes it a sheaf on X with projection π.

This sheaf is called the *sheaf of germs of continuous functions on* X. Each stalk $\mathscr{C}_x$ is a *local ring*: it has the unique maximal ideal $\mathfrak{m}_x = \{[f]_x \mid f(x) = 0\}$.

Example 1.3.2.14 Consider $\mathscr{C}(\mathbb{R})$ and the two functions $g = 0$ and

$$f(t) = \begin{cases} 0, & t \leq 0, \\ t, & t > 0\,. \end{cases}$$

Then $[f]_0 \neq [g]_0$ but $[f]_t = [g]_t$ for all $t < 0$. It follows that the two germs at 0 cannot be separated whence $\mathscr{C}(\mathbb{R})$ is not Hausdorff.

Ex 1.3.2.15 Let $\mathscr{S}$ be a sheaf of K-modules. If two sections agree at a point, then they agree in a neighborhood of the point. Thus the set of points where a section is not 0 is closed. [Hint: complement.]

Ex 1.3.2.16 A continuous (real-valued) function f on X determines a section $[f]$ of $\mathscr{C}(X)$. The set where a continuous function is not 0 is open, but according to the previous Ex the set where the section is not 0 is closed. Reconcile. [Hint: germs.]

Finally, it will be convenient to have available this result.

{corresp} **Theorem 1.3.2.17** *The* $\mathbb{R}$*-algebras* $C(X)$ *and* $\Gamma(\mathscr{C}X)$ *are isomorphic.*

The map is the obvious one $f \mapsto [f]$ pointwise.

2 *Manifolds*

Throughout the Age of Exploration, accuracy in navigation became increasingly important. Prince Henry created an entire school devoted to this. The mariners and navigators knew that the Earth's surface was not a plane surface. Thus the Earth's surface was the first nonlinear (or curved) surface of practical interest, the first *manifold* required for an application (navigation).

Already, an *atlas* was known as a book of maps, and the special maps used in navigation were called *charts.* Hence marine navigators required an *atlas of charts* to ply their craft. It was known that all charts were created by *projecting* part of the surface of the Earth onto a piece of paper. Mercator's projection became the great standard because it facilitated navigation by *constant heading:* draw a straight line from the origin to the destination, then merely sail along that constant compass direction.

Nowadays Mercator's projection is little used and navigation no longer relies on constant heading, but the terminology of an atlas of charts survives in the theory of manifolds.

2.1 Definitions

Intuitively, a manifold is a space built of open blobs in $\mathbb{R}^n$ glued together according to some precise specifications. Let M be a *locally Euclidean* topological space. Locally Euclidean means that every point in M has a neighborhood that is homeomorphic to an open set in $\mathbb{R}^n$, for some $n \in \mathbb{N}$. This explains the "open blobs."

Suppose that U_i and U_j are two such open sets in M with nonempty intersection U_{ij}, and that $\varphi_i : U_i \to \mathbb{R}^n$ and $\varphi_j : U_j \to \mathbb{R}^n$ are the locally Euclidean maps. Then $\varphi_i(U_{ij}) \subseteq \varphi_i(U_i)$, $\varphi_j(U_{ij}) \subseteq \varphi_j(U_j)$ and $\varphi_j|_{U_{ij}} \circ \varphi_i|_{U_{ij}}^{-1}$ maps $\varphi_i(U_{ij})$ homeomorphically to $\varphi_j(U_{ij})$. This map is known as a *transition map*. We call the pair (U_i, φ_i) a *chart.* Without loss of generality, we may assume that some $p \in U_i$ maps to $0 \in \mathbb{R}^n$, and say that the chart is *centered* at $p \in M$. A covering of M by charts with smooth overlap is known as a smooth *atlas* and denoted by $\mathfrak{A}$. All this explains the "precise specifications."

As coverings are partially ordered by inclusion and the union of a chain of atlases is another atlas, we obtain by Zorn's Lemma that there exist maximal atlases. We refer to a maximal atlas as a *differential structure* on M. Clearly, any atlas is contained in some differential structure.

{s7} **Remark 2.1.1** There are 28 distinct differential structures on S^7 [40].

In order to do Calculus on the manifold, it suffices to require all the transition maps to be differentiable of the same order. This gives rise to C^k-, C^∞- and C^ω-manifolds when the transition maps are C^k, C^∞ and C^ω, respectively. A C^∞-manifold is called a *smooth* manifold, while a C^ω-manifold is called an *analytic* manifold.

To minimize pathology, we shall further assume that M is Hausdorff and paracompact. Recall that in a Hausdorff space, limits (of nets or filters) are unique when they exist. Paracompactness gives us the eminently desirable partition of unity which lets us put together locally defined objects into a global object. Recall that for connected spaces, paracompact, second countable, and σ-compact are equivalent.

{le} **Remark 2.1.2** The line with two origins is not Hausdorff and the long line is not paracompact; but both are locally Euclidean.

{mfld} **Definition 2.1.3** A *smooth manifold* is a locally Euclidean, Hausdorff, paracompact topological space with a smooth differential structure.

Henceforth we require all manifolds to be smooth unless explicitly otherwise stated.

{dim} As M is locally Euclidean, it has a well-defined *dimension* at each point. It follows easily that this dimension is constant on connected components. We shall further require all connected components of a manifold have the same dimension n. We write $\dim(M) = n$ and say that M is *n-dimensional* or an *n-manifold.*

2.2 Examples

At least one is required to establish that we have a viable theory.

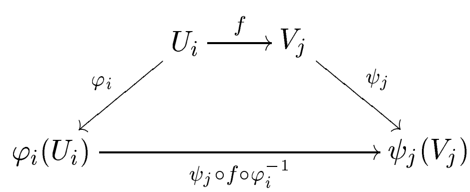
{Rn} **Example 2.2.1** Each $\mathbb{R}^n$, for $n \geq 0$ is a manifold: the chart $(\mathbb{R}^n, 1_{\mathbb{R}^n})$ is also an atlas. This provides the *usual* differential structure on $\mathbb{R}^n$.

{open} **Example 2.2.2** Any open $\Omega \subseteq \mathbb{R}^n$ is an n-manifold by restriction of the differential structure on $\mathbb{R}^n$. More generally, any open subset of a manifold is also a manifold.

Example 2.2.3 Any finite-dimensional real vector space is isomorphic to {fdvs} some $\mathbb{R}^n$ and thus an n-manifold. So $\mathbb{C}^n$ is a $2n$-manifold.

Remark 2.2.4 A theorem of Kirby-Siebenmann [41] shows that the usual {ks} differential structure on $\mathbb{R}^n$ is unique except possibly when $n = 4$. Freedman and Donaldson [23, 18] furthered this result by proving the existence of *exotic* differential structures on $\mathbb{R}^4$. Continuing, Gompf [25] obtained a "doubly infinite" family of distinct exotic structures on $\mathbb{R}^4$. Hence one needs to be cautious when working in dimension four. We consider only manifolds with the usual differential structure on $\mathbb{R}^4$.

We fix the dimension of the manifold M as n and may denote that by M^n. Let N be another smooth manifold of dimension m. Suppose that $f : M \to N$ such that for a chart (U_i, φ_i) on M, we have $f(U_i) \cap V_j \neq \varnothing$ where (V_j, ψ_j) is a chart on N. The following diagram illustrates the scenario.



Definition 2.2.5 A map of manifolds $f : M \to N$ is *smooth* if and only if {smfcn} its *local representative* $\psi_j \circ f \circ \varphi_i^{-1}$ in each such pair of charts is smooth.

Example 2.2.6 The set of all invertible $n \times n$ real matrices is denoted by {gln} GL_n and may be regarded as a subset of $\mathbb{R}^{n^2}$. The determinant is a polynomial function, hence continuous (and smooth), so GL_n is an open subset of $\mathbb{R}^{n^2}$. Thus GL_n is a smooth manifold of dimension n^2.

Ex 2.2.7 The product $M \times N$ of two smooth manifolds is also a smooth {prod} manifold. (Hence for any finite product by the usual induction argument.)

2.3 Partitions of unity

The basic ingredient is a *bump* function: a smooth function h on $\mathbb{R}$ which is {pu} identically 1 on $[-1, 1]$ and identically 0 outside $(-2, 2)$.

To construct one, let

$$f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0, \\ 0\,, & t \leq 0. \end{cases}$$

This function is nonnegative and positive for $t > 0$.

{sul} is below; **Ex 2.3.1** Verify that f is smooth.

Then define

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

whence g is smooth, nonnegative, identically 1 for $t \geq 1$, and identically 0 for $t \leq 0$. Finally, define

$$h(t) = g(2+t)g(2-t)$$

and we have the desired h.

To get a cubical bump function λ on $\mathbb{R}^n$ we take

$$\lambda(x) = h(x^1) \cdot h(x^2) \cdots h(x^n)\,.$$

Similarly, $\lambda(x) = h(|x|)$ produces a spherical bump function. Other variations are easily constructed.

As a manifold M is paracompact, it is easy to use λ to construct smooth partitions of unity subordinate to any open cover on M.

{sul} **Ex 2.3.2** Prove the smooth Urysohn Lemma: given two disjoint closed sets C_0 and C_1 in M, there exists a smooth function $f : M \to [0,1]$ such that $f(x) = 0$ for $x \in C_0$, $f(x) = 1$ for $x \in C_1$, and $0 < f(x) < 1$ elsewhere.

2.4 The function algebra

{fa}

The set of all smooth, real-valued functions on a manifold M is commonly called the *function algebra* of M. Indeed, pointwise definitions of sums, products, and scalar multiples as in elementary analysis provide it with the structure of an $\mathbb{R}$-algebra. It is denoted by $C^\infty(M)$ or $\mathfrak{F}(M)$ with the latter shortened to $\mathfrak{F}$ when M is fixed or clear from context.

{fads} **Ex 2.4.1** Giving the subalgebra $C^\infty(M)$ of $C(M)$ is equivalent to giving M a differential structure.

We begin with a theorem about approximation of continuous functions by smooth functions; equivalently, a theorem about density of the space of smooth functions in the space of continuous functions. The proof provides a typical example of a "partition of unity argument."

{dense} **Theorem 2.4.2** *On a smooth manifold, $C^\infty(M)$ is dense in $C(M)$.*

Proof: Let $\{V_\alpha\}$ be a locally finite open cover of M and for each α let $\varepsilon_\alpha > 0$. Let $f : M \to \mathbb{R}$ be continuous. For each $x \in M$, let W_x be a neighborhood of x intersecting only finitely many V_α. Set $\delta_x = \min\{\varepsilon_\alpha \mid x \in V_\alpha\}$ and let $U_x \subseteq W_x$ be a sufficiently small open neighborhood of x so that $|f(y) - f(x)| < \delta_x$ for all $y \in U_x$.

Define constant functions $g_x : U_x \to \mathbb{R} : y \mapsto f(x)$. Relabeling the cover $\{U_x\}$ and the functions $\{g_x\}$, we have an open cover $\{U_i\}$ of M and functions $\{g_i : M \to \mathbb{R}\}$ such that if $y \in U_i \cap V_\alpha$, then $|g_i(y) - f(y)| < \varepsilon_\alpha$.

Now choose a partition of unity $\{\lambda_i\}$ subordinate to $\{U_i\}$ and define $g : M \to \mathbb{R}$ by

$$g(y) = \sum_i \lambda_i(y) g_i(y) \,.$$

Clearly g is smooth and

$$|g(y) - f(y)| = \left|\sum\nolimits_i \lambda_i(y) g_i(y) - \sum\nolimits_i \lambda_i(y) f(y)\right| \leq \sum_i \lambda_i(y)\, |g_i(y) - f(y)| \,.$$

Hence if $y \in V_\alpha$, then $|g(y) - f(y)| \leq \sum_i \lambda_i(y)\varepsilon_\alpha = \varepsilon_\alpha$. □

Note that the proof actually shows that $C^\infty(M)$ is *uniformly* dense in $C(M)$; in other words, smooth functions are dense in the space of continuous functions with the *strong (or Whitney)* topology [31]. This topology is mainly used in differential topology; elsewhere it is rarely seen. Note also that the proof requires only trivial modification to apply to functions with values in any normed linear space. In fact, similar changes establish that C^s is dense in C^r for any $0 \leq r < s \leq \infty$. A deep theorem of Grauert and Remmert [26] proves that C^ω is also dense in C^r for $0 \leq r \leq \infty$.

Ex 2.4.3 Recall that $C^\infty(M, \mathbb{R}^k) \cong \mathfrak{F} \mathbin{\widehat{\otimes}} \mathbb{R}^k$ and $C(M, \mathbb{R}^k) \cong C(M) \mathbin{\widehat{\otimes}} \mathbb{R}^k$. Conclude that $C^\infty(M, \mathbb{R}^k)$ is uniformly dense in $C(M, \mathbb{R}^k)$. {densex1}

Ex 2.4.4 Show that $C^\infty(M, N)$ is uniformly dense in $C(M, N)$. [Hint: Use a partition of unity and start locally.] {densex2}

We now examine the function algebra from the perspective of functional analysis.

Example 2.4.5 Let M be a connected manifold. Then $\mathfrak{F}$ is a Fréchet space. To prove this, first take an atlas $\mathfrak{A}$; this provides an open cover. Write M as a countable union of compact sets K_j. Then each K_j has a finite subcover $\mathfrak{A}_j$ by charts intersected with K_j. Now we take the seminorms {fsx2}

$$|f|_{m,\mathfrak{A}_j} = \sup_{|\alpha|\leq m} \sup_{\mathfrak{A}_j} \left|D^\alpha f(x)\right|$$

as x varies first in each chart of $\mathfrak{A}_j$, and then over all the finitely many charts of $\mathfrak{A}_j$. The rest of the proof now follows as in Example 1.1.5. This topology on $\mathfrak{F}$ is also called the *Schwartz (or weak C^∞)* topology.

Also using a similar proof to that for $C^\infty(\Omega)$ for Ω open in $\mathbb{R}^n$ [79, p. 530], one may show that $\mathfrak{F}$ is nuclear. It then follows immediately that

$$C^\infty(M \times N) \cong C^\infty(M) \mathbin{\widehat{\otimes}} C^\infty(N)\,.$$

Just as we did for the continuous functions $C(M)$ in Example 1.3.2.10, we can consider germs of the smooth functions $\mathfrak{F}(M)$. We continue with the same notation so that the germ of f at p is denoted by $[f]_p$. Similarly, as in Example
{shsmf} 1.3.2.11, we form the *sheaf of germs of smooth functions* $\mathscr{F}(M) = \mathscr{C}^\infty(M)$ on M. Each stalk $\mathscr{F}_p = \mathscr{C}^\infty_p$ is again a local ring.

Ex 2.4.6 Prove that $\mathfrak{F}_{\mathfrak{m}_p} \cong \mathscr{F}_p$.

The topology on $\mathscr{F}$ is the "sheafication" of the Schwartz topology on $\mathfrak{F}$. We also continue with the notation for sections as $\Gamma(\mathscr{F})$ when M is clear from context. The correspondence theorem continues to hold.

{corresp1} **Theorem 2.4.7** *The $\mathbb{R}$-algebras $\mathfrak{F}(M)$ and $\Gamma(\mathscr{F}M)$ are isomorphic.*

Proof: The key idea is that continuity in germs of a section $[f]$ is equivalent to continuity of f and all its derivatives in the usual sense. The details are left to the reader as an exercise. □

Note that this encodes the *differential* information about a smooth function f as the *continuity* of the section $[f]$ of $\Gamma(\mathscr{F})$. In algebraic geometry, $\mathscr{F}$ is called the *structure sheaf* of M; indeed, it follows from Ex 2.4.1 above that specifying $\mathscr{F}$ is equivalent to specifying the differential structure. See also Theorem 2.5.10 and "sheafify" it for more validation.

2.5 Ideals and near places

One might wonder about the relation to algebraic geometry over $\mathbb{R}$ or $\mathbb{C}$. It turns out that the *regular* parts of algebraic varieties are smooth manifolds. Varieties are characterized by prime ideals and points by maximal ideals. An example of a maximal ideal in $\mathfrak{F}$ is $\mathfrak{m}_p = \{f \mid f(p) = 0\}$.

Ex 2.5.1 Prove that $\mathfrak{F}/\mathfrak{m}_p \cong \mathbb{R}$ for every $p \in M$.

We now consider ideals of the $\mathbb{R}$-algebra $\mathfrak{F} = C^\infty(M)$. We call points p where $\mathfrak{F}/\mathfrak{m}_p \cong \mathbb{R}$ *real points of* M (in two senses, as will become apparent).

Ex 2.5.2 If M is compact, all maximal ideals are $\mathfrak{m}_p$ for some $p \in M$. {micpt}

More generally, let K be an $\mathbb{R}$-algebra and consider epimorphisms $e : \mathfrak{F} \twoheadrightarrow K$. Two such, e_1 and e_2, are *equivalent* if and only if there exists an $\mathbb{R}$-algebra isomorphism $\alpha : K_1 \to K_2$ such that $e_2 = \alpha e_1$. We call an equivalence class $[e]$ under this relation a K*-place of* M.

Example 2.5.3 Let K be a field and $[e]$ be a K-place of M; then there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{F}/\mathfrak{m} \cong K$. If $K = \mathbb{R}$, then $\mathfrak{m}$ represents a real point of M and $\mathfrak{m} = \mathfrak{m}_p$ for some $p \in M$. For other fields K, the maximal ideal $\mathfrak{m}$ represents a *virtual point of* M. {mikpl}

In 1948, Hewitt [30] studied these fields for the algebra of continuous functions on a completely regular space. His results extend readily to smooth functions on manifolds. All the fields arising from virtual points are those he called *hyperreal*. They are ordered, real-closed, non-Archimedean extensions of $\mathbb{R}$ that contain infinitely large ($u > n$ for all $n \in \mathbb{N}$) and infinitely small ($0 < u < \frac{1}{n}$ for all $n \in \mathbb{N}$) numbers, of transcendence degree at least $2^{\aleph_0}$ over $\mathbb{R}$. A modern, well-known example of such a field is A. Robinson's nonstandard real numbers $^*\mathbb{R}$.

Ex 2.5.4 No real manifold M has any complex points; *i.e.,* virtual points with $K = \mathbb{C}$. Indeed, if $e : \mathfrak{F} \twoheadrightarrow \mathbb{C}$ and $e(f) = i$, then $e(1+f^2) = 0$, but $1+f^2$ is invertible (a unit) in $\mathfrak{F}$. {norcp}

Even if M is compact, prime ideals can still be surprising.

Ex 2.5.5 Consider the ideal {iff}

$$\mathfrak{p}_p = \{f \in \mathfrak{F} \mid D^\alpha f(p) = 0 \text{ for all } \alpha\}$$

in one, hence any, local coordinate chart at $p \in M$. One says that such an f is *(infinitely) flat at* p. Show that $\mathfrak{p}_p$ is prime. [Hint: $\mathfrak{p}_p = \bigcap_{k=1}^\infty \mathfrak{m}_p^k$.]

If M is not compact, then $\mathfrak{F}_c = C_c^\infty(M)$, the set of smooth functions on M with compact support, is an ideal in $\mathfrak{F}$. All the maximal ideals $\mathfrak{m}$ that contain $\mathfrak{F}_c$ are not $\mathfrak{m}_p$ for any $p \in M$; *i.e.*, all of the corresponding points are virtual.[30]

{1ptcptf} **Example 2.5.6** Consider S^1 as the smooth one-point compactification of $\mathbb{R}$, with the extra point ∞ being the north pole of S^1. Now consider the set

$$\mathfrak{m} := \left\{ f \in \mathfrak{F}(\mathbb{R}) \;\middle|\; \lim_{t\to\pm\infty} f(t) = 0 \text{ and } \lim_{t\to+\infty} D^k f(t) = \lim_{t\to-\infty} D^k f(t),\ k \geq 1 \right\}.$$

Then $\mathfrak{m}$ is a maximal ideal containing $\mathfrak{F}_c$ and represents the single virtual point ∞. Extend this to the one-point compactification of $\mathbb{R}^n$ as S^n. Can you determine/identify $\mathfrak{F}/\mathfrak{m}$ for any n?

{2ptcptf} **Example 2.5.7** Decouple $+\infty$ and $-\infty$ in the previous example to get two virtual points $\mathfrak{m}_{+\infty}$ and $\mathfrak{m}_{-\infty}$. These represent the two added points in the two-point compactification $\mathbb{R} \hookrightarrow [-1, 1]$ of $\mathbb{R}$ as the extended reals. Can you identify either quotient field?

Ex 2.5.8 It would not be surprising if these two fields were isomorphic. Are they? Is either (or are both) isomorphic to the field in the prior example of $\mathbb{R}$ and S^1?

Ex 2.5.9 More generally, it seems that each point in any smooth compactification is already present as a virtual point. In particular, this applies to attaching smooth boundary components. Can you prove these? How far can you extend this idea?

The next theorem implies that the theory of smooth manifolds can be made purely algebraic; see Nestruev [61] for an elementary treatment. It was also part of the inspiration for so-called "noncommutative geometry."

{fiso} **Theorem 2.5.10** *A manifold M is diffeomorphic to another manifold M' if and only if $C^\infty(M)$ is isomorphic to $C^\infty(M')$.*

Here is an outline of the proof.

Ex 2.5.11 First, observe that a smooth $\varphi : M \to M'$ induces a morphism $\varphi^* : \mathfrak{F}' \to \mathfrak{F} : f' \mapsto f' \circ \varphi$. Thus $\mathfrak{F}$ is a cofunctor from *Mfld* to the category of $\mathbb{R}$-algebras; in fact, to the category of nuclear Fréchet $\mathbb{R}$-algebras or of complete nuclear $\mathbb{R}$-algebras.

{fs1} 1. Show that a ring morphism $\mathfrak{F}' \to \mathfrak{F}$ is also an algebra morphism.

{fs2} 2. Show that the map $C^\infty(M, M') \to \mathrm{Hom}(\mathfrak{F}', \mathfrak{F}) : \varphi \mapsto \varphi^*$ is injective: that if $\varphi_1^* = \varphi_2^*$ then $\varphi_1 = \varphi_2$.

3. Using paracompactness, show that the map in the preceding part is also surjective (thus bijective). [Hint: Recalling the theory of duality for K-modules, treat $\mathfrak{F}$ as the dual of M and look at the bidual.]

The definition of a K-place (right after Ex 2.5.2) can be modified in various ways. The most useful and fruitful is to weaken the condition on the morphism $\mathfrak{F} \to K$ to not require epi while tightening up the conditions on K to be a local $\mathbb{R}$-algebra (a local ring that is also an algebra over $\mathbb{R}$).

Definition 2.5.12 A *Weil algebra* A is a local algebra of finite dimension over $\mathbb{R}$ with maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m} \cong \mathbb{R}$. {walg}

The A-places of a manifold M are precisely the "contact elements" of classical analysis and geometry as studied by Lie [47] and others; see [59] for a complete verification. Weil's formulation is ultimately based on the viewpoint of Fermat [83, p. 113]. Weil called them *points proches*, which translates as points that are near, nigh, approaching, or infinitely neighboring[1] (to) points of M. I have called them places for consistency, and because some are better conceptualized as extended entities.

Here are some basics about Weil algebras. There is a canonical imbedding $\mathbb{R} \hookrightarrow A : r \mapsto r1$ where this 1 denotes the unit of A. Identify $\mathbb{R}$ with its image so $\mathbb{R} \hookrightarrow A$ becomes an inclusion and regard $A/\mathfrak{m} = \mathbb{R}$. Then this inclusion splits the short exact sequence

$$0 \to \mathfrak{m} \to A \to \mathbb{R} \to 0 \qquad (2.5.1)$$ {sses}

whence $A = \mathbb{R} \oplus \mathfrak{m}$. For each $a \in A$, the scalar a_0 defined by $a \equiv a_0 \pmod{\mathfrak{m}}$ is called the *finite part* of a. Then the algebra morphism $A \to \mathbb{R} : a \mapsto a_0$ is just the natural projection $A \to A/\mathfrak{m} = \mathbb{R}$. Since A is finite-dimensional, $\dim \mathfrak{m}/\mathfrak{m}^2 = w$ is a natural number called the *width* of A, and $\mathfrak{m}^{h+1} = 0$ for some natural number h. The smallest such h is called the *height* of A. The width corresponds to the dimension and the height to the *order of contact* of a classical contact element. As well, $\mathfrak{m}$ is the nilradical (all the nilpotent elements) of A.

[1]In the literature, they are occasionally called "infinitely near" points, but that name is already reserved for points on blow-ups. The simplest blow-up is replacing a point in an n-dimensional space with a copy of $\mathbb{P}^{n-1}$.

Ex 2.5.13 Show subalgebras, quotient algebras, and tensor products of Weil algebras are Weil algebras and specify the height and width as well as possible. To get started, if A has height h and width w, then any subalgebra has height and width at most h and w, respectively. The maximal ideal of $A \otimes B$ is $A \otimes \mathfrak{m}_B + \mathfrak{m}_A \otimes B$.

{wpe} **Definition 2.5.14** The object $W^A M := \mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}M, A)$ is called the *Weil prolongation* or *extension* of M of *type* A. It is the set of all A-places of M.

Consider a typical element α of $W^A M$. Note that if it is composed with the projection $A \to A/\mathfrak{m} = \mathbb{R}$, one obtains a real point p of M. It follows that $W^A M$ fibers over M. The inclusion $\mathbb{R} \hookrightarrow A$ allows real points of M to be regarded as A-places of M. Thus one may regard $M \hookrightarrow W^A M$.

{nearp} **Definition 2.5.15** Any such A-place α is said to be *near* p and p is called the *projection* of α.

When it helps to keep track of base points (points of M that places are near), I may write places as (p, α) or α_p.

Now consider smooth structures on A and $W^A M$. Because A is a finite-dimensional real vector space, it has a standard smooth structure, in which it is also a smooth algebra. It may be somewhat unexpected that $W^A M$ is not only finite-dimensional, it is a smooth manifold: there are no singular points. To show that $W^A M$ is smooth, the claim of [59] is that this prescription suffices: for all $f \in \mathfrak{F}M$, declare the function $W^A f : W^A M \to A : \alpha \mapsto \alpha(f)$
{wamsm} to be smooth. Then this determines a unique smooth structure on $W^A M$. One can verify this using Nestruev's theory of C^∞-closures and smooth envelopes [61, 3.32–38].

Ex 2.5.16 I suspect there is a simpler way. Can you find one?

Ex 2.5.17 It can be shown that $\mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}M, A)$ is a finite-dimensional real affine variety in $\mathrm{Hom}_{\mathsf{Vec}}(\mathfrak{F}M, A)$. Can you? [Hint: RealSpec and relative dimension of schemes.]

If $\varphi : M \to N$ is smooth, then $W^A \varphi : W^A M \to W^A N : \alpha \mapsto \alpha \circ \varphi^*$ is smooth. Indeed, if $f \in \mathfrak{F}N$ then $\varphi^* f \in \mathfrak{F}M$ and the smooth structures on $W^A M$ and $W^A N$ require $\alpha \mapsto \alpha(\varphi^* f)$ to be smooth for all $f \in \mathfrak{F}N$. It follows
{wfct} that W^A is a (covariant) endofunctor of *Mfld* called a *Weil functor*. Note that this implies there is a group morphism $\mathrm{Aut}(M) \to \mathrm{Aut}(W^A M)$.

One important and useful property of Weil functors is that they preserve products.

Proposition 2.5.18 *If M and N are smooth manifolds and A is a Weil algebra, then $W^A(M \times N) \cong W^A M \times W^A N$.*

Proof: Recall that $\mathfrak{F}M$ is a nuclear algebra, so $\mathfrak{F}(M \times N) \cong \mathfrak{F}M \mathbin{\widehat{\otimes}} \mathfrak{F}N$, and that $\widehat{\otimes}$ is the coproduct. Thus $W^A(M \times N) = \mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}(M \times N), A) \cong \mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}M \mathbin{\widehat{\otimes}} \mathfrak{F}N, A) \cong \mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}M, A) \times \mathrm{Hom}_{\mathsf{Alg}}(\mathfrak{F}N, A) = W^A M \times W^A N$ *via* universality of the coproduct [51, p. 151]. $\square$

Now consider an (internal) binary operation $b : M \times M \to M$ and its prolongation (or extension) $W^A b : W^A M \times W^A M \to W^A M$, following [83].

Ex 2.5.19 Show that $W^A b$ prolongs all algebraic and smooth properties of b. For example, if b is associative, commutative, has an identity element (or unit), or has inverses, then so does $W^A b$. If b is distributive over another binary operation b', so is $W^A b$ over $W^A b'$.

In addition, we also need that the *external* binary operation (a sort of action) $m : \mathbb{R} \times E \to E$ of scalar multiplication on a left $\mathbb{R}$-module E prolongs to scalar multiplication on the left A-module $W^A E$. But first, it is necessary to verify that $W^A \mathbb{R} = A$.

Let α be an A-place of $\mathbb{R}$ near $x \in \mathbb{R}$. Recall the canonical embedding $\mathbb{R} \hookrightarrow A$ as a subalgebra. Note that $1_{\mathbb{R}} \in \mathfrak{F}\mathbb{R}$ and that any α is completely determined by $\alpha(1_{\mathbb{R}}) \in A = \mathbb{R} \oplus \mathfrak{m}$. If α is not x itself, then $\alpha_x(1_{\mathbb{R}})$ must be in the copy of $\mathfrak{m}$ opposite x. Hence $W^A \mathbb{R} = A$ as *sets*. But it is easy to verify that the prolongations of addition and multiplication from $\mathbb{R}$ to A coincide with those of A. Therefore $W^A \mathbb{R} = A$ as *algebras*. The splitting of (2.5.1) now shows that the two canonical inclusions, the one $\mathbb{R} \hookrightarrow A$ and the other $\mathbb{R} \hookrightarrow W^A \mathbb{R} = A$, coincide.

It follows immediately that $W^A m : A \times W^A E \to W^A E$ is scalar multiplication; that is, if E is a left $\mathbb{R}$-module, then its prolongation $W^A E$ is a left A-module. Thus when E is of finite rank over $\mathbb{R}$, so is $W^A E$ over A. Hence $W^A \mathbb{R}^n \cong A \otimes \mathbb{R}^n \cong A^n$ and $W^A B \cong A \otimes B$, considering $\mathbb{R}^n$ and B as left $\mathbb{R}$-modules and applying Change of Rings [51, p. 332] to get left A-modules.

This gives us one of the most important properties of Weil functors: that they are *transitive,* as in the next result [83].

Theorem 2.5.20 *If A and B are Weil algebras, then $W^A W^B M \cong W^{A \otimes B} M$.* {wft}

Proof:

to be done

$\square$

The relations among Weil functors are quite enriched.

Theorem 2.5.21 *There is a bijective correspondence between* $\mathrm{Hom}_{Alg}(A, B)$ *and* $\mathrm{Nat}(W^A, W^B)$ *that is natural in* A *and* B. {wbc}

Proof: Any algebra morphism $\mu : A \to B$ induces a map $\mu_M : W^A M \to W^B M$ by composition. Indeed, for any $\alpha \in \mathrm{Hom}_{Alg}(\mathfrak{F}M, A)$, one has $\mu \circ \alpha \in \mathrm{Hom}_{Alg}(\mathfrak{F}M, B)$. It is easy to verify that for any smooth $\varphi : M \to N$, this diagram commutes.

$$\begin{array}{ccc} W^A M & \xrightarrow{\mu_M} & W^B M \\ {\scriptstyle W^A\varphi}\downarrow & & \downarrow{\scriptstyle W^B\varphi} \\ W^A N & \xrightarrow[\mu_N]{} & W^B N \end{array}$$

Therefore each such algebra morphism μ induces a natural transformation $W^A \rightsquigarrow W^B$.

Conversely, let $\tau \in \mathrm{Nat}(W^A, W^B)$ and consider $\tau_{\mathbb{R}} : W^A\mathbb{R} \to W^B\mathbb{R}$. Since $W^A\mathbb{R} = A$ and $W^B\mathbb{R} = B$, we get an algebra morphism $\tau_{\mathbb{R}} : A \to B$.

Naturality in A and B follows from the Parameter Theorem [51, p. 514]. □

Ex 2.5.22 The bijective correspondence $\mathrm{Hom}_{Alg}(A, B) \leftrightarrow \mathrm{Nat}(W^A, W^B)$ appears to be a Yoneda-type situation. Can the whole theorem be proved *via* the Parameter Theorem and related results?

Something about enrichment?

{walgse} **Example 2.5.23** Here are the standard examples of Weil algebras. Define $\mathbb{D}_w^{(\infty)} := \mathbb{R}[[x_1, \ldots, x_w]]$ and let $\mathfrak{m}$ denote the maximal ideal of this ring of formal power series. Then define $\mathbb{D}_w^{(h)} := \mathbb{D}_w^{(\infty)}/\mathfrak{m}^{h+1}$. This is a Weil algebra of height h and width w.

If (p, α) is a $\mathbb{D}_w^{(h)}$-place of a smooth manifold M, then $\ker \alpha \supseteq \mathfrak{m}_p^{h+1}$. In a cubical chart (U, x) centered at p, using Taylor expansions yields $\mathfrak{F}(U) = \mathbb{R}[x^1, x^2, \ldots, x^n] \oplus \mathfrak{m}_p^{h+1}$. Thus α is completely determined by its restriction to $\mathbb{R}[x^1, \ldots, x^n]$, hence is an Ehresmann h-jet of a local map from an open neighborhood of $0 \in \mathbb{R}^w$ to a neighborhood of $p \in M$. Using only these particular Weil algebras, one recovers Ehresmann's theory of jets of smooth functions [22, 70].

For later use, define $\mathbb{D} := \mathbb{D}_1^{(1)}$. These are Study's *dual numbers*, generated by 1 and δ with $\delta^2 = 0$. Trying to keep the notation under control, when $A = \mathbb{D}_w^{(h)}$ write $W^A M$ as $W_w^{(h)} M$. Since the major point of Weil's construction is to include *all* classical contact elements, in addition to the $\mathbb{D}_w^{(h)}$ we also include their subalgebras, quotient algebras, and tensor products.

Remark 2.5.24 It is known that in fact all Weil algebras are obtained this way, up to isomorphism of course. A proof can be extracted from [42].

The Weil algebra $\mathbb{D}_{w_1}^{(h_1)} \otimes \cdots \otimes \mathbb{D}_{w_k}^{(h_k)}$ can be denoted more compactly as $\mathbb{D}_{w_1,\ldots,w_k}^{(h_1,\ldots,h_k)}$ and the corresponding Weil functor as $W_{w_1,\ldots,w_k}^{(h_1,\ldots,h_k)}$.

Finish this.

Include tangent vectors to $W^A M$
(after derivations and tangent vectors).

Rounding out this excursion, here are two miscellaneous exercises.

{fs4} **Ex 2.5.25** A manifold M is connected if and only if $\mathfrak{F}$ is not a direct product of nontrivial rings.

Ex 2.5.26 What happens when you localize a ring of smooth functions at a prime ideal? A hyperreal maximal ideal?

2.6 Derivations

{der}

For the next part, we need another algebraic structure. Here is the general definition of a Lie algebra. As usual, K denotes a commutative ring with 1.

{la2} **Definition 2.6.1** A *Lie algebra* is a K-module L with a *Lie bracket* product satisfying

1. $[a,b] = -[b,a]$ (skew), and

{jid} 2. $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$ (Jacobi identity).

Also as usual, skew is equivalent to alternating for characteristic not 2.

{lax1} **Example 2.6.2** Let A be a K-algebra. The *commutator* $[a,b] := ab - ba$ satisfies the requirements for a Lie bracket on A. In particular, one may take $A = \mathrm{End}_K(M)$ for any K-module M.

{drvn2} **Definition 2.6.3** Let A be a K-algebra and let $\mathcal{D} \in \mathrm{End}_K(A)$. We say $\mathcal{D}$ is a *derivation* if and only if $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ for all $a, b \in A$.

{lax2} **Proposition 2.6.4** *If $\mathcal{D}$ and $\mathcal{D}'$ are derivations then so is the commutator $[\mathcal{D}, \mathcal{D}']$. Thus the derivations of any K-algebra A form a Lie subalgebra $\mathrm{Der}_K(A)$ of $\mathrm{End}_K(A)$ over K.*

Proof: For any $a, b \in A$ we have

$$\begin{aligned}
[\mathcal{D}, \mathcal{D}'](ab) &= \mathcal{D}\mathcal{D}'(ab) - \mathcal{D}'\mathcal{D}(ab) \\
&= \mathcal{D}(\mathcal{D}'a \cdot b + a\mathcal{D}'b) - \mathcal{D}'(\mathcal{D}a \cdot b + a\mathcal{D}b) \\
&= \mathcal{D}\mathcal{D}'a \cdot b + \mathcal{D}'a\mathcal{D}b + \mathcal{D}a\mathcal{D}'b + a\mathcal{D}\mathcal{D}'b - \mathcal{D}'\mathcal{D}a \cdot b \\
&\quad - \mathcal{D}a\mathcal{D}'b - \mathcal{D}'a\mathcal{D}b - a\mathcal{D}'\mathcal{D}b \\
&= \mathcal{D}\mathcal{D}'a \cdot b - \mathcal{D}'\mathcal{D}a \cdot b + a\mathcal{D}\mathcal{D}'b - a\mathcal{D}'\mathcal{D}b \\
&= (\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D})a \cdot b + a(\mathcal{D}\mathcal{D}' - \mathcal{D}'\mathcal{D})b \\
&= [\mathcal{D}, \mathcal{D}']a \cdot b + a[\mathcal{D}, \mathcal{D}']b\,.
\end{aligned}$$

□

We note that $\mathcal{D}(k) = 0$ for any $k \in K$ because for any $a \in A$, $\mathcal{D}(ka) = \mathcal{D}k \cdot a + k\mathcal{D}a$ by definition of derivation while $\mathcal{D}(ka) = k\mathcal{D}(a)$ as k is a scalar. This means that $\mathcal{D}k \cdot a = 0$ whence $\mathcal{D}k = 0$ for any scalar $k \in K$ since a was arbitrary.

We shall apply this to the topological vector space $\mathfrak{F}(M)$. Thus $\mathrm{End}(\mathfrak{F})$ and $\mathrm{Der}(\mathfrak{F})$ will denote the spaces of *continuous* linear maps and derivations, {cdrvn} respectively. The topology is that of bounded convergence [79, p. 337] so that both are complete nuclear spaces. For now, we focus on $\mathrm{Der}(\mathfrak{F})$.

If $\mathcal{D} \in \mathrm{Der}(\mathfrak{F})$, then we may apply it to germs of smooth functions in the obvious way: $\mathcal{D}[f]_p := [\mathcal{D}f]_p$. This suggests that we should consider germs of derivations, and the sheaf $\mathscr{D}$ thereof. We also note that since each stalk $\mathscr{F}_p$ is an $\mathbb{R}$-algebra, it has a Lie algebra of derivations, $\mathrm{Der}(\mathscr{F}_p)$.

In general, we may consider germs of smooth maps of manifolds $M \to N$, germs of charts, *etc.* We leave the construction and examination of associated sheaves to the sufficiently motivated reader, as we won't use them here.

Note that germs of smooth maps $\varphi : M \to N$ continue to have the same functorial properties: $[\varphi]^* \circ [\psi]^* = [\psi \circ \varphi]^*$ and $1_M^* = 1_{\mathfrak{F}}$. Thus we may define $[\varphi]_p^*[f]_{\varphi(p)} := [\varphi^* f]_p$ so that germs of smooth maps of manifolds act on germs of smooth functions.

As any point $p \in M$ can be the center of a chart, thus of a chart germ, we look at the algebra $\mathscr{E}_0$ of germs of smooth functions at $0 \in \mathbb{R}^n$. Clearly, $\mathscr{E}_0$ is a model for any and all stalks $\mathscr{F}_p$. Indeed, if (U, φ) is a chart centered at p, then $[\varphi]_p^* : \mathscr{E}_0 \to \mathscr{F}_p$ is an isomorphism.

Ex 2.6.5 Prove the preceding claim. Note that $[\varphi]_p$ is invertible and use functoriality. {germx2}

Lemma 2.6.6 (Hadamard) *Let B be an open ball centered at $0 \in \mathbb{R}^n$. If $f : B \to \mathbb{R}$ is smooth, then there exist smooth functions $f_1, \ldots, f_n : B \to \mathbb{R}$ such that $f(x) = f(0) + x^i f_i(x)$.* {germl}

Proof: Note that

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx^1, \ldots, tx^n)\, dt \\ &= x^i \int_0^1 \partial_i f(tx^1, \ldots, tx^n)\, dt \end{aligned}$$

and define f_i by the last integral. □

Observe that each ∂_i is a derivation of $\mathscr{E}_0$ *via* the map $[f]_0 \mapsto [\partial_i f]_0$.

Corollary 2.6.7 *The set of all the ∂_i constitute a basis of the vector space* $\mathrm{Der}(\mathscr{E}_0)$. {germc1}

Proof: If the derivation $a^i\partial_i = 0$, then $a^i\partial_i[x^j]_0 = 0$ for all j, therefore the ∂_i are linearly independent.

Now, let $\mathcal{D}$ be the derivation given by $\mathcal{D}[x^i]_0 = a^i$ and consider $\mathcal{D}' = \mathcal{D} - a^i\partial_i$. Clearly $\mathcal{D}'$ is a derivation. By construction, $\mathcal{D}'[x^j]_0 = 0$ for all j. For $[f]_0 \in \mathscr{E}_0$, use the Lemma and write $f = f(0) + x^i f_i$. Applying $\mathcal{D}'$ we obtain $\mathcal{D}'[f]_0 = \mathcal{D}'[x^i]_0 f_i(0) = 0$. Thus $\mathcal{D}' = 0$ and $\mathcal{D} = a^i\partial_i$ as desired. □

That is, the dimension of M is the same as the dimension of the stalks $\mathrm{Der}(\mathscr{F}_p)$.

{loc} **Definition 2.6.8** Let (U, φ) be a chart centered at $p \in M$; that is, $\varphi(p) = 0 \in \mathbb{R}^n$. We take U such that $\varphi(U)$ is an open cube in $\mathbb{R}^n$. We write *local coordinates* in U as $x = (x^j) = (x^1, \ldots, x^n) \in M$, where each x^k, $1 \le k \le n$, is a real-valued function on U. We think of the x^j as living on M and a way of decomposing φ into n local coordinate functions. To emphasize this, we may write (U, x) for the chart.

We use the invertible chart map φ and the associated φ^* to lift the basis (∂_i) to M locally. This amounts to taking a chart (U, x) centered at p, regarding the (x^i) as local coordinates on M near p, and then (∂_i) as a local basis of derivations near p.

{germx1} **Ex 2.6.9** Prove that $\mathrm{Der}(\mathscr{F}_p) \cong \mathscr{D}_p/\mathfrak{m}_p\mathscr{D}_p$.

Given a derivation $\mathcal{D} \in \mathrm{Der}(\mathscr{F}_p)$ at a point $p \in M$, we can expand it in this basis as $\mathcal{D} = a^i\partial_i$. On the other hand, a germ of this derivation at the point p is given by $[\mathcal{D}]_p = [a^i]_p\partial_i \in \mathscr{D}_p$ for suitable smooth functions a^i. This illustrates the subtle difference between a derivation $\mathcal{D}$ at a point p and a germ of a derivation $[\mathcal{D}]_p$ at p.

We next examine some functorial properties of Der. Let $\mathcal{D} = a^i\partial_i \in \mathrm{Der}(\mathscr{F}_p)$ be a derivation at a point, and consider a smooth map $\varphi : M \to N$ with germ $[\varphi]_p$ at p. If $q = \varphi(p)$, then $[\varphi]_p^* : \mathscr{F}_q(N) \to \mathscr{F}_p(M)$. For a real-valued function f on N, consider the germ $[f]_q$ and write

$$\begin{aligned}
\mathrm{Der}_p(\varphi)(\partial_i)[f]_q(q) &= \partial_i\left([\varphi]_p^*[f]_q\right)(p) \\
&= \partial_i[f \circ \varphi]_p(p) \\
&= [\bar{\partial}_j f]_q(q) \cdot [\partial_i \varphi^j]_p(p)
\end{aligned}$$

where $(\bar{\partial}_j)$ form a basis for $\mathrm{Der}(\mathscr{F}_q)$. The last equality follows from the chain rule, and the summation convention is obeyed. This gives us the formula for how $\mathrm{Der}_p(\varphi)$ acts on the basis (∂_i) of $\mathrm{Der}(\mathscr{F}_p)$.

$$\mathrm{Der}_p(\varphi)(\partial_i) = \left(\partial_i \varphi^j(p)\right) \bar{\partial}_j$$

Note that if $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$, then $\partial_i \varphi^j(p)$ is the Jacobian matrix $D\varphi$ of φ evaluated at p. This means that for any smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{Der}(\varphi) = D\varphi$. This tells us how to write the result for a local representative with respect to charts.

Let $f : M^n \to N^m$ be a smooth map with $q = f(p)$ and choose charts (U, φ) centered at $p \in M$ and (V, ψ) centered at $q \in N$ such that $f(U) \subseteq V$. Then $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \psi(V) \subseteq \mathbb{R}^m$ is smooth with $\psi \circ f \circ \varphi^{-1}(0_n) = 0_m$. We rewrite the preceding formula in this setting.

$$\mathrm{Der}_p(f) = D(\psi \circ f \circ \varphi^{-1})(0_n)$$

In particular, if $a = (a^i) \in \mathbb{R}^n$ and $b = (b^j) \in \mathbb{R}^m$, then (suppressing composition circles)

$$D(\psi f \varphi^{-1})(0_n) \cdot a = b \tag{2.6.1}$$ {deqn}

so that the derivation $a^i \partial_i$ of $\mathscr{F}_p$ maps to the derivation $b^j \bar{\partial}_j$ of $\mathscr{F}_q$. Similarly, let $[a^i]_p$ and $[b^j]_q$ be germs of smooth functions taking the values a^i at p and b^j at q, respectively. Form column vectors $[a]_p$ and $[b]_q$ of these germs and ∂ and $\bar{\partial}$ of the bases. To apply Der_p to a map germ $[f]_p$, simply replace a by $[a]_p$ and b by $[b]_q$.

$$\mathrm{Der}_p([f]_p) : \left[{}^t a\right]_p \partial \mapsto D\left([\psi]_q [f]_p [\varphi^{-1}]_{0_n}\right) [a]_p \cdot \bar{\partial} = \left[{}^t b\right]_q \bar{\partial}$$

The preceding versions are all up on the manifold (more or less). Alternatively, one could use local charts to carry out the same calculation down in the model spaces. For a smooth function $f : M \to N$ with $q = f(p)$, choose charts (U, φ) centered at p and (V, ψ) centered at q. Repeating the calculation yet again yields

$$D(\psi f \varphi^{-1})(0_n) \cdot a = b \;\mapsto\; b^j \bar{\partial}_j = b^T \bar{\partial} \tag{2.6.2}$$ {deqn1}

where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are column vectors; for germs, *mutatis mutandis.*

2.7 Ends of manifolds

{ends}

Recall that a relation $\leq$ is *directed* if and only if for all $a, b \in D$ there exists a $c \in D$ such that $a \leq c$ and $b \leq c$. A *directed set* $(D, \leq)$ is a set D together with a reflexive, transitive, and directed relation $\leq$. A directed set $(D, \leq)$ can be thought of as a category D with elements of D as objects, and arrows $a \to b$ if and only if $a \leq b$.

Let C be any category. An *inverse system* in C is a cofunctor $\mathcal{F} : D \to C$. Define another cofunctor by

$$\mathcal{G}(S) := \left\{ (g_a) \in \prod_{a \in D} \hom(S, \mathcal{F}a) \;\middle|\; \mathcal{F}(a \to b) g_b = g_a \right\}.$$

A universal $(L, (u_a))$ for this $\mathcal{G}$ is called the *inverse limit* of the inverse system, denoted $\varprojlim \mathcal{F}D$. The object L is a *classifying object* of $\mathcal{G}$: $\hom(S, L) \cong \mathcal{G}(S) : g \mapsto (u_a g)$.

Example 2.7.1 Subobjects, equalizers, and products are examples of inverse limits. The inverse limit of any descending chain $U_1 \supseteq U_2 \supseteq \cdots$ is given by $\cap_{i=1}^{\infty} U_i$.

Ex 2.7.2 Define *direct systems* in C and *direct limits via* duality.

Example 2.7.3 Quotients, coequalizers, and coproducts are examples of direct limits. The direct limit of any ascending chain $U_1 \subseteq U_2 \subseteq \cdots$ is given by $\cup_{i=1}^{\infty} U_i$.

Remark 2.7.4 In category theory, inverse limits are examples of limits, and direct limits examples of colimits, of functors.

In a concrete category it is possible to construct the direct limit as a quotient of the coproduct of the objects in the direct system, and the inverse limit as a subobject of the product.

Ex 2.7.5 Consider $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ as $\{x^{n+1} = 0\}$ in *Set*. This defines an ascending chain with direct limit denoted by $\mathbb{R}^\infty$. If considered in *Top* instead of *Set*, how does the topology carry through?

Example 2.7.6 Similarly, consider the ascending chain defined by $\mathbb{Z}_{p^k} \hookrightarrow \mathbb{Z}_{p^{k+1}}$ in *Grp* in the obvious way. The direct limit is denoted by $\mathbb{Z}_{p^\infty}$ and is isomorphic to a subgroup of the roots of unity.

Let X be a locally path-connected topological space and consider the collection $\{K_i, \leq\}$ of all compact subsets, partially ordered by inclusion. Notice that this partial ordering is directed. Moreover, $K_i \hookrightarrow K_j$ implies that $X \neg K_j \hookrightarrow X \neg K_i$, which in turn implies that $\pi_0(X \neg K_j) \hookrightarrow \pi_0(X \neg K_i)$. The set of *ends (of X)* is defined as the inverse limit $\varprojlim \pi_0(X \neg K_i)$.

If X is also σ-compact, then there is one end for each sequence $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$, where U_i is a (connected or path-) component of $X \neg K_i$.

Remark 2.7.7 An end represents a way to "go off to infinity" on a topological space. One says that a space X is *asymptotically something* if and only if all ends of X are *something.*

We now assume the space is a manifold, denoted by M.

Definition 2.7.8 A *manifold M with boundary ∂M* is one on which some charts take their values in the closed half-space $\{x^n \geq 0\}$ of $\mathbb{R}^n$. The *boundary* ∂M consists of those points whose images lie in the boundary hyperplane in $\mathbb{R}^n$. {bdry}

That the boundary is well defined follows from the Inverse Function Theorem 1.1.1. It is a fact [31] that ∂M is a (possibly disconnected) codimension-1 embedded submanifold (Definition 4.1.1) of M. One may also consider the even more general *manifolds with corners* in which some charts take values in closed subspaces $\{x^k \geq 0, x^{k+1} \geq 0, \ldots, x^n \geq 0\}$.[56] It is an old tradition to call a manifold *closed* if it is compact and without boundary (or corners), *open* if noncompact and without boundary.

Since each component of a smooth manifold is σ-compact, we may assume that each end is represented by a descending chain $U_1 \supseteq U_2 \supseteq \cdots$ as above.

Definition 2.7.9 An end $U_1 \supseteq U_2 \supseteq \cdots$ is *tame* if and only if some U_k (hence all U_i for $i \geq k$) is of the form $S \times (0, \infty)$ for $S \leq M$ a closed submanifold of codimension 1. Ends that are not tame are said to be *wild*. {tame}

Thus a tame end looks like a simple topological cylinder. A manifold M is the interior of a compact manifold with boundary if and only if all ends of M are tame.

Example 2.7.10 *A wild end.* Consider an infinite binary tree, thickened by replacing each edge with a tube and each vertex with a "pair of pants". The resulting space has a Cantor set of ends.

Ends of manifolds are studied in three general areas of mathematics: (differential) topology, differential geometry, and partial differential equations. Topologists study general properties of ends; for example, what kinds of ends are possible? In particular, they are interested in obtaining tameness theorems, for which they have good results in high dimensions ($\dim M \geq 6$).[71]

Geometers are also interested in tameness: they use asymptotic geometric properties to help prove theorems about tameness. When a manifold has a finite number of ends, geometers also study the geometry of the ends themselves and how the geometry of ends affects the geometry of the manifold as a whole, or *vice versa*. The main advantage of the geometric approach is that they are able to obtain results for all dimensions (≥ 2). For example, if a manifold M has finitely many ends, $\dim M \geq 3$, and sectional curvature is everywhere negative on M, then every end of M has the form of an infranilmanifold cross $\mathbb{R}$.[27, 21] So far, all the geometric study of ends has been done for Riemannian manifolds only.

In partial differential equations, researchers usually consider manifolds with a finite number of tame ends. If there is a notion of "circumference" c of a tame end, such as a volume element on S, ends are classified by the growth rate of c as one goes out an end. If c is (eventually) constant along a tame end, then the end is called *cylindrical*. If c grows linearly along an end it is called *conical*, and if c grows exponentially it is called *hyperbolic*. Researchers tend to study scattering theory on spaces with approximately cylindrical or conical ends, the heat kernel on spaces with hyperbolic ends, and symbol calculus on spaces with various kinds of ends. For example, it is proved in [35] that if all ends are almost cylindrical and one end is exactly cylindrical, then scattering data at the cylindrical end determines the isometry class of a Riemannian manifold with these ends.

Definition 2.7.11 Consider an n-dimensional manifold K with one tame end of type $S \times (0, 1)$, and an n-dimensional manifold H with two tame ends of type $S \times (0, 1)$. We may replace K by a compact manifold with one boundary component S, and H by a compact manifold with two diffeomorphic boundary components $S_1 \cong S \cong S_2$. Finally, form an infinite adjunction $M = K \cup_S H \cup_S H \cup_S \cdots$, and smooth afterward as necessary. We say that M has a *periodic end (in the sense of Taubes)*.

Using gauge theory and periodic ends satisfying additional conditions, Taubes proved [76] this major result.

Theorem 2.7.12 *There are uncountably many distinct differential structures on* $\mathbb{R}^4$.

3 Bundles

We mostly follow [8] with modifications and deviations as in [17]. A few results come from [34].

Suppose that $\pi : E \to M$ is a smooth surjection such that for every $p \in M$, we have each *fiber* $E_p := \pi^{-1}(\{p\})$ diffeomorphic to a specific manifold F. This is called a (smooth) *fiber bundle* over (the *base space*) M with *total spce* E and *model fiber* F, or just *a fiber bundle over* M if the precise identity of F is not relevant. One may pick a particular fiber as the model if convenient. A *trivial* bundle is a simple product $M \times F$. One thinks of fiber bundles as "twisted" products, because the *set* $E \cong M \times F$ while the topology and differential structure *need not* be products.

3.1 Chart, atlas, cocycle

To introduce bundle charts, we need to have $\pi^{-1}(U_i)$ diffeomorphic to $U_i \times F$. Let $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$ be one such diffeomorphism: a *bundle chart*. This is *local triviality*, implied by the paracompactness of M. As with (coordinate) charts on manifolds, we form the *transition functions* by setting $U_{ij} = U_i \cap U_j$, extending the notation in the obvious way, and defining

$$\varphi_{ij} := \varphi_i \circ \varphi_j^{-1} : U_{ij} \times F \to U_{ij} \times F.$$

Without loss of generality, we may assume that each $\varphi_{ij}|_{U_{ij}} = 1_{U_{ij}}$. Note that the denoted restriction is actually a restriction composed with a projection, and the projection has been suppressed. By analogy with manifolds, we have a *bundle atlas* $\mathfrak{B}$.

Observe that $\varphi_{ij}(x, \,)$ is a diffeomorphism of F for each fixed $x \in M$ by definition. Thus we may consider φ_{ij} as a map from U_{ij} to the automorphism group of F, $\mathrm{Aut}(F)$. We want this map to be "smooth" in some reasonable sense. As $\mathrm{Aut}(F)$ may not possess any smooth structure, we have to decide *a priori* what we shall mean by "smooth." By definition, $\varphi_{ij}(x)f$ is smooth in

f for each fixed x and smooth in x for each fixed f (as a map $U_{ij} \to F$). This will be sufficient for our purposes here.

Then on U_{ijk} we have $\varphi_{ij}\varphi_{jk} = \varphi_i\varphi_j^{-1} \circ \varphi_j\varphi_k^{-1} = \varphi_i\varphi_k^{-1} = \varphi_{ik}$. Hence $\varphi_{ii} = 1$ and $\varphi_{ij} = \varphi_{ji}^{-1}$.

The essence of being a bundle lies in the *transition cocycle* which is defined as the collection of all the transition functions for a bundle atlas $\mathfrak{B}$ (or for an equivalence class of such atlases). Thus the transition functions satisfy the *cocycle conditions*

(i) $\varphi_{ij} = \varphi_{ji}^{-1}$ and

(ii) $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.

It is common to call a transition cocycle a bundle cocycle and just condition (ii) the cocycle condition, since (ii) implies (i). We say that two such cocycles $\{\varphi_{ij}\}$ and $\{\psi_{ij}\}$ are *equivalent* (or *cohomologous*) if and only if there exists $\{h_i : U_i \to \mathrm{Aut}(F)\}$ such that $\psi_{ij} = h_i^{-1}\varphi_{ij}h_j$. An equivalence class of bundle cocycles defines the *bundle structure*.

{bdlm} **Definition 3.1.1** A bundle *morphism* is a pair (u, f) of (smooth) maps, $u : E \to E'$ which preserves fibers and $f : M \to M'$ such that the diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

Note that u completely determines f. If the fibers have enriched structure (*e.g.*, vector space), the induced maps on fibers $u_p : E_p \to E'_{f(p)}$ are usually required to preserve it (*e.g.*, be linear).

As usual for subobjects, a *subbundle* is one for which the morphism (u, f) has u an inclusion; in addition, one also requires that $M = M'$ and that $f = 1_M$. This means that each fiber E_p is a subobject of the fiber E'_p (*e.g.*, a vector subspace).

{sbdlx} **Example 3.1.2** If E and E' are bundles of vector spaces, then $\ker(u)$ is a subbundle of E. If also f is surjective, then $\mathrm{im}(u)$ is a subbundle of E'.

{bdleq} **Theorem 3.1.3** *Two bundles with the same fiber over the same base space are isomorphic if and only if their bundle classes are equivalent.* □

The proof is left as an exercise.

{lineovrs1} **Example 3.1.4** Let us consider S^1 as a smooth manifold (by embedding in $\mathbb{R}^2$ or in $\mathbb{C}$ and inheriting the smooth structure). We take $\mathbb{R}$ to be the model fiber and take two overlapping charts U_1 and U_2 to cover S^1 which give rise to two disjoint intersections, U_{12} and U'_{12}. Now we consider the subgroup of the diffeomorphism group of $\mathbb{R}$ that assigns $+1$ to U_{12} and ± 1 to U'_{12}. As a set this construction is $S^1 \times \mathbb{R}$, but as a smooth bundle we also have the "twisting" by the cocycle. If U'_{12} is assigned $+1$, then we have the manifold $S^1 \times \mathbb{R}$, an infinite cylinder. On the other hand if U'_{12} is assigned -1, then we have a smooth Möbius band.

Example 3.1.5 A covering space is a bundle with discrete fibers. How about the converse? {csbdf}

As the definition stands, the domain of a bundle chart is not necessarily linked in any manner to the domain of any existing manifold chart. On the other hand, it is possible to have common domains for the manifold and bundle charts. We can achieve this by first considering the union of the covers in a manifold and a bundle atlas; then we take a common refinement and create a new, locally finite cover such that each open set in the new cover is contained in some chart domain of each of the original atlases. This is possible by the paracompactness of the underlying manifold. Thus for this common refinement, we get an atlas that is simultaneously a manifold atlas and a bundle atlas, and is also a locally finite cover.

Any right inverse to a surjection is called a *section*. Sections of the map $\pi : E \to M$ are called (F-valued) *fields* (on M). Any section σ of π has the property that $\pi \circ \sigma = 1_M$. We focus on the space $\Gamma(E)$ of smooth sections, of course. Note that for each bundle E and section space $\Gamma(E)$, there is a corresponding sheaf of germs of (smooth) sections $\mathscr{E}$ and its (continuous) sections $\Gamma(\mathscr{E})$.

Let us reflect a bit before proceeding. Semidirect products are algebraically twisted and topologically trivial. Bundles and sheaves are algebraically trivial and topologically twisted. We shall see that section spaces are twisted both ways. One may think of bundles as generalized topological/smooth products or as generalized parametrized families of spaces, with the base space as the parameter space.

3.2 Constructions with bundles

{cwb}

{prbdl} Given two bundles $F \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} M$ and $F' \overset{\iota'}{\hookrightarrow} E' \overset{\pi'}{\twoheadrightarrow} M'$, the *product bundle* $E \times E'$ over $M \times M'$ is described by this diagram.

$$\begin{array}{ccc} F \times F' & \xhookrightarrow{\iota \times \iota'} & E \times E' \\ & & \Big\downarrow{\scriptstyle \pi \times \pi'} \\ & & M \times M' \end{array}$$

The model fiber of a product bundle is the product of the model fibers of each factor. By inducting on the number of factors, one may construct product bundles with any finite number of factors.

{cbdl} Now suppose that $E \overset{\pi}{\twoheadrightarrow} M$ is a bundle with model fiber F and $E' \overset{\pi'}{\twoheadrightarrow} E$ is a bundle with model fiber F'. The *composite bundle* $\pi \circ \pi' : E' \twoheadrightarrow M$ has model fiber $F \times F'$.

Ex 3.2.1 Explain how the model fiber $F \times F'$ is included in E'.

Again, one may induct on the number of bundles in the composition to form *towers* of bundles over M.

{pbdl} **Definition 3.2.2** Let $f : M \to N$ be a smooth map between two smooth manifolds and E a smooth bundle over N. Then the *pullback bundle* f^*E over M is given pointwise by $(f^*E)_x := E_{f(x)}$.

{pbdlx1} **Ex 3.2.3** Prove that if E is smooth over N, then f^*E is smooth over M with projection $f^*\pi$, where $f^*\pi$ is the pullback of π along f as in algebra [51].

A pullback bundle is a universal: algebraically, it's a *fibered product* [51]. The general concept of a pullback is categorical. The exercise proves that the category of smooth vector bundles has pullbacks.

{pbdlx2} **Ex 3.2.4** Show that f^*E can be identified as a subset of $M \times E$.

{rstbdl} **Example 3.2.5** If $\iota : M \hookrightarrow N$ is an inclusion, then ι^*E is called the *restriction* of E to M and is also denoted by $E|_M$.

{cbdlx} **Example 3.2.6** Given a bundle $E \overset{\pi}{\twoheadrightarrow} M$, the pullback π^*E of E along π is a composite bundle over M.

{dm} Let $\Delta : M \to M \times M : p \mapsto (p, p)$ be the *diagonal map*. Suppose E and E' are bundles over M and consider the pullback $\Delta^* E \times E'$ of the product bundle along Δ.

{ws} **Definition 3.2.7** When the model fibers F and F' are vector spaces, we define the *Whitney sum* to be $E \oplus E' := \Delta^* E \times E'$ with model fiber $F \oplus F'$ over M.

Thus $(E \oplus E')_p = E_p \oplus E'_p$ for every $p \in M$.

Proposition 3.2.8 *If F is a vector space, then the composite bundle $\pi^* E$ is naturally isomorphic to $E \oplus E$.* {cbws}

Proof: Compute $\pi^* E = \{(v, w) \in E \times E \mid \pi v = \pi w\} \cong \Delta^* E \times E = E \oplus E$. $\square$

Suppose $e \in (f^* E)_x$. Then the *pushforth* is defined by $f_\natural e := e \in E_{f(x)}$. In effect, the pushforth $f_\natural$ acts as an inclusion does: it changes the set in which we regard the element as being. {pf}

Ex 3.2.9 In the context of Ex 3.2.4, $f_\natural$ coincides with pr_2.

Ex 3.2.10 If f is injective, the pushforth $f_\natural$ extends to sections $\Gamma(f^* E) \to \Gamma(E|_{\mathrm{im}(f)})$.

Note that this pushforth is *not* a pushout or pushforward as used in categorical algebra and sheaf theory; it is a more natural entity, hence the notation.

One may show that operators on sections of E also pull back to operators on sections of $f^* E$; for instance, an operator $\mathcal{D}$ on $\Gamma(E)$ induces $f^* \mathcal{D}$ on $\Gamma(f^* E)$, *i.e.*, $\mathcal{D} \mapsto f^* \mathcal{D}$; this yields "restrictions"—operators *along* f.

3.3 G-bundles

Let us consider F to be a (left) right G-space. That is, there exists a morphism $G \to \mathrm{Aut}(F)$ (a [left] right action of G on F) which is *effective* (the kernel is the identity). Then we may regard the map $G \to \mathrm{Aut}(F)$ as an inclusion. If a bundle cocycle takes values only in $G \leq \mathrm{Aut}(F)$ then we label the maps with their image points in the group G. So, instead of φ_{ij} we write g_{ij} and call it a *G-cocycle*, with values in G regarded as a subgroup of $\mathrm{Aut}(F)$ by a monomorphism. We say that G is the *structure group* of the G-bundle. {gbdl}

Definition 3.3.1 A G-bundle *morphism* is a pair (u, f) of maps, $u : E \to E'$ {gbdlm}
is equivariant and $f : M \to M'$ such that the diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

Note that an equivariant u is fiber-preserving, and completely determines f as before.

If $F = V$ is a (real) vector space and $G \leq GL(V)$, a G-bundle is called a (real) *vector bundle*. Those most interesting for us will be when $V \cong \mathbb{R}^k$ for some $k \in \mathbb{N}$ and $G \leq GL_k = GL_k(\mathbb{R})$. When $k = 1$ it is a *line* bundle, for $k = 2$ a *plane* bundle, and in general a k-*plane* bundle. The (infinite) cylinder and Möbius band are the only two line bundles over S^1 up to isomorphism.

{nfs2} **Ex 3.3.2** If E is a vector bundle over a connected M, then $\Gamma(E)$ is a nuclear Fréchet space. [Hint: the proof is the same as that for $\mathfrak{F}$, *mutatis mutandis.*]

In fact, it suffices that the model fiber vector space be complete and nuclear [79, p. 533]. On a trivializing open set U, it follows that $\Gamma(E|_U) \cong \mathfrak{F} \widehat{\otimes} V$ where V is the model fiber.

{shfsec} **Example 3.3.3** When E is a finite-dimensional vector bundle, we again obtain an isomorphism of real vector spaces $\Gamma(E) \to \Gamma(\mathscr{E})$. Note that $\mathfrak{F} \cong \Gamma(\mathbb{R} \times M)$, the trivial line bundle over M.

{affbdl} **Example 3.3.4** There are also *affine bundles* where $G \leq GA_n \cong \mathbb{R}^n \ltimes_\theta GL_n$, the affine group of Example 1.2.1.13, and the model fiber is the n-dimensional affine space $\mathbb{A}^n$ of Example 1.2.3.10. Just as each affine space has an associated vector space of translations, each affine bundle E has an associated vector bundle $E^\sharp$ of translations. From Example 1.3.1.16, it follows that the total spaces are homeomorphic, $E \cong E^\sharp$. Moreover, $\Gamma(E)$ is an affine space with translations $\Gamma(E^\sharp)$ so $\Gamma(E)$ is nonempty and is an affine space with translations a nuclear Fréchet space, or an "affine nuclear Fréchet space" for short.

When we speak of G-bundles hereinafter, we shall always assume that G
{lg1} is a *Lie group*: a smooth manifold for which the binary operation $G \times G \to G$
and the unary operation $g \mapsto g^{-1}$ are both smooth; *viz.* Chapter 7.

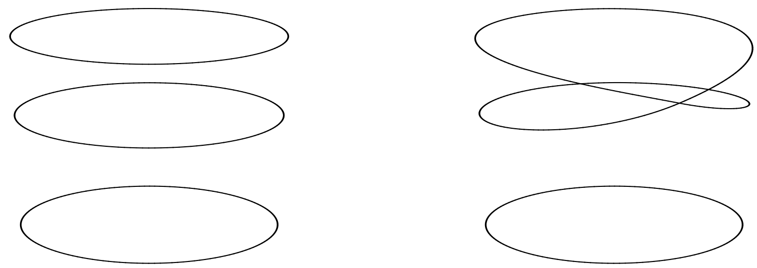


Figure 3.1: principal $\mathbb{Z}_2$-bundles over S^1

{pz2ovrs1}

3.4 Principal G-bundles

{pgbdl} Now assume that the model fiber of a G-bundle P is the group G itself and that the action of G on itself is the right regular representation. In other words, the right action of G on itself is $G \times G \to G : (x, g) \mapsto xg$ and the induced action of G on P is *free*, short for *fixed-point free*. In this case we call P a *principal* G-bundle. Note that the total space P itself is a principal G-set as after Definition 1.2.3.3, and that each fiber is a G-torsor as in Definition 1.2.3.7.

Ex 3.4.1 The only principal G-bundle morphisms are isomorphisms. {mpgb}

Example 3.4.2 Continuing from Example 3.1.4, note that there are four O_1-cocycles: $\{1,1\}, \{-1,1\}, \{1,-1\}$ and $\{-1,-1\}$. Observe that $\{1,1\}$ and $\{-1,-1\}$ are equivalent, and that $\{-1,1\}$ and $\{1,-1\}$ are equivalent. Since there are two equivalence classes of O_1-cocycles over S^1, we obtain two principal $\mathbb{Z}_2$-bundles over S^1 (up to isomorphism) as shown in Figure 3.1. {z2ovrs1}

Ex 3.4.3 Is either a covering space of S^1?

Ex 3.4.4 How many isomorphism classes of principal $\mathbb{Z}_n$-bundles are there over S^1?

If P is a principal H-bundle which maps to a principal G-bundle Q, we say that P is obtained from Q by a *reduction* (think of $H \leq G$) or *lifting* (think of H covering G) of the structure group, or that Q is obtained from P by an *extension* or *prolongation* of the structure group. From the definitions we immediately obtain the next result.

Proposition 3.4.5 *For a subgroup H of G, a principal G-bundle admits a reduction to H if and only if there exists a bundle atlas for which the transition functions are H-valued.* {reduc}

Proof: Translating from classical into cocycle language, a "bundle atlas with H-valued transition functions" is a "G-cocycle which is also an H-cocycle." □

{affpro} **Example 3.4.6** A principal affine bundle may be regarded as resulting from the prolongation of the structure group from GL_n to GA_n.

Now we come to possibly the single most important example of a principal bundle.

{L} **Example 3.4.7** On an atlas $\{(U_i, \varphi_i)\}$ of M, define

$$l_{ij}(x) = D\left(\varphi_i \circ \varphi_j^{-1}\right)(\varphi_j(x)) \tag{3.4.1}$$

for all $x \in M$. Here $\varphi_i\varphi_j^{-1}$ is a diffeomorphism between two open sets in $\mathbb{R}^n$. The derivative is the linear transformation $D\,\varphi_i\varphi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ which we identify with its matrix on the standard basis, the Jacobian matrix of $\varphi_i\varphi_j^{-1}$. We evaluate it at the point $\varphi_j(x) \in \mathbb{R}^n$ and obtain $l_{ij}(x)$ for each $x \in M$. It is easy to see that l_{ij} satisfies the cocycle condition. Hence l_{ij} as defined above is a GL_n-cocycle. We call it the *Leibniz cocycle* in honor of his association with the chain rule. The equivalence class of this cocycle defines the principal GL_n-bundle called the *(linear) frame bundle* or *bundle of linear frames LM*, or just L when only one manifold is being considered. A local or global section of L is a *local* or *global frame field*. When the context allows, *frame field* is usually shortened to *frame*.

In order to understand the working of a linear frame, we take two coordinate charts that overlap, compute the function $\varphi_i \circ \varphi_j^{-1}$, differentiate it, and evaluate the derivative at a point in $\mathbb{R}^n$ to get an invertible (as $\varphi_i \circ \varphi_j^{-1}$ is a diffeomorphism) GL_n matrix, say A. This matrix acts on each fiber of the bundle by right translation. In other words, pick any point in any fiber of the frame bundle and just right multiply by A. Notice that every fiber of the principal bundle is an isomorphic copy of the group GL_n but in no particular way, because every right translation of GL_n by A moves the identity to A and moves A^{-1} to the identity. This denies us a natural or canonical way to identify any fiber with GL_n. On the other hand, as soon as we pick local coordinates and bundle charts, we immediately obtain a fiberwise isomorphism with GL_n in each bundle chart.

{frm} **Remark 3.4.8** In linear algebra, we learned that GL_n is the set of all isomorphisms of $\mathbb{R}^n$ with itself. It is not always explained there that GL_n also is the set of all frames (bases) for $\mathbb{R}^n$; the columns of any matrix in GL_n give a basis for $\mathbb{R}^n$, and the matrix itself is the change-of-basis matrix from this basis to the standard basis. In this view, l_{ij} acts as a change-of-basis matrix at each $p \in M$. In addition, a translation moves the identity. Thus a section of L over U gives us a local field of frames, one at every point in U. The fiber above each point is the set of all possible frames at that point of $\mathbb{R}^n$.

Example 3.4.9 A reduction of L from GL_n to O_n determines a Riemannian structure, to O_p^q with $p+q=n$ a pseudoRiemannian structure, to Sp_n when n is even a symplectic structure, to SL_n a volumetric structure, *etc.* {reduc1}

It turns out that every fiber bundle can be thought of as associated to some principal bundle, possibly more than one. This may be not of much interest if the fiber is F and the principal bundle is the $\mathrm{Aut}(F)$-bundle. On the other hand if the group G is smaller than $\mathrm{Aut}(F)$, it may force the bundle to have significantly restricted structure and reveal important information. But if G is *too* small we learn nothing: if G is the trivial group, then the bundle will be the trivial bundle $M \times F$.

Ex 3.4.10 (on the orientation bundle) The map $GL_n \to O(1) = \{\pm 1\}$ given by {or1}

$$g \longmapsto \frac{\det g}{|\det g|}$$

induces a map from GL_n-bundles to $O(1)$-bundles. The image under this map of a principal GL_n-bundle P is called the *orientation bundle* $\mathrm{or}(P)$. The bundle P is said to be *orientable* or *nonorientable* according as $\mathrm{or}(P)$ is trivial or twisted, respectively. The classical case of course is when P is the frame bundle LM of a manifold; we then write $\mathrm{or}(M) = \mathrm{or}_M$ for this orientation bundle and say that M is orientable or not.

If M is not orientable, then the total space or_M of the orientation bundle is orientable and called the *orientation double covering space* of M.

This brings us to a connection with representation theory, the preceding being one of the most celebrated instances. In general, any morphism $G \to H$ is called a *representation* of G in H. When H is the general linear group of some (possibly infinite-dimensional) vector space, the representation is said to be *linear*. We shall be most interested in representations $G \to \mathrm{Aut}(F)$ in the group of self-diffeomorphisms of a manifold F which are smooth in some reasonable sense. We say then that F is a *G-space* and the representation is *on* F. Linear representations are on vector spaces.

3.5 Associated bundles

{asbdl} Our immediate objective is to obtain a categorical method of changing the fibers in a principal bundle into any (reasonable) space, while preserving all the essential information of the principal bundle. Somewhat more intuitively, we want to think of a principal bundle as some kind of holder into which we can insert various fibers which may be changed at will. If we think of the fibers as 'vertical' and the base space as 'horizontal', then what we wish to do is to make a vertical change while preserving the horizontal information. Bundle cocycles tell us how to accomplish this.

In order to make parity consistent, we shall always consider only *left* G-spaces F and write gf to indicate that $g \in G$ is applied to $f \in F$ by means of the representation $\tau : G \to \mathrm{Aut}(F)$ which makes F a left G-space. Clearly, what we must do is to collapse each fiber of P to a point. Thus, define a right action of G on $P \times F$ by

$$(p, f)g = (pg, g^{-1}f)$$

and then define

$$P[F] := (P \times F)/G\,,$$

the orbit space of this action. We denote the equivalence class or orbit of (p, f) by $[p, f]$ and define a projection

$$\pi : P[F] \twoheadrightarrow M : [p, f] \longmapsto \pi(p)\,,$$

where the second $\pi : P \twoheadrightarrow M$ comes from the principal bundle as before. We call $P[F]$ an *associated* bundle of P with *structure group* G. If $\{g_{ij}\}$ is a representative cocycle of P, then $\{\tau(g_{ij}) = \tau \circ g_{ij}\}$ is a representative cocycle of the associated bundle $P[F]$. Note that associated bundles of linear representations are vector bundles. Also note that an alternative expression of the equivalence relation is $(pg, f) \sim (p, gf)$.

{band} **Ex 3.5.1** Work out the details for the plain band and the Möbius band as associated bundles of the two principal $\mathbb{Z}_2$-bundles over S^1. Use $\mathbb{R}$ or $[-1, 1]$ for the fiber F according as you want open or closed bands, respectively.

{smab} **Ex 3.5.2** Prove that if P and F are smooth, then so is $P[F]$.

{apb} **Ex 3.5.3** Let $\rho : P \times F \twoheadrightarrow P[F]$ be the quotient map in the construction of the associated G-bundle $P[F]$. As a manifold, $\rho : P \times F \twoheadrightarrow P[F]$ is a principal G-bundle over $P[F]$.

If we take the principal G-bundle cocycle $\{g_{ij}\}$ and suppress the representation of G on F, then all associated bundles $P[F]$ of a principal G-bundle P have the "same" equivalence class of bundle cocycles. In fact, P is a functor from the category of left G-spaces to the category of G-bundles over the base space of P.

Ex 3.5.4 Verify this. {pbfct}

All G-bundles are obtained by this process.

Proposition 3.5.5 *If E is a G-bundle with model fiber F, then $E = P[F]$ for some principal bundle P.* {Gbdlassoc}

Proof: Let $\mathfrak{B}$ be a bundle atlas for E with cocycle $\{\psi_{ij}\}$. By definition, F is a left G-space for some $\tau : G \to \operatorname{Aut}(F)$. Then $\psi_{ij} = \tau \circ \varphi_{ij}$ for some G-valued cocycle $\{\varphi_{ij}\}$. Take P to be the principal G-bundle with cocycle $\{\varphi_{ij}\}$. □

The next few results concern section spaces.

Theorem 3.5.6 *Let P be a principal G-bundle over M, F a left G-space, and $P[F]$ the associated G-bundle. Then for each $s \in \Gamma(P[F])$ there exists a unique equivariant $\phi_s : P \to F$ such that $\phi_s(pg) = g^{-1}\phi_s(p)$ for all $p \in P$ and $g \in G$.* {unisct}

Conversely, given $\phi \in \hom^{\mathrm{eqv}}(P, F)$ there exists a unique section $s_\phi \in \Gamma(P[F])$. Therefore $\Gamma(P[F]) \cong \hom^{\mathrm{eqv}}(P, F)$.

Proof: Given a section s, define ϕ_s implicitly by the relation $s(\pi p) = [p, \phi_s p] \in P[F] = (P \times F)/G$ where $\pi : P \to M$ is the projection. Since $[p, \phi_s p] = [pg, g^{-1}\phi_s p] = [pg, \phi_s(pg)]$, and since s is a section, the map ϕ_s is well defined and satisfies $\phi_s(pg) = g^{-1}\phi_s(p)$ for all $p \in P$ and $g \in G$.

Now, for $\phi \in \hom^{\mathrm{eqv}}(P, F)$ define $s_\phi(\pi p) = [p, \phi p] \in P[F]$. As $[pg, \phi(pg)] = [pg, g^{-1}\phi p] = [p, \phi p]$ in $P[F] = (P \times F)/G$, it follows that s_ϕ is well defined. Since s_ϕ is the factorization of $p \mapsto [p, \phi p]$ by the projection π, it follows that s_ϕ is smooth. Temporarily denoting the projection of $P[F]$ by π', one verifies that $\pi' s_\phi(\pi p) = \pi'[p, \phi p] = \pi p$ so s_ϕ is a section of $P[F]$. □

Ex 3.5.7 Verify that ϕ_s is smooth.

The first corollary is one of the most salient properties of smooth (or merely continuous) principal bundles.

Corollary 3.5.8 *A principal bundle has a smooth section if and only if it is trivial.* {triv}

A trivial bundle obviously has at least one smooth section (and usually, many).

Ex 3.5.9 Use the Theorem to prove the other direction.

Ex 3.5.10 Determine if and when a k-affine bundle is a principal $\mathbb{R}^k$-bundle.

This corollary is a key step in the homotopy classification of bundles.

{prnmrph} **Corollary 3.5.11** *If* (P, π, M) *and* (Q, π', N) *are principal* G*-bundles, then all principal morphisms* $P \to Q$ *are of the form* (ϕ_s, f) *where* s *is a section of* $P[Q]$*. Moreover,* $f = \pi' s$*.* □

The proof is another simple application of the Theorem and is left to the reader.

If E is a G-bundle, then there is a natural representation of G on $\Gamma(E)$ given by $sg(x) = (s(x))g$. Since $E = P[F]$ for some P and F, one may regard the representation of G on $\Gamma(E)$ as being decomposed by means of P into that of G on F. This can be generalized even more by gluing together bundles over various spaces which are themselves strata of some larger stratified space, as in Davis [15].

The generic situation here is for F to be finite dimensional and $\Gamma(E)$ to be infinite dimensional, so this is potentially of great importance in analyzing these most intractable representations. It is not known which representations can be so decomposed. In addition, there are large numbers of secondary problems with regard to the efficiency and beauty of any particular such decomposition, the minimum necessary class of fibers and base spaces, *etc.*

{qft} We should also note here the vital use of bundles in quantum field theories. All bundles are considered over one base space called the *spacetime.* The bundles themselves are the *quantum fields* and come in two types: a principal bundle is an *interaction* field, and certain of its associated vector bundles are the *matter* fields for that interaction. Sections of the vector bundles are *wavefunctions* for those types of matter. The interaction *potential* is a connection (parallelism or parallel transport structure) on the principal bundle, whose curvature is the *intensity* of the interaction. Associated with the connection on the principal bundle is an induced covariant derivative on each vector bundle, which gives the *coupling* of that matter field and the interaction field. The structure group is called the *gauge* group and expresses the *local symmetry* of the interaction. Elements of the individual fibers are called *phases* or *internal states*, and entire fibers are *points* of the fields. Each of the four known interactions (gravitational, electromagnetic, weak, and strong) has been expressed in this form, and some of them have been unified *via* prolongation of structure groups. Paraphrasing t'Hooft [33], while we don't have the master yet, at least all the keys are cut from the same blank.

3.6 The tangent bundle

We now define the most important vector bundle over a smooth manifold.

Definition 3.6.1 The *tangent bundle* of the manifold M with frame bundle L is $TM := L[\mathbb{R}^n]$. {tbdl}

Sections of the tangent bundle are called *vector fields* and the space of all such is denoted by $\Gamma(TM) = \mathfrak{X}(M)$, or just $\mathfrak{X}$ when M is fixed (or clear from context).

Ex 3.6.2 What happens to the pullback of vector fields along f if $E = TN$? {pbkvf}

Example 3.6.3 A reduction of structure group of TM from GL_n to O_n determines a Riemannian metric tensor, *etc.* {reduc2}

Next we consider equation (2.6.1) and the discussion leading up to it. Suppose we replace $f : M \to N$ by the identity map $1_M : M \to M$, allow $p \neq q$, and assume that $p, q \in U \cap V$. Then the equation becomes

$$D(\psi \circ \varphi^{-1})(0_n) \cdot a = b$$

with $a, b \in \mathbb{R}^n$. This shows that the vector bundle with fibers $\mathrm{Der}(\mathscr{F}_p) \cong \mathbb{R}^n$ has the Leibniz cocycle $D(\psi\varphi^{-1})$ as its transition cocycle. As any two vector bundles with the same model fiber and the same transition cocycle are isomorphic, we conclude that $\mathrm{Der}(\mathscr{F}) \cong TM$. This proves that point derivations are tangent vectors and (global) derivations are smooth vector fields. It also implies that $\mathrm{Der}\,\mathfrak{F} \cong \mathfrak{X}$.

Another popular characterization of tangent vectors is in terms of what are effectively physical velocity vectors. Consider the germ $[\gamma]_0$ of a path $\gamma : I \to M$ with $\gamma(0) = p$, and $I = (-a, a)$, $a > 0$. Denote the set of all such path germs at p by $\mathscr{P}_p$. Define an equivalence relation $[\gamma_1]_0 \sim [\gamma_2]_0$ if and only if

$$D(f \circ \gamma_1)(0) = D(f \circ \gamma_2)(0)$$

for all function germs $[f]_p$ at p. An equivalence class in $\mathscr{P}_p$ is a *velocity vector* at p and is denoted by $[\gamma]_\sim$. {velvec}

Proposition 3.6.4 *The map* $\mathscr{P}_p/\sim \to \operatorname{Der} \mathscr{F}_p$ *defined by*

$$\mathcal{D}_\gamma[f]_p := D(f \circ \gamma)(0)$$

is a bijection.

Proof: The map is injective by construction. It is surjective since $\gamma(t) := (ta^1, \ldots, ta^n)$ in local coordinates (U, x) yields $\mathcal{D}_\gamma = a^i \partial_i$. □

We can now transport the vector space and (commutative) Lie algebra structures back to $\mathscr{P}_p$ to make it a vector space and a Lie algebra.

Ex 3.6.5 In general one need only check equality of derivations on the coordinate functions (x^i) of any chart (U, x).

Proposition 3.6.6 *The induced tangent morphism acts as follows: for any* $\varphi : M \to N$, $[\gamma]_\sim \mapsto [\varphi \circ \gamma]_\sim$. □

Ex 3.6.7 Verify that this version of the tangent map agrees with $T_p\varphi$ and $\operatorname{Der}_p \varphi$.

{ch1} **Ex 3.6.8** If $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $\varphi(tx) = t\varphi(x)$ for all $t \in \mathbb{R}$, then φ is linear.

We recall from Corollary 2.6.7 and Definition 2.6.8 that the ∂_j form a local basis for the vector fields on U or for $TM|_U = TU$. Our goal is to obtain local coordinates on TM *induced* in some sense by the extra structure a tangent bundle provides us. As the total space, the tangent bundle is a smooth manifold in its own right and all the properties of a manifold hold true. In particular, a tangent bundle has its own charts and atlases. The set underlying the tangent bundle is $M \times \mathbb{R}^n$, but the topology need *not* be the product topology.

{tw} **Remark 3.6.9** If the cocycle's class is nontrivial then it *twists* the topology of $M \times \mathbb{R}^n$ and we do not obtain the product topology on the *set* $M \times \mathbb{R}^n$. On the other hand, if the cocycle's class is trivial then we do obtain the product topology on $M \times \mathbb{R}^n$.

As a manifold, the tangent bundle TM has dimension $2n$. For convenience, we may denote a point in TM as $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$, where we consider the x^j and y^j as functions on TM near some point, existing in a neighborhood around it. Since the ∂_i form a basis for vector fields on U, we write $v = y^i \partial_i$ for a $v \in \mathfrak{X}(U)$. Note that the y^i are real-valued functions on TM but are constant on fibers and thus the $y^i \partial_i$ are (local) vector fields on U. If we require y^i to be defined in this way, then we obtain *induced* local
{ilc} coordinates on TM. They are induced by the local coordinates (chart) (U, x).

There are advantages in dealing with induced coordinates. For example, the change of coordinate matrix from (x, y) to (x', y') is written as

$$\begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & 0 \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{bmatrix}$$

where $\frac{\partial x'}{\partial y} = 0$ since x' depends only on x, by definition of induced coordinates. Note that if $v \in T_pM$, then $v = (0, y) = (0, \ldots, 0, y^1, \ldots, y^n)$.

Example 3.6.10 The tangent bundle of $\mathbb{R}$ looks like $\mathbb{R}^2$ differentially, but *not* {tr2}
algebraically. The usual xy-coordinates of $\mathbb{R}^2$ are induced local coordinates for the identity chart (centered at $x = 0$) on $\mathbb{R}$. The only vector addition defined is that of tangent vectors at the same point: $(x, y_1) + (x, y_2) = (x, y_1 + y_2)$.

We now generalize the notion of derivatives in $\mathbb{R}^n$. A map $f : M \to N$
induces a map $f_* : TM \to TN$. This induced map is also denoted Tf by {tmap}
other authors.[1] Consider local charts (U_i, φ_i) in M and (V_j, ψ_j) in N, with $f(U_i) \subseteq V_j$. We insert f into a mixed Leibnizian cocycle as follows:

$$D(\psi_j \circ f \circ \varphi_i^{-1})\, \varphi_i(p)$$

meaning that the Jacobian matrix of the composition $\psi_j \circ f \circ \varphi_i^{-1}$ is evaluated at $\varphi_i(p)$ thereby inducing a map from $T_pM \to T_{f(p)}N$. In other words,

$$(p, v) \mapsto (f(p), f_*(v)) = (f(p), D(\psi_j \circ f \circ \varphi_i^{-1})\, \varphi_i(p) \cdot v)\,.$$

Ex 3.6.11 The functor T preserves products: $T(M \times N) \cong TM \times TN$ and {tprod}
$T(f \times g) = Tf \times Tg$.

Ex 3.6.12 Let E be a smooth vector bundle. Write a_{f} for fiberwise scalar multiplication by a and $+_{\mathrm{f}}$ for fiberwise vector addition. Compute $a_{\mathrm{f}*}$ and $+_{\mathrm{f}*}$. [Hint: E is a groupoid.] [Hint$^+$: look in [52] at the parts about Lie groupoids.]

[1] In modern functorial notation, only Tf is correct. I use f_* as it was functorial notation when I learned it. Some still use the old, wrong notation *df* even now; worse, they call it a differential.

3.7 The vertical bundle

For any bundle $F \hookrightarrow E \xrightarrow{\pi} M$, the map $\pi_* = T\pi$ exists and makes this diagram commute.

$$\begin{array}{ccc} TE & \xrightarrow{\pi_*} & TM \\ \downarrow{\scriptstyle \pi_T} & & \downarrow{\scriptstyle \mathrm{pr}} \\ E & \xrightarrow[\pi]{} & M \end{array} \qquad (3.7.1)$$ {tpi1}

We use this map π_* to define an important subbundle of TE.

{vbdl} **Definition 3.7.1** The *vertical bundle* over E is

$$\mathscr{V}E := \ker \pi_* = \ker T\pi$$

for $\pi : E \twoheadrightarrow M$ the natural projection.

{vbdlf} **Proposition 3.7.2** *If f^*E is the pullback of E along $f : N \to M$, then $\mathscr{V}f^*E$ and $f_\natural^*\mathscr{V}E$ are isomorphic, where $f_\natural$ is the pushforth of f, as shown.*

$$\begin{array}{ccccc} \mathscr{V}f^*E & \xrightarrow{\cong} & f_\natural^*\mathscr{V}E & \xrightarrow{f_{\natural\natural}} & \mathscr{V}E \\ & \searrow{\scriptstyle \pi_{\mathscr{V}}} & \downarrow{\scriptstyle f_\natural^*\pi_{\mathscr{V}}} & & \downarrow{\scriptstyle \pi_{\mathscr{V}}} \\ & & f^*E & \xrightarrow{f_\natural} & E \\ & & \downarrow{\scriptstyle f^*\pi} & & \downarrow{\scriptstyle \pi} \\ & & N & \xrightarrow[f]{} & M \end{array}$$

In summary, the vertical bundle is functorial.

Proof: By a computation based on Ex 3.2.4,

$$Tf^*E = \{(u,v) \in TN \times TE \mid f_*u = \pi_*v\}.$$

Thus if $(u,v) \in Tf^*E$, then $\mathrm{pr}_*(u,v) = u = 0$ if and only if $v \in \mathscr{V}E$. The map $f_\natural : (u,v) \mapsto v$ is an isomorphism of fibers of $\mathscr{V}f^*E \to \mathscr{V}E$ along $f_\natural$. □

{vbpb} **Proposition 3.7.3** *If $E \xrightarrow{\pi} M$ is a vector bundle, then $\mathscr{V}E$ is isomorphic to π^*E over E.*

Proof: If $(u,v) \in \pi^*E$, then $\pi(u+tv)$ is constant in t. Thus the map $(u,v) \mapsto \frac{d}{dt}(u+tv)|_{t=0}$ maps π^*E into $\mathscr{V}E$. Clearly this is a vector bundle isomorphism. □

Using Proposition 3.2.8, the next result is immediate.

Corollary 3.7.4 *For a vector bundle E over M, the composite vector bundle* {vrtws}
$\pi\pi_{\mathscr{V}} : \mathscr{V}E \twoheadrightarrow M$ is isomorphic to $E \oplus E$ over M. □

Definition 3.7.5 Given a vector bundle E over M, denote the isomorphism {cptj}
$\pi^*E \to \mathscr{V}E$ by

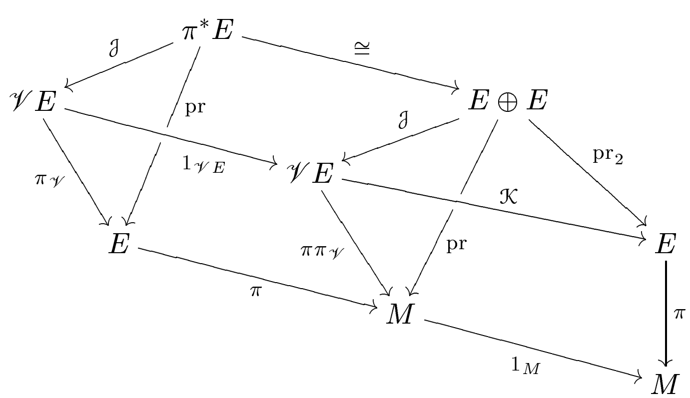
$$\mathcal{J}_u v = \mathcal{J}(u,v) := \frac{d}{dt}(u+tv)|_{t=0},$$

and the fiber isomorphism $\mathscr{V}E \to E$ along π by

$$\mathcal{K} : \mathscr{V}E \to E : \mathcal{J}_u v \mapsto v.$$

Ex 3.7.6 Both $\mathcal{J}$ and $\mathcal{K}$ are bundle versions of canonical parallel translation on a vector space.

Summarizing the relationships of these bundles and their bundle morphisms, we have this diagram in which all bundle morphisms are isomorphisms of fibers.



(3.7.2) {jk1}

Definition 3.7.7 Given a vector bundle $E \xrightarrow{\pi} M$ one may define a vector bundle structure on $TE \xrightarrow{\pi_*} TM$ by

$$\begin{aligned} u + v &:= \alpha_*(u,v) \quad \text{for } \pi_* u = \pi_* v \text{ in } TM, \\ c \cdot v &:= m_{c*} v \quad \text{for scalar } c \text{ and } v \in TE, \end{aligned}$$

where $\alpha(e,e') = e + e' \in E \oplus E$ and $m_c e = ce$ for scalar c and all $e \in E$.

Ex 3.7.8 Verify this is indeed a vector bundle structure.

Proposition 3.7.9 *The vertical bundle* $\mathscr{V}E$ *is a vector subbundle with respect to* π_* *of* TE *over* TM. □ {vrtsbb}

Ex 3.7.10 Consider $\mathscr{V}E$ over M when M is embedded in TM as the 0-section.

There is a canonical or natural parallel translation on a vector space V. It may be regarded as a system of diffeomorphisms between tangent spaces at distant points in V. We want to generalize this to *parallel transport* on a manifold M. This can be accomplished in at least two ways.

{pcm} **Definition 3.7.11** A *parallelism* (*structure*) on M is the specification of which vector fields are (*absolutely* or *globally*) *parallel*. Equivalently, one provides a suitable subbundle $\mathscr{H}$ of TTM that is complementary to $\mathscr{V}$ so that $\mathscr{H} \oplus \mathscr{V} = TTM$, a *horizontal* bundle or *connection* on M.

Here, $\mathscr{H}$ being suitable means that it determines a system of parallel transport, which not all complementary subbundles do; see the companion volume on connection geometry. If E is a G-bundle over M and the complementary bundle $\mathscr{H} \leq TE$ is G-equivariant, then $\mathscr{H}$ is called a *G-connection* on E over M.

This occurs naturally in one special case.

{||zbl} **Definition 3.7.12** A manifold M^n is called *parallelizable* if and only if its tangent bundle is trivial: $TM = M \times \mathbb{R}^n$.

{||zblx1} **Ex 3.7.13** Show that this is equivalent to LM being trivial.

{can||sm} **Example 3.7.14** Define the parallel vector fields on a parallelizable manifold M to be the constant ones. This is the *canonical* or *natural* parallelism on M. This yields parallel transport by $1_{\mathbb{R}^n}$ that is independent of path in M, and also extends along submanifolds.

{||smx1} **Ex 3.7.15** There is also a parallelism for each global frame on M. The parallel vector fields are those that have constant coefficients with respect to the global frame.

The equivalent canonical or natural connection may be seen thusly:

$$TTM = T(M \times \mathbb{R}^n)$$

$$= TM \times T\mathbb{R}^n$$
$$= TM \times (\mathbb{R}^n \oplus \mathbb{R}^n).$$

The vertical bundle is $\mathscr{V} = TM \times (0 \oplus \mathbb{R}^n)$, and has the canonical complement $\mathscr{H} = TM \times (\mathbb{R}^n \oplus 0)$; *cf.* induced local coordinates on TM.

3.8 Affine bundles

We have already encountered affine spaces and affine bundles in Examples 1.2.3.10 and 3.3.4. Now we shall take a closer look at them.

Let V be a (real) vector space. Recall that an affine space A on V is a V-torsor. (Since V is an abelian group, we need not distinguish between left and right.) If V is finite dimensional, define $\dim A := \dim V$. Further recall that A being a V-torsor means that V acts simply transitively on A. From Remark 1.2.3.8, this yields a V-valued binary operation on A denoted $a - b$ for $a, b \in A$. For $v \in V$, denote the V-action by $a + v = v + a$. {affsp1}

Example 3.8.1 Regarded simply as a set, any vector space V may be considered as an affine space $V^\flat$ on itself. Similarly, any affine space A on V may be regarded as its own group of translations $A^\sharp \cong V$. {affsp1x1}

Example 3.8.2 Let $0 \to V_1 \to V_2 \to V_3 \to 0$ be a short exact sequence of (real) vector spaces, and let $\mathrm{spl}(V_3, V_2)$ be the set of splittings. Then $\mathrm{spl}(V_3, V_2)$ is affine on $\mathrm{Hom}(V_3, V_1)$. {affsp1x2}

Example 3.8.3 If A is affine on V and V is either finite-dimensional or smooth, then A is a smooth manifold diffeomorphic to V. {affsp1x3}

As expected, there is a category Aff with affine spaces as objects and affine morphisms as arrows. A function $f : A_1 \to A_2$ is *affine* if and only if there is a (unique) linear map $\mathcal{L}f : V_1 \to V_2$ such that $f(a + v) = f(a) + \mathcal{L}f(v)$ for all $a \in A_1$, $v \in V_1$. {affcat}

Ex 3.8.4 Verify that $\mathcal{L}f$ is well defined (unique) *via* the usual algebraic swindle. {Lwd}

It follows that $\mathcal{L} : \mathsf{Aff} \to \mathsf{Vec}$ defined by $\mathcal{L}A = V$ on objects, and as above on arrows, is a functor. The linear map $\mathcal{L}f : V_1 \to V_2$ is called the *linear part* of the affine map $f : A_1 \to A_2$. {linfct}

Here is another construction of interest.

Definition 3.8.5 The *mixed dual* of an affine space A is $A^\dagger := \hom_{\mathsf{Aff}}(A, \mathbb{R})$. {mxd}

It is indeed "mixed" as we are taking affine morphisms of an affine space to a linear space; the codomain being a vector space means that $A^\dagger$ is also a vector space. The proof of this lemma is left as an exercise.

{mxdl} **Lemma 3.8.6** *If* $\dim A = \dim V$ *is finite, then* $\dim A^\dagger = \dim A + 1$. □

The mixed dual defines a cofunctor $(\ )^\dagger : \mathsf{Aff} \to \mathsf{Vec}$ that acts on arrows as in a hom-cofunctor. If $f : A_1 \to A_2$ is an affine morphism, then $f^\dagger : A_2^\dagger \to A_1^\dagger : \alpha \mapsto \alpha \circ f$.

Let V be a vector space with dual V^* or strong dual V'.

{mxdp1} **Proposition 3.8.7** *The map* $i : A \to (A^\dagger)' : a \mapsto \langle\ , a\rangle$ *is an affine injection.*

Proof: The linear part is $\mathcal{L}i : V \to (A^\dagger)' : v \mapsto \langle \mathcal{L}(\ ), v\rangle$, and indeed $\ker \mathcal{L}i = 0$. □

{mxdp2} **Proposition 3.8.8** *The map* $\tau : \mathbb{R} \to A^\dagger$ *that sends the real number* t *to the constant affine function* t *on* A *extends to the short exact sequence*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\tau} A^\dagger \xrightarrow{\mathcal{L}} V' \longrightarrow 0.$$

Proof: If $\varphi \in A^\dagger$ and $\mathcal{L}\varphi = 0$, then $\varphi(a + v) = \varphi(a)$, so $\varphi \in \operatorname{im} \tau$. The rest is obvious. □

{mxdc} **Corollary 3.8.9** *If* A *is finite dimensional, then we obtain the short exact sequence*

$$0 \longrightarrow V \xrightarrow{\mathcal{L}i} (A^\dagger)^* \xrightarrow{\tau^*} \mathbb{R} \longrightarrow 0$$

with $\tau^* V = 0$ *and* $\tau^* A = 1$.

Proof: One need only check that $\mathcal{L}(\ )^* = \mathcal{L}i$. □

{mxdx1} **Ex 3.8.10** Verify this claim, then do so.

{mxdx2} **Ex 3.8.11** Every finite-dimensional affine space A appears this way as a co-dimension-1 hyperplane in a vector space that is then $(A^\dagger)^*$. What happens when the dimension is infinite?

Now we are ready to consider affine bundles. Let A be affine on V and consider two bundles E^A with model fiber A and E^V with model fiber V over the same base. Let $\{\varphi_{ij}^A\}$ be a representative cocycle for E^A with $\varphi_{ij}^A(p)$ an affine automorphism of A. Then $\varphi_{ij}^V(p) := \mathcal{L}\varphi_{ij}^A(p)$ is a linear automorphism of V, and $\{\varphi_{ij}^V\} = \{\mathcal{L}\varphi_{ij}^A\}$ is a vector bundle cocycle.

{affbdl1} **Definition 3.8.12** We say that E^A with (affine bundle) cocycle $\{\varphi_{ij}^A\}$ is an *affine bundle on the vector bundle* E^V with (vector bundle) cocycle $\{\varphi_{ij}^V\} = \{\mathcal{L}\varphi_{ij}^A\}$.

The next two examples should be compared with 3.8.1 and 3.8.2.

Example 3.8.13 Any vector bundle E^V may be regarded as an affine bundle $(E^V)^\flat$ on itself. Similarly, any affine bundle E^A on E^V may be regarded as its own bundle of translations $(E^A)^\sharp \cong E^V$. {affbdl1x1}

As an exercise, compare the bundles $(E^V)^\flat$ and $(E^A)^\sharp$ with $(E)^{V^\flat}$ and $(E)^{A^\sharp}$, respectively.

Example 3.8.14 Let $0 \to E_1 \to E_2 \to E_3 \to 0$ be a short exact sequence of vector bundles and let $\operatorname{spl}(E_3, E_2)$ be the bundle of splittings. Then $\operatorname{spl}(E_3, E_2)$ is affine on the hom-bundle $\operatorname{Hom}(E_3, E_1)$. {affbdl1x2}

Example 3.8.15 One particular case of the latter is of significant interest. Let P be a principal G-bundle over M and let $\mathfrak{g}$ be the Lie algebra of G; *viz.* Section 7.2 *et seq.* We need only note here that $\mathfrak{g}$ is a real vector space and a left G-space. Thus $P[\mathfrak{g}]$ is a vector bundle over M. Next, recall that P itself is a right G-space and that $P/G \cong M$. If we look at $TP \twoheadrightarrow P$, then TP is also a right G-space. {affbdl1x3}

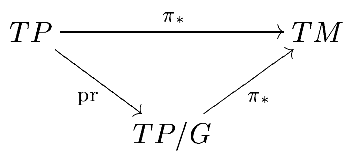
Ex 3.8.16 If the action is $P \times G \to P$, then we get $TP \times TG \to TP$ by the functoriality of T. The 0-section σ_0 embeds G into TG, hence induces an action of G on TP by restricting to $\operatorname{im}\sigma_0$, as claimed. {affbdl1x4}

Thus TP/G is a vector bundle over M, and there is a short exact sequence of vector bundles (over M)

$$0 \longrightarrow P[\mathfrak{g}] \longrightarrow TP/G \xrightarrow{\pi_*} TM \longrightarrow 0.$$

Remark 3.8.17 This short exact sequence is an example of a *Lie algebroid*; consult [48, 49] as the standard references.

Calling this arrow π_* is justified by the following diagram.



This diagram commutes as $\pi_* : TP \to TM$ respects the orbit equivalence relation for the action of G on TP, whence it factors uniquely through the quotient TP/G.

According to Atiyah [2], splittings of this short exact sequence correspond bijectively with (principal) G-connections on P. From Example 3.8.14, $\mathrm{spl}(TM, TP/G)$ is affine on $\mathrm{Hom}(TM, P[\mathfrak{g}])$, providing a much richer structure on the space of (principal) G-connections than is first apparent.

Affine bundles form a category denoted by *AffBdl*. The arrows are bundle morphisms that are affine on fibers. The functor $\mathcal{L}$: *Aff* $\to$ *Vec* of spaces extends to a functor *AffBdl* $\to$ *VecBdl* as follows. Let A and B be affine spaces modeled on the vector spaces V and W, respectively, and consider the affine bundles E^A and E^B over M and N, respectively. If $(u, f) : E^A \to E^B$ is an affine-bundle morphism, then $\mathcal{L}E^A = E^V$, $\mathcal{L}E^B = E^W$, and $\mathcal{L}(u, f) := (\mathcal{L}u, f)$ maps E^V over M to E^W over N.

The mixed dual cofunctor $(\ )^\dagger$ also extends to affine bundles as $(E^A)^\dagger = E^{(A^\dagger)}$. This yields a dual pairing $(E^A)^\dagger \times E^A \to M \times \mathbb{R}$. Mixed duality also extends to section spaces. The space ΓE^A is affine on ΓE^V over $\mathbb{R}$ (thinking of ΓE^V as an $\mathbb{R}$-vector space) and over $\mathfrak{F}$ (thinking of ΓE^V as a projective $\mathfrak{F}$-module). In this case, the dual pairing is given by $\Gamma(E^A)^\dagger \times \Gamma E^A \to \mathfrak{F}$.

Ex 3.8.18 Verify the claims and supply any missing details in the preceding two paragraphs.

Ex 3.8.19 The section space $\Gamma(E^A)^\dagger = \Gamma E^{(A^\dagger)}$ by definition. Can you say anything about the space $(\Gamma E^A)^\dagger$?

More about affine spaces and bundles *via* torsors can be found in [46]. The interest of affine bundles in applications is illustrated by a recent article in quantum field theory [5] and an older article in mechanics of Lagrangian systems [54].

4 Local and Global Properties

Now we consider various properties of manifolds and bundles. Some are *local*, meaning near each point, some are *global*, on the whole manifold at once, and some have aspects of both.

4.1 Submanifolds and embeddings

Definition 4.1.1 The map $f : M \to N$ is {sub}

an *immersion* if and only if each $f_{*p} = T_p f$ is injective, so $\dim M \leq \dim N$;

a *submersion* if and only if each $f_{*p} = T_p f$ is surjective, so $\dim M \geq \dim N$;

a *submanifold* if and only if $f : M \hookrightarrow N$ is an injective immersion;

an *embedding* if and only if the *corestriction* $f : M \to \mathrm{im}(f)$ is a diffeomorphism.

Many texts define *submanifold* to be an embedding, but as Warner [82] illustrates this leaves out some Lie subgroups.

Example 4.1.2 These should help clarify the preceding definitions. {subx}

1. The inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ for $n \leq m$ is an embedding with the first n coordinates coming from the domain and the remaining $m - n$ coordinates set to zero.

2. A figure eight is an immersion in $\mathbb{R}^2$ but is not injective.

3. The real line looping around, approaching itself to intersect in the limit, but not actually intersecting, is a submanifold but not an embedding.

4. The *skew* (or *winding* or *irrational*) line on the torus $\mathbb{T}^2 = S^1 \times S^1$ given by $t \mapsto (e^{2\pi it}, e^{2\pi\xi it})$, where ξ is irrational, is a submanifold but not an embedding as every neighborhood of a point in the image contains infinitely many distinct pieces of the image.

5. The curve $x \mapsto x \log|x|$ for $x \neq 0$, $0 \mapsto 0$, is an embedding of the real line into the real plane. This curve often appears in calculus courses as a singular graph obtained *via* extension by continuity.

6. More generally, the graph of any smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$ is an embedded submanifold of $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

7. Even more generally, the image of any smooth section σ of a bundle E over M is an embedded submanifold of E, diffeomorphic to M.

Note that it is sufficient for the inclusion map of an injective immersion to be open in order that M be homeomorphic with its image, hence smoothly embedded.

Ex 4.1.3 The set of all smooth functions vanishing on a submanifold is an ideal in $\mathfrak{F}$. Is it of any particular kind?

The next result provides us with another example of a construction using a partition of unity.

{clemb} **Theorem 4.1.4** *A submanifold $\iota : M \hookrightarrow N$ is a closed embedding if and only if the following is true: a function is smooth on M if and only if it is the restriction of a smooth function on N.*

Proof: For each $p \in M$, choose an open cubical chart $O_p \subseteq N$ at p such that $M \cap O_p$ is a single slice S_p. For $g \in \mathfrak{F}M$, define $\widetilde{g}_p$ to be $g \circ \mathrm{pr}_p$ where $\mathrm{pr}_p : O_p \twoheadrightarrow S_p$ is the natural projection. Then $\widetilde{g}_p$ is a smooth extension of g from $O_p \cap M = S_p$ to all of O_p. The collection $\{O_p \mid p \in M\} \cup (N \neg M)$ is an open cover of N, so there exists a subordinate partition of unity $\{\varphi_i\}$. Take the subset $\{\varphi_j\}$ for which $\operatorname{supp} \varphi_j \cap M \neq \varnothing$ and choose a p_j for each j such that $\operatorname{supp} \varphi_j \subseteq O_{p_j}$. Then $f = \sum_j \varphi_j \widetilde{g}_{p_j}$ is smooth on N and $\iota^* f = f|_M = g$.

For the other direction, ι^* a surjection implies that M is both closed and embedded. □

Ex 4.1.5 Give examples to show that the result fails if either closed or embedded is omitted from the hypotheses.

{csbmfl} **Theorem 4.1.6** *Let $\varphi : M \to N$, $\iota : P \hookrightarrow N$ be a submanifold, and φ factor through P;* i. e., *$\varphi(M) \subseteq \iota(P)$. Let $\widetilde{\varphi} : M \to P$ be the unique factor map. If $\widetilde{\varphi}$ is continuous, then it is smooth.*

Proof: It suffices to show that P can be covered by open cubical charts (U, x) such that $x \circ \widetilde{\varphi}$ restricted to the open set $\widetilde{\varphi}^{-1}(U)$ is smooth.

Let $p \in P$ and (V, y) be an open cubical chart in N at p. Then *locally*, $P \cap V$ is represented by a single slice S. Take $m = \dim N$ and $k = \dim P$. Letting $\mathrm{pr} : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^k$ be the natural projection, we find $x = \mathrm{pr} \circ y \circ \varphi$ provides a chart on an open neighborhood U of p with $U \subseteq S$. Hence

$$\begin{aligned} x \circ \widetilde{\varphi}|_{\widetilde{\varphi}^{-1}(U)} &= \mathrm{pr} \circ y \circ \iota \circ \widetilde{\varphi}|_{\widetilde{\varphi}^{-1}(U)} \\ &= \mathrm{pr} \circ y \circ \varphi|_{\widetilde{\varphi}^{-1}(U)} \end{aligned}$$

which is smooth. □

As a historical note, manifolds may have originated with navigators' charts and atlases, but they were first studied as embedded submanifolds of $\mathbb{R}^3$—curves and surfaces in space—and later of $\mathbb{R}^n$. The next theorem [86] tells us that the two approaches are equivalent.

Theorem 4.1.7 (Whitney Embedding) *Given any smooth manifold M^n, there exists a smooth injection $w : M \to \mathbb{R}^{2n+1}$ such that the topology on M is the subspace topology and the differential structure on M is the one induced from $\mathbb{R}^{2n+1}$.* {whit}

A relatively accessible proof occupies Chapter 6 of [3]. The only introductory book following the embedded route is [29].

4.2 Some major theorems

We recall that $\mathfrak{F}$ is a nuclear Fréchet algebra. It follows that if E is a smooth vector bundle over M then $\Gamma(E)$ is also a nuclear Fréchet space and in fact, $\Gamma(E)$ is a module over $\mathfrak{F}$. If E is finite dimensional, then $\Gamma(E)$ is a finitely generated projective module over $\mathfrak{F}$.

Remark 4.2.1 A module P over a commutative ring K is called projective if P is a direct summand of a free module over K; equivalently, every short exact sequence $0 \to M \to N \to P \to 0$ *splits; i.e.*, there is a morphism of modules $P \to N$ such that $(N \to P) \circ (P \to N) = 1_P$. {prjmod}

A vector bundle is of *finite type* if it is a direct summand of a trivial bundle. All finite-dimensional vector bundles over finite-dimensional manifolds are of finite type (by a typical partition of unity argument). This next result comes from [75].

{swan} **Theorem 4.2.2 (Swan)** *There is a bijection between isomorphism classes of (finite-dimensional) vector bundles of finite type over M and isomorphism classes of (finitely generated) projective $\mathfrak{F}$-modules.*

Ex 4.2.3 Is the bijection natural? In which category?

There are two IFTs that are most important in differential theories: the Inverse Function Theorem and the Implicit Function Theorem.

{InFT} **Theorem 4.2.4 (InFT)** *Let $f : M \to N$ be smooth and $p \in M$. If $f_{*p} = T_p f$ is an isomorphism $T_pM \to T_{f(p)}N$, then f is a local diffeomorphism near p (and $\dim M = \dim N$).*

Proof: This is a local theorem so we may work in $\mathbb{R}^n$ and assume $Df(p)$ is an isomorphism. This is equivalent to $\det Df(p) \neq 0$. Since det is continuous (being a polynomial function), we have $\det Df \neq 0$ in an open neighborhood of p. Apply the InFT 1.1.1 from elementary analysis. □

See Warner [82, pp. 24–5] for six corollaries. Here are two corollaries of the proof that we shall use.

{rkt} **Corollary 4.2.5** *If $f : M \to N$ is smooth and $T_p f$ has rank r, then there is a neighborhood U of $p \in M$ such that $\operatorname{rk} T_q f \geq r$ for all $q \in U$.* □

{sbmrk} **Corollary 4.2.6** *If $f : M \to N$ is a submersion, then the tangent map $T_p f : T_pM \to T_{f(p)}N$ has maximal rank $\dim N$ at each $p \in M$.* □

The next theorem is not quite so local.

{ImFT} **Theorem 4.2.7 (ImFT)** *Let $f : M \to N$ be a differentiable map, $q \in N$ and $f^{-1}(q) \neq \varnothing$ in M. If f is a submersion on $P = f^{-1}(q)$, then $P \hookrightarrow M$ is an embedded submanifold of M with $\dim P = \dim M - \dim N$.*

Define the *codimension* of P as $\operatorname{codim} P := \dim M - \dim P$. Thus in the ImFT, $\operatorname{codim} P = \dim N$.

Proof: Let $p \in P$ and choose charts centered at p and q. We may take the chart at p to be cubical, decomposed as $\mathbb{R}^n \times \mathbb{R}^m$ where $n + m = \dim M$ and $m = \dim N$. Furthermore, we may assume that T_pM is similarly decomposed and that $\ker(T_pf) = \mathbb{R}^n \times \{0\}$. Applying Theorem 1.1.2 we obtain that P is a graph near p, thus locally embedded with $\dim P = n$. Since this holds at every $p \in P$, it follows that P is embedded in M. □

Example 4.2.8 Each fiber E_p of a smooth bundle E over M is an embedded submanifold of E. [Hint: the projection is a submersion.] {imftx1}

Example 4.2.9 Observe that $f : \mathbb{R}^3 \neg 0 \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$ is a submersion. Thus $S^2 = f^{-1}(0)$ is an embedded submanifold of $\mathbb{R}^3$ of dimension $3 - 1 = 2$, and similarly for S^{n-1} in $\mathbb{R}^n$. {imftx2}

Among the several variants of ImFT, the following is useful in many applications. This version is found in Bröcker and Jänich and in a somewhat disguised manner in Warner.

Definition 4.2.10 Let M, N be differentiable manifolds and $L \subseteq N$ a k-dimensional submanifold. A differentiable map $f : M \to N$ is called *transverse to* L if and only if the *transversality condition* {trn}

$$T_pf(T_pM) + T_{f(p)}L = T_{f(p)}N \tag{4.2.1}$$

is satisfied for all $p \in M$ with $f(p) \in L$.

Note that $T_pf(T_pM) + T_{f(p)}L$ need not be a direct sum as nonzero overlap is allowed.

As a consequence we obtain

Theorem 4.2.11 (ImFT with transversality) *If* $f : M \to N$ *is transverse to the submanifold* $L \subseteq N$ *with* $\operatorname{codim} L = k$ *and* $f^{-1}(L) \neq \varnothing$*, then* $f^{-1}(L) \hookrightarrow M$ *is a submanifold with codimension* k. {iftrn}

Proof: Take $p \in M$ with $q = f(p) \in L$ and (V, x) a cubical chart at $q \in N^m$. Then there is a neighborhood U of p such that $f(U) \cap V = \{x^1 = x^2 = \cdots = x^k = 0\}$. The transversality hypothesis then implies that $U \to f(U) \cap V \to \mathbb{R}^k$ is a submersion at $0 \in \mathbb{R}^k$, where the second arrow is $\mathrm{pr} \circ x$ and pr is the natural projection $\mathbb{R}^m \twoheadrightarrow \mathbb{R}^k$. Then by Theorem 4.2.7, $f^{-1}(L)$ is a submanifold near p, hence globally a submanifold, of codimension k. □

Ex 4.2.12 Prove that if L is embedded in N, then $f^{-1}(L)$ is embedded in M.

Ex 4.2.13 Transversality of images is sufficient for the existence of *fibered products* [51, p. 153], or *pullback squares*, in the category of smooth manifolds.

Remark 4.2.14 It is important to know that transversality is *generic* in that the set of all such maps is open and dense in $C^\infty(M, N)$ when this space is given the *strong* C^∞ or *Whitney* topology [31, p. 74].

4.3 Peetre's Theorem

It is a fact due to Milnor [57] that every connected manifold has a finite atlas. Of course, this may require that (at least) one chart has multiple (possibly even infinitely many) connected components.

Recall (Example 1.1.5) that a *multi-index* is an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{N}$. The *length* of α is $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let $x = (x^1, x^2, \ldots, x^n)$ and define $x^\alpha := (x^1)^{\alpha_1}(x^2)^{\alpha_2} \cdots (x^n)^{\alpha_n}$. The *multinomial coefficients* are

$$\binom{r}{\alpha} := \frac{r!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!},$$

where $|\alpha| = r$. The *multinomial expansion*

$$(x^1 + x^2 + \cdots + x^n)^r = \sum_{|\alpha|=r} \binom{r}{\alpha} x^\alpha$$

is the extension of the binomial expansion to n-nomials. Note that

$$\sum_{|\alpha|=r} \binom{r}{\alpha} = n^r.$$

Pascal's triangle ($n = 2$) also extends to Pascal's n-simplex.

Ex 4.3.1 Draw and explicate Pascal's 3-simplex.

{DO} **Definition 4.3.2** A *(linear) differential operator* of order k is a map $P : \mathfrak{F}M \to \mathfrak{F}M$ given locally by

$$Pf = \sum_{|\alpha| \le k} a_\alpha D^\alpha f\,,$$

where $a_\alpha \in \mathfrak{F}M$ are the *coefficients*.

In order to prove Peetre's Theorem, we need a few preliminaries.

{MT} **Definition 4.3.3** The *Maclaurin-Taylor polynomial* at $0 \in U \subseteq \mathbb{R}^n$ of order k of $f \in \mathfrak{F}U$ is given by

$$p^k f := \sum_{r=0}^{k} \sum_{|\alpha|=r} \binom{r}{\alpha} D^\alpha f(0) x^\alpha.$$

This may also be used when (U, x) is a chart centered at some point $p \in M$.

Ex 4.3.4 Verify the last claim.

Definition 4.3.5 If $f \in \mathfrak{F}M$ and $p \in M$, let (U, x) be a chart centered at p. {kjet} The *k-jet* of f at p, denoted by $j^k f(p)$ is represented in local coordinates by $p^k f$. Two such functions are *k-jet equivalent*, or *represent* the *same* k-jet, at p if and only if $p^k f = p^k g$.

Ex 4.3.6 This is independent of the choice of chart at p.

Ex 4.3.7 Extend to k-jets of (local) sections of a vector bundle. [Hint: $U \times$ {MTkjvb} $\mathbb{R}^m = U \times (\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R})$, m times.] At this point, it becomes better to use jet bundles and sections thereof [70].

A complete proof of this next theorem first appeared in [65]. Our proof is partially a completion of the first attempted proof in the article preceding the one cited, using some variants adduced later.

Theorem 4.3.8 (Peetre) *Let E_1 and E_2 be vector bundles over the same* {PT} *connected, base manifold M and suppose $P : \Gamma E_1 \to \Gamma E_2$ is linear, a morphism of sheaves, and satisfies* $\operatorname{supp} P\sigma \subseteq \operatorname{supp} \sigma$ *for all $\sigma \in \Gamma E_1$. Then P is a differential operator.*

Proof: It suffices to consider an open domain $\Omega \subseteq \mathbb{R}^n$ (coming from a chart, say) and to assume that E_1 and E_2 are both the trivial line bundle $\Omega \times \mathbb{R}$.

Ex 4.3.9 Justify this reduction.

We claim it will suffice to prove that the hypotheses imply the following estimate: for every $x \in \Omega$ and $C > 0$, there exists a neighborhood V of x and $k \in \mathbb{N} \neg \{0\}$ such that for any $y \in V \neg \{x\}$, and any $s \in \mathfrak{F}\Omega$ with $j^k s(y) = 0$, then $|Ps(y)| < C$.

Indeed, if $|Ps(y)| = C' > 0$, rescale s by $2C/C'$. But as $C' \neq 0$ and P is linear, we then get $|Ps(y)| = 2C > C$, contradicting the estimate. This will prove the theorem in $V \neg x$. Now, P is linear with smooth coefficients and, being a sheaf morphism, it maps germs at x to germs at x. Therefore the coefficients will be smooth at x, hence on all of V.

Now, assume the estimate does not hold at $x \in \Omega$. Then there exists a sequence $x_k \to x$ with $x_k \in B_k$ such that the coordinate (Euclidean) distance between any two of these balls is strictly positive. There are also smooth functions s_k on B_k such that $j^k s_k(x_k) = 0$ but $|Ps_k(x_k)| \geq C > 0$. Let h be a standard bump function such that $h(y) = 1$ on $B_{\frac{1}{2}}(0)$ and $\operatorname{supp} h = B_1(0)$.

Consider the even-indexed functions. We have $j^{2k} s_{2k}(x_{2k}) = 0$, so there exists a ball $B_\delta(x_{2k})$ on which

$$\sum_{|\alpha|=k} \sup_{y \in B_\delta(x_{2k})} |D^\alpha s_{2k}(y)| \leq \frac{1}{M_k} \left(\frac{\delta}{2}\right)^k,$$

where

$$M_k = \sum_{|\alpha| \leq k} \sup |D^\alpha h|.$$

Observe that

$$h_{2k}(y) := h\left(\frac{y - x_{2k}}{\delta}\right)$$

is a standard bump function supported in $B_\delta(x_{2k})$, and that the derivative of $h_{2k}s_{2k}$ is bounded.

$$\max_{|\alpha| \leq k} \sup_{y \in B_\delta(x_{2k})} |D^\alpha(h_{2k}s_{2k})(y)| \leq \frac{1}{2^k}$$

Then the series $g(y) := \sum_{k=1}^\infty h_{2k}(y)s_{2k}(y)$ converges uniformly, as do all of its derivatives, whence g is smooth on V.

Finally, since s_{2k} and $h_{2k}s_{2k}$ coincide in a neighborhood of x_{2k},

$$\lim_{k \to \infty} |Pg(x_{2k})| \geq C,$$

so $|Pg(x)| \geq C > 0$. But

$$\lim_{k \to \infty} |Pg(x_{2k+1})| = 0$$

because g vanishes identically in B_{2k+1} and $\operatorname{supp} Pg \subseteq \operatorname{supp} g$: contradiction.

□

Remark 4.3.10 One can define the infinity-jet of a function $f \in \mathfrak{F}M$ at $p \in M$ as the direct limit

$$j^\infty f(p) := \varinjlim j^k f(p).$$

While this encodes information about infinitely many derivatives of f at p, it still describes f *only at* p. Compare this with the germ $[f]_p$ of f at p which contains all of this information, but also infinitely more information about f "infinitely near" p. As an example, consider a typical bump function with support $[-1, 1]$. The infinity-jets at both 1 and -1 are 0, but neither germ is the 0-germ.

Peetre's Theorem has been extended to nonlinear operators in several inequivalent ways. A recent version with a similar proof, and a brief summary of the previous, more highly technical versions, is [60].

4.4 Vector fields and flows

{f+ic}

Observe that $\mathfrak{X} = \Gamma(TM)$ is the nuclear Fréchet space of all vector fields on M.

The induced map on tangent bundles can be used to define an action of $\mathfrak{X}$ on $\mathfrak{F} = C^\infty(M)$. Let $V \in \mathfrak{X}$ and $f \in \mathfrak{F}$. Define $V(f)$ pointwise by $V(f)(x) := f_*(V(x))$. One can think of this number as the directional derivative of f in the direction V at x. In particular, $V(f) \in \mathfrak{F}$. Since $V(f)(x)$ is a real number, it can be regarded as the result of applying an element of $T^*_x M$ to $V(x)$. This element is denoted $df(x)$; $df \in \Gamma(T^*M)$ and is called the *differential* of f. {df}
Thus one also writes $V(f) = df(V) = \langle df, V\rangle$, the last to emphasize duality.

Continuing with the action of $\mathfrak{X}$ on $\mathfrak{F}$, it follows from the definitions that $V(fg) = fV(g)+gV(f)$. Thus V can be considered as a continuous first order linear differential operator (analysis) or *derivation* (algebra) on $\mathfrak{F}$, and in fact the correspondence with the latter is bijective as we saw in Section 2.4.

Indeed, consider the continuous derivations on the $\mathbb{R}$-algebra $\mathfrak{F}$. Recall from Section 2.4 that derivations are $\mathbb{R}$-linear and obey the product rule $V(fg) = V(f)g + fV(g)$, where V is a derivation. From Definition 2.6.8, a derivation V on a manifold can be given locally as $V = v^i\partial_i$. Thus for a vector field V, we write $V(f) = v^i\partial_i f$, assuming the Einstein summation convention as usual.

Summing up, vector fields on M are continuous derivations on $\mathfrak{F}$ and conversely. Also, they are continuous first-order linear differential operators on $\mathfrak{F}$ over M.

Definition 4.4.1 The *Lie bracket* of vector fields V and W is defined as $[V, W] = V \circ W - W \circ V$ using composition of derivations. {lb}

{lbloc} **Ex 4.4.2** The Lie bracket is well defined. Locally, with $V = v^i \partial_i$ and $W = w^j \partial_j$, apply $[V, W]$ to $f \in \mathfrak{F}$ and observe that all second-order terms cancel. (Or see the proof of Proposition 2.6.4.)

Ex 4.4.3 Verify that with this bracket, $\mathfrak{X}$ is a (real) Lie algebra as in Definition 2.6.1.

{prel} **Definition 4.4.4** Let M and M' be two manifolds, $\mathcal{A} \in \mathrm{End}(\mathfrak{F})$, $\mathcal{A}' \in \mathrm{End}(\mathfrak{F}')$, and $\varphi : M \to M'$ smooth. We say $\mathcal{A}$ and $\mathcal{A}'$ are *φ-related* if and only if

$$\begin{array}{ccc} \mathfrak{F}' & \xrightarrow{\varphi^*} & \mathfrak{F} \\ {\scriptstyle \mathcal{A}'}\downarrow & & \downarrow{\scriptstyle \mathcal{A}} \\ \mathfrak{F}' & \xrightarrow[\varphi^*]{} & \mathfrak{F} \end{array}$$

commutes.

If φ is a diffeomorphism then $\mathcal{A}$ and $\varphi_* \mathcal{A}$ are φ-related. In general, $\mathcal{A}$ neither determines such an $\mathcal{A}'$ nor is $\mathcal{A}'$ unique if it exists.

{endeqv} **Proposition 4.4.5** *Let $\varphi : M \to M'$ be smooth and $\mathcal{A}_1, \mathcal{A}_2$ be φ-related to $\mathcal{A}'_1, \mathcal{A}'_2$, respectively. Then the following are also φ-related:*

1. *$\lambda \mathcal{A}_1$ and $\lambda \mathcal{A}'_1$ for all $\lambda \in \mathbb{R}$;*
2. *$\mathcal{A}_1 + \mathcal{A}_2$ and $\mathcal{A}'_1 + \mathcal{A}'_2$;*
3. *$\mathcal{A}_1 \circ \mathcal{A}_2$ and $\mathcal{A}'_1 \circ \mathcal{A}'_2$;*
4. *$[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{A}'_1, \mathcal{A}'_2]$.* □

{vfeqv} **Corollary 4.4.6** *Vector fields V and V' are φ-related if and only if $\varphi_* \circ V = V' \circ \varphi$ globally on M.* □

We now turn to integrability of vector fields, in particular to flows.

{intc} **Definition 4.4.7** Let V be a smooth vector field on M and $\gamma : I \to M$ be a smooth curve on M, where I is some convenient interval in $\mathbb{R}$. We say that γ is an *integral curve* if and only if $\dot{\gamma}(t) = V(\gamma(t))$ for all $t \in I$.

Here $\dot{\gamma}(t)$ is the velocity lift of γ into TM with $\pi\dot{\gamma} = \gamma$, defined *via* $\gamma^*TM \cong I \times \mathbb{R}^n$. This gives $\dot{\gamma}(t) = \gamma_*(D\gamma)(t)$.

Ex 4.4.8 Write out $\dot{\gamma}(t)$ in local coordinates, a chart centered at $\gamma(t_0)$ for some $t_0 \in I$ [WLOG, $t_0 = 0$ in I].

The existence of such integral curves is guaranteed by the Fundamental Existence and Uniqueness Theorem (FEUT) for Ordinary Differential Equations [32, pp. 162f, 167f], which states that there exist integral curves through every point of M. Moreover, they depend smoothly on initial conditions [32, pp. 299–302]. {feut}

Definition 4.4.9 A vector field V is *complete* if and only if all domains of integral curves are $\mathbb{R}$. {cvf}

Lemma 4.4.10 *Let $\varepsilon > 0$, $I_\varepsilon = (-\varepsilon, \varepsilon)$, and $\Phi : I_\varepsilon \times M \to M$ be a local 1-parameter group (*i. e.*, a local flow) which induces the vector field $V \in \mathfrak{X}(M)$. Then Φ extends to a global 1-parameter group and V is complete.* {vfc}

Proof: Without loss of generality, assume that Φ_t is a diffeomorphism for $|t| \leq \varepsilon$. Write $t = k\frac{\varepsilon}{2} + r$, where $k \in \mathbb{Z}$ and $|r| < \frac{\varepsilon}{2}$. For $k > 0$, define $\widetilde{\Phi}_t := \left(\Phi_{\varepsilon/2}\right)^k \circ \Phi_r$. For $k < 0$, define $\widetilde{\Phi}_t := \left(\Phi_{-\varepsilon/2}\right)^{-k} \circ \Phi_r$. It follows readily that $\widetilde{\Phi}$ satisfies the desired criteria. □

Ex 4.4.11 Prove that every vector field on a compact manifold is complete; equivalently, every compactly supported vector field is complete.

Ex 4.4.12 Prove that every vector field along an immersion has *local* smooth extensions; that is, if $f : M \to N$ is an immersion then *locally* we have $\Gamma(f^*TN) \to \Gamma(TN) : V \mapsto f_*V$.

For these next remarks, see [32, pp. 167–176]; in particular, extension of solutions is found on p. 171f.

Let V be a vector field on M and I be maximal open interval(s) and γ be smooth curve(s) on M. Then for each $p \in M$, there exists a unique integral curve $\gamma_p : I_p \to M$ with $0 \mapsto p$ [so $0 \in I_p$] and I_p maximal with γ_p inextendible. This condition yields $\dot{\gamma}_p(t) = V(\gamma(t))$.

Define the open set $\mathcal{D}_t = \{p \in M \mid t \in I_p\}$ and the map $\Phi_t : \mathcal{D}_t \to \mathcal{D}_{-t} : p \mapsto \gamma_p(t)$. Clearly $\bigcup_{t>0} \mathcal{D}_t = M$. For each $p \in M$ there exists $\varepsilon_p > 0$ and an open neighborhood U_p of p, such that $\Phi : (-\varepsilon_p, \varepsilon_p) \times U_p \to M$ is a smooth map called the *local flow* of V. Clearly, each Φ_t is a local diffeomorphism and $\Phi_s \circ \Phi_t = \Phi_{s+t}$ whenever it is well defined. Associated to Φ we have $\dot{\Phi} : M \to TM : p \mapsto \dot{\gamma}_p(0)$, known as the *velocity field,* which is a vector field on M. {flow}

Thus starting with an arbitrary vector field V we realize that $V = \dot{\Phi}$ for some (local) flow Φ on M. We then say Φ is a *global* flow if and only if V is complete.

4.5 Smooth functors and vector bundles

{sf+vb}

We now define *Vec* to be the category of finite-dimensional vector spaces over $\mathbb{R}$ as objects and linear transformations as arrows. Thus $\mathrm{Hom}(E, F) = L(E, F)$ is in *Vec* for any $E, F \in$ *Vec*.

{smf} **Definition 4.5.1** A functor $\mathcal{F}$: *Vec* $\to$ *Vec* is *smooth* if and only if the induced map $\mathrm{Hom}(E, F) \to \mathrm{Hom}(\mathcal{F}E, \mathcal{F}F) : t \mapsto \mathcal{F}t$ is a smooth map.

Ex 4.5.2 Extend the definition to (co)functors of several variables.

We state the following theorem, whose proof is left as an exercise for the reader.

{smfvb} **Theorem 4.5.3** *A smooth (co)functor $\mathcal{F}$, when applied fiberwise, extends to smooth finite-dimensional vector bundles.* □

Ex 4.5.4 If E and F are finite-dimensional vector bundles over a base space M, we obtain the following bundles over M:

1. $E \oplus F$, the Whitney (or direct) sum, with fiber dimension the sum of the fiber dimensions of E and F;
2. $E \otimes F$, the tensor product, with fiber dimension the product of the fiber dimensions of E and F;
3. E^*, the dual of E;
4. $\mathrm{Hom}(E, F)$;
5. E/F, the quotient bundle when $F \leq E$;
6. $\bigwedge^k E$, the exterior algebra bundle.

Observe that for $\Delta : M \hookrightarrow M \times M$ the diagonal inclusion, $E \oplus F \cong \Delta^* E \times F$, the latter over $M \times M$.

Returning temporarily to $E, F \in \mathit{Vec}$, consider linear transformations $s : E \to E'$ and $t : F \to F'$. Then $s \oplus t$ is represented by the block-diagonal matrix

$$\begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix},$$

$s \otimes t$ by the Kronecker product, s^* by the transpose, and so on.

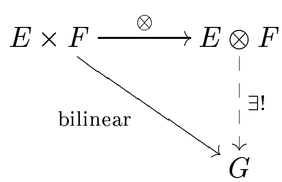
Ex 4.5.5 For $s : E \to E$, determine the matrix representation of $\wedge^k s$.

Note that the matrix formula for $\mathcal{F}$ on $E \in \mathit{Vec}$ is also the formula for changing a cocycle for the bundle E to one for the bundle $\mathcal{F}E$. When $E = TM$, this means apply $\mathcal{F}$ to the Leibniz cocycle to obtain a cocycle for $\mathcal{F}TM$.

4.6 Tensors and forms {t+f}

Having studied sections of TM, it is logical to consider sections of T^*M, or what one might expect to be called covector fields. It turns out that exterior powers of the cotangent bundle are some of the most useful bundles of which to take sections. Thus we shall digress for a *brief* review of tensor and exterior algebra. For more details specific to real vector space theory consult [82]; for K-module theory look in any good algebra book such as [51].

Let E, F and G be finite-dimensional $\mathbb{R}$-vector spaces. Recall that $E \otimes F$ solves the universal mapping problem

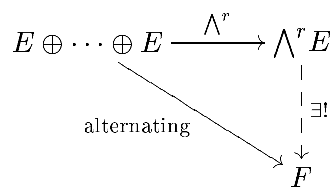


and that $\otimes$ is commutative, associative and distributive over $\oplus$ up to natural isomorphisms. We recall that $E^* \otimes F \cong L(E, F)$ naturally and so $\dim(E \otimes F) = \dim E \cdot \dim F$. In this connection, recall that $\dim(E \times F) = \dim E + \dim F$ and that $E \times F$ is also written as $E \oplus F$. If $\{e_i\}$ and $\{f_j\}$ are bases for E and F, respectively, then $\{e_i \otimes f_j\}$ is a basis for $E \otimes F$.

The *tensor space of type* (r, s) or the *space of homogeneous* (r, s)-tensors over E is $\bigotimes^r_s(E) := E \otimes \cdots \otimes E \otimes E^* \otimes \cdots \otimes E^*$, where there are r copies of E and s copies of E^*. The tensor algebra of E is $\bigotimes E := \bigoplus_{r,s \geq 0} \bigotimes^r_s E$, with

$\bigotimes_0^0 E := \mathbb{R}$. Elements of $\bigotimes_s^r E$ of the form $e_1 \otimes \cdots e_r \otimes e_1^* \cdots \otimes e_s^*$ are called *decomposable*.

Let $C(E)$ be the subalgebra $\bigoplus_{r\geq 0} \bigotimes_0^r(E)$ and $I(E)$ the 2-sided ideal generated by elements of the form $e \otimes e$. Set $I_r E := I(E) \cap \bigotimes_0^r E$ so that $I(E) = \bigoplus_{r\geq 0} I_r E$. Define the *exterior algebra* of E, $\bigwedge E := C(E)/I(E)$. Observe that if one defines $\bigwedge^r E := \bigotimes_0^r E/I_r E$ for $r \geq 2$, $\bigwedge^1 E := E$ and $\bigwedge^0 E := \mathbb{R}$, then $\bigwedge E = \bigoplus_{r\geq 0} \bigwedge^r E$. Recall that (char $\neq 2$) *alternating* (repeating an argument implies vanishing) and *skew-symmetric* (interchanging two arguments changes signs) are equivalent for multilinear functions. Exterior powers solve the universal mapping problem



If $u \in \bigwedge^r E$ and $v \in \bigwedge^s E$ then $u \wedge v = (-1)^{rs} v \wedge u \in \bigwedge^{r+s} E$. If $\{e_i\}$ is a basis of E, then $\{e_{i_1} \wedge \cdots \wedge e_{i_j}\}$ with $i_1 < \cdots < i_j$ where $\{i_1, \ldots, i_j\} \subseteq \{1, \ldots, \dim E\}$, and the number 1 if this set is the empty subset, forms a basis of $\bigwedge E$. Thus $\bigwedge^{\dim E} E \cong \mathbb{R}$ and $\bigwedge^r E = 0$ for $r > \dim E$, $\dim \bigwedge E = 2^{\dim E}$ and $\dim \bigwedge^r E = \binom{\dim E}{r}$ for $0 \leq r \leq \dim E$.

Ex 4.6.1 $\bigotimes_s^r E^* \cong (\bigotimes_s^r E)^*$ naturally *via* the nonsingular pairing $\bigotimes_s^r E^* \times \bigotimes_s^r E \to \mathbb{R}$ given by contraction: on decomposable elements,

$$\langle e^* \otimes e, f \otimes f^* \rangle \mapsto e^*(f) \cdot f^*(e).$$

Ex 4.6.2 $\bigwedge E^* \cong (\bigwedge E)^*$ naturally *via* $\bigwedge^r (E^*) \times \bigwedge^r E \to \mathbb{R} : (e^*, f) \to \det[e_i^*(f_j)]$.

Now let M be a (smooth) manifold. One defines the *bundle* of (r,s)-tensors

$$\textstyle\bigotimes_s^r TM = \bigcup_{p\in M} \bigotimes_s^r (T_p M),$$

the *space* of (r,s)-tensors

$$\mathfrak{T}_s^r(M) := \Gamma\left(\textstyle\bigotimes_s^r TM\right),$$

the *space of differential r-forms*

$$\Omega^r(M) := \Gamma\left(\textstyle\bigwedge^r T^*M\right),$$

and the *space of differential forms*

$$\Omega^*(M) := \bigoplus_{r\geq 0} \Omega^r(M).$$

Ex 4.6.3 Given an atlas on M, construct atlases for each bundle.

Ex 4.6.4 Write each down in local coordinates if you haven't yet.

Observe that $\Omega^0(M) = \mathfrak{F}(M)$ and $\Omega^1(M) = \Gamma(T^*M)$.

Ex 4.6.5 If $f \in \mathfrak{F}$, then $df \in \Omega^1$.

The wedge product can be extended to forms pointwise (fiberwise). If $\alpha \in \Omega^r$ and $\beta \in \Omega^s$, define $\alpha \wedge \beta$ via

$$(\alpha \wedge \beta)(p)\,(v_1, \ldots, v_{r+s}) := \sum_{r,s \text{ shuffles}} (\operatorname{sgn}\sigma)\,\alpha(p)(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\,\beta(p)(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}),$$

where $v_1, \ldots, v_{r+s} \in T_pM$ and $\sigma \in S_{r+s}$ is an r, s shuffle if and only if $\sigma(1) < \cdots < \sigma(r)$ and $\sigma(r+1) < \cdots < \sigma(r+s)$.

Ex 4.6.6 If $\alpha, \beta \in \Omega^r(M), v_1, v_2 \in T_pM$, then

$$(\alpha \wedge \beta)(p)\,(v_1, v_2) = \alpha(p)v_1\,\beta(p)v_2 \,-\, \alpha(p)v_2\,\beta(p)v_1.$$

Ω^* forms an associative (but not commutative) $\mathbb{R}$-algebra under $\wedge$.

WARNING: there is another definition of $\wedge$ which differs precisely by the factor $r!s!/(r+s)!$.

The action of $\Omega^r(M)$ can be extended to vector fields *via*

$$\alpha(V_1, \ldots, V_r)(p) := \alpha(p)(V_1(p), \ldots, V_r(p)).$$

Ex 4.6.7 Observe that a smooth $f : M \to N$ induces {f*dfs}

$$f^* : \Omega^r(N) \to \Omega^r(M) : \alpha \to \alpha \circ f_*;$$

explicitly,

$$(f^*\alpha)(p)(v_1, \ldots, v_r) := \alpha(f(p))(f_*v_1, \ldots, f_*v_r).$$

Also, $d : \mathfrak{F}(M) = \Omega^0(M) \to \Omega^1(M)$ can be extended uniquely to $d : \Omega^r(M) \to \Omega^{r+1}(M)$ such that: (1) d is $\mathbb{R}$-linear; (2) $d^2 = 0$; and (3) for $\alpha \in \Omega^r, \beta \in \Omega^s$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$. In terms of vector fields $V_0, \ldots, V_r$, for $\alpha \in \Omega^r$, d is defined by

$$\begin{aligned} d\alpha(V_0, \ldots, V_r) = & \sum_{i=0}^{r} (-1)^i V_i(\alpha(V_0, \ldots, \hat{V}_i, \ldots, V_r)) \\ & + \sum_{i=j} (-1)^{i+j} \alpha([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_r) \end{aligned}$$

where the hatted terms are to be omitted. In local coordinates, $T^*M|_U$ has (dx^i) as a *local coframe*. With

$$\alpha|_U = \sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r} \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_r} \, ,$$

we have

$$d\alpha|_U = \sum_{i_1 < \cdots < i_r} d\alpha_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} \, .$$

We call d the *exterior derivative* (or differential).

{1d} Let E be a vector bundle over M, V be a vector field on M and $\sigma \in \Gamma(E)$. Then the *Lie derivative*

$$\pounds_V \sigma := \frac{d}{dt} \left[\Phi_t^* \sigma\right]_{t=0} = \lim_{t \to 0} \frac{1}{t} \left[\Phi_t^* \sigma - \sigma\right]$$

is well defined and exists in the nuclear Fréchet space $\Gamma(E)$.

In general, however, we only know how to compute the limit pointwise. This means we must get everything in the same fiber. If $E = \mathcal{F}(TM)$, we adjust for the correct fiber thus:

$$\lim_{t \to 0} \frac{1}{t} \left[\mathcal{F}\left(\Phi_{-t\,*}\right) \circ \sigma \circ \Phi_t - \sigma\right](p) \, .$$

We shall use this formula to compute the Lie derivative of differential forms.

It is a sort of "directional derivative" for forms. As the formula and examples make clear, however, there is also some differentiation of the vector field involved. If $\alpha \in \Omega^r(M)$ and $V \in \mathfrak{X}(M)$ with (local) flow Φ, then the Lie derivative of α with respect to V is given by

$$(\pounds_V \alpha)(p) = \frac{d}{dt} \, \Phi_t^*(\alpha(\Phi_t(p))|_{t=0}$$

so $\pounds_V : \Omega^r \to \Omega^r$.

{ld1} **Ex 4.6.8** If $f \in \mathfrak{F}$ and $U, V \in \mathfrak{X}$, then $\pounds_V f = V(f)$ and $\pounds_V U = [V, U]$.

There is one last operation on forms to be considered. For V, $V_1, \ldots, V_r$ and $\omega \in \Omega^{r+1}$ define the *contraction of* ω *by* V or the *interior product of* ω *by* V as $V \lrcorner\, \omega(V_1 \cdots V_r) = i_V \omega(V_1, \ldots, V_r) := \omega(V, V_1, \ldots, V_r)$. Thus $V \lrcorner = i_V : \Omega^{r+1} \to \Omega^r$.

Ex 4.6.9 Here are several useful formulas relating the various operations defined so far. Let $f : M \to N$ and $g : N \to P$ be smooth. {uf}

1. If f is a diffeomorphism, define $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N) : V \mapsto f_* \circ V \circ f^{-1}$ and $f^* : \mathfrak{X}(N) \to \mathfrak{X}(M) : V \mapsto (f^{-1})_* \circ V \circ f$. Then $f_*[V, U] = [f_* V, f_* U]$ and $(g \circ f)_* V = g_* f_* V$. This last property is true in general provided V is replaced by a tangent vector. It is the chain rule. {f*vfs}

2. If $\alpha, \beta \in \Omega^*(N)$, then $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$ and if $\gamma \in \Omega^*(P)$, then $(g \circ f)^*\gamma = f^* g^* \gamma$. Also $f^* d\alpha = df^*\alpha$ and for $\alpha \in \Omega^r(N)$, $V_1, \ldots, V_r \in \mathfrak{X}(M)$, $f^*\alpha(V_1, \ldots, V_r)(p) = \alpha(f(p))(f_* V_1(p), \ldots, f_* V_r(p))$.

3. $V \in \mathfrak{X}(M)$, $\alpha \in \Omega^r(M)$, $\varphi \in \mathfrak{F}(M)$ imply $\varphi V \lrcorner\, \alpha = \varphi(V \lrcorner\, \alpha) = V \lrcorner\, \varphi\alpha$. Also, $(i_V)^2 = 0$, $V \lrcorner\, \alpha$ is $\mathbb{R}$-bilinear in V and α, and for $\beta \in \Omega^s(M)$, $i_V(\alpha \wedge \beta) = i_V \alpha \wedge \beta + (-1)^r \alpha \wedge i_V \beta$.

4. If $V \in \mathfrak{X}(N)$, $\alpha \in \Omega^*(N)$, and f is a diffeomorphism, then $f^*(V \lrcorner\, \alpha) = f^*V \lrcorner\, f^*\alpha$ and $f^* \pounds_V \alpha = \pounds_{f^*V} f^*\alpha$.

5. If $V, U, V_1, \ldots, V_r \in \mathfrak{X}$, $\omega \in \Omega^r$, and $\varphi \in \mathfrak{F}$, then $\pounds_V \omega(V_1, \ldots, V_r) = \langle \omega(V_1, \ldots, V_r) - \sum_{i=1}^r \omega(V_1, \ldots, [V, V_i], \ldots, V_r) \rangle$.

For the additional properties below (P1–P6), one could have $\omega \in \Omega^*$.

P1 $\pounds_{\varphi V} \omega = \varphi \pounds_V \omega + d\varphi \wedge V \lrcorner\, \omega$.

P2 $\pounds_V \omega = d(V \lrcorner\, \omega) + V \lrcorner\, d\omega$; brutally short, $\pounds = d \circ i + i \circ d$. {frcv}

P3 $\left\{ \begin{array}{c} \pounds_{[V,U]}\, \omega = [\pounds_V, \pounds_U]\, \omega \\ [V, U] \lrcorner\, \omega = [\pounds_V, i_U]\, \omega \end{array} \right\}$ brackets on RHS are genuine commutators.

P4 $\pounds_V i_V \omega = i_V \pounds_V \omega$

P5 $\pounds_V d\omega = d \pounds_V \omega$

P6 Also, $\pounds_V \omega$ is $\mathbb{R}$-bilinear in V and ω, and $\alpha \in \Omega^*(M)$ implies $\pounds_V(\omega \wedge \alpha) = \pounds_V \omega \wedge \alpha + \omega \wedge \pounds_V \alpha$.

4.7 Integration and orientation

We begin by recalling the change of variables formula for multiple integrals. Let $U \subseteq \mathbb{R}^n$ be open and $f \in C^\infty(U)$. If g is a diffeomorphism of U, then

$$\int_{g(U)} f = \int_U f \circ g \, |\det Dg| \, .$$

{null} **Ex 4.7.1** Sets of Lebesgue measure zero are well defined on manifolds.

Thus the appropriate things to study in order to have invariant integrals are those that transform according to the absolute value of the determinant of the Jacobian of the change of variables (or coordinates) diffeomorphism. Such objects are called *densities*. On a manifold M with charts (U, φ) and (V, ψ) a typical density μ then transforms like

$$\mu_\psi = |\det D(\psi \circ \varphi^{-1})| \, \mu_\varphi \, .$$

If we think of μ as real-valued, then a density must be a section of some line bundle, called the *volume bundle* because densities are actually like measures.

As an illustration of a process that is frequently left as an exercise, we give an invariant construction of the volume bundle. Observe that $\bigwedge^n T^*M$ has the transition cocycle $\det D(\psi \circ \varphi^{-1})$ for charts as above. Construct the *orientation bundle* with model fiber $\mathbb{R}$ and transition cocycle $\operatorname{sgn} \det D(\psi \circ$
{or2} $\varphi^{-1})$, where sgn denotes the sign function (*signum*), and denote it by $\operatorname{or}(M)$. Then if $\operatorname{vol}(M)$ denotes the volume bundle, $\operatorname{vol}(M) \cong \operatorname{or}(M) \otimes \bigwedge^n T^*M$.

Ex 4.7.2 What is the relation between $\operatorname{or}(M)$ defined here and as previously defined in Ex 3.4.10?

Now we can define the integral of a density. Choose a locally finite atlas $\{(U_i, \varphi_i)\}$ and a *subordinate* partition of unity $\{f_i\}$. *Subordinate* means $\operatorname{supp} f_i \subseteq U_i$ for each i. Let $\mu_i = \mu|_{U_i}$. Define

$$\int_M \mu := \sum_i \int_{U_i} f_i \mu_i,$$

where if μ does not have compact support we must ask for convergence of the sum *and* require that each U_i be relatively compact (*i. e.*, have compact closure).

If μ is a density which is nonnegative and has compact support we can use it to define an integral of functions.

$$I_\mu(f) := \int_M f\mu.$$

This satisfies the usual linearity and nonnegativity properties of integrals, and the continuity property: $f_i \searrow 0$ implies $I(f_i) \to 0$. This means that we have a Daniell integral. One could also switch the compact support from μ to f to get a regular Borel measure.

It turns out that the volume bundle is trivial. It is not *naturally* trivial, however, and this is important for its applications.

More generally, one can consider objects μ which transform like

$$\mu_\psi = |\det D(\psi \circ \varphi^{-1})|^\alpha \mu_\varphi$$

where $\alpha \in \mathbb{R}$. These are α-*densities*; our densities already defined are 1-densities, but are usually still referred to as just densities. Denote the space of α-densities by $\mathfrak{F}_\alpha$.

Ex 4.7.3 Verify that 0-densities are smooth functions, that we can multiply α-densities and β-densities to obtain $(\alpha + \beta)$-densities, and that nonnegative α-densities can be exponentiated. These results can also be obtained for complex-valued α-densities defined in the obvious way.

It turned out that $\frac{1}{2}$-densities are very useful in the theory of partial differential equations and in geometric quantization. As the reason is the same for both and it requires only minimal digression to do so *via* the former, we follow that route.

To begin, go back to the model space $\mathbb{R}^n$. Following L. Schwartz, the space of all smooth functions on $\mathbb{R}^n$ is denoted by $\mathscr{E}$ and those of compact support by $\mathscr{D}$. It is customary in linear PDE theory for these to be complex valued, but for the present purpose you may take them as real valued if you wish. Further recall that Lebesgue measure λ on $\mathbb{R}^n$ has quite strong uniqueness properties: *e. g.,* the theorems of Haar and of Riesz and Markoff. It provides a canonical pairing between $\mathscr{E}$ and $\mathscr{D}$ given by

$$\langle f, \varphi \rangle := \int_{\mathbb{R}^n} f \varphi \, d\lambda \, .$$

This extends continuously to the dual pairing between $\mathscr{D}$ and its strong dual $\mathscr{D}'$. Elements of $\mathscr{D}'$ are called *(Schwartz) distributions* or *generalized functions.* They may have *singularities*: points where they are not smooth, or not even a function; the Dirac δ is neither smooth nor a function. (It is a measure.) Elements of $\mathscr{D}$ are called *test functions*: multiply a distribution f by a test function φ to see if f is singular on the support of φ. Distributions are *necessary* for a complete theory of linear PDEs.

If we try to replicate this on a manifold M, there is no counterpart to Lebesgue measure in $\mathbb{R}^n$: no canonical density. The strong dual of $C_c^\infty(M)$ is the space of *generalized densities*, while the space of *generalized functions* is the strong dual of $\mathfrak{F}_1(M)$. Note there is a canonical pairing between $1-\alpha$ and α-densities, that extends continuously to the dual pairing of each with the strong dual of the other. Therefore, choosing $\alpha = \frac{1}{2}$ yields generalized $\frac{1}{2}$-densities as the strong dual of $\frac{1}{2}$-densities, telling us that this is the proper way to approach linear PDE theory (or any other linear theory that needs generalized functions) on manifolds.

Remark 4.7.4 For nonlinear problems, there are Rosinger's 1978 algebras of generalized functions. His book [67] takes advantage of a decade's experience to clarify and simplify the presentation. The most popular of these algebras is Colombeau's from 1982 [14]. See [28] for an exposition of the geometric theory and [62] for recent applications in General Relativity.

Let E be an m-dimensional vector space and $V(E)$ the set of frames of E. Denote by $|\bigwedge^m| E$ the space of all real-valued functions f on $V(E)$ such that $f(b_2) = |\det g| f(b_1)$ for $b_1, b_2 \in V(E)$ and $g \in GL(E)$ with $b_1 = g(b_2)$.

Ex 4.7.5 $|\bigwedge^m| E$ is a 1-dimensional $\mathbb{R}$-vector space.

Thus if E is a vector bundle with fiber dimension m one can form the line bundle $|\bigwedge^m| E$. In particular, $|\bigwedge^n| T^*M$ is the *bundle of densities* over M and its space of smooth sections, also denoted Ω_1, is the space of *densities* on M.

Ex 4.7.6 Replace $|\det g|$ with $|\det g|^\alpha$ to obtain the α-densities Ω_α.

Recall that frames of T^*M are called coframes. Thus one has $L^*(M)$, the *coframe bundle*.

Ex 4.7.7 Express densities in terms of $L^*(M)$. Do the same for α-densities.

M is said to be *orientable* or *nonorientable* according as the orientation bundle $\mathrm{or}(M)$ is trivial or not. In the former case a choice of trivialization is called an *orientation* of M, which is then said to be *oriented*.

Ex 4.7.8 M is orientable if and only if $L(M)$ consists of precisely two connected components. A choice of component is then an orientation.

Since $L(M)$ is a principal GL_n-bundle, any atlas allows us to define a determinant function on $L(M)$. Although this function is not invariant, its *sign* is if M is orientable. Thus we can divide atlases on orientable manifolds into two classes: positive and negative.

Ex 4.7.9 (on orientation)

1. An orientation is a choice of one of these classes.

2. $\mathbb{R}^n$ with the standard basis is oriented by the usual or standard orientation. [Hint: $\det I_n > 0$.]

3. Find some more equivalent formulations of orientation.

Although it will not be discussed here, there is a theory of integration of r-forms over oriented r-dimensional submanifolds. It is in this theory where one finds generalizations of Stokes theorem, the divergence theorem, *etc.* More details can be found, *e.g.,* in [82].

For additional examples and/or exercises, see any of the following (all but Hirsch are especially suitable for beginners): Brickell and Clarke [7], Bröcker and Jänich [8], Guillemin and Pollack [29], Hirsch [31], Spivak [73], Wallace [81], and Warner [82].

5 The Cotangent Bundle: Symplectic Mechanics

In classical mechanics, a manifold M is a *configuration* space and its cotangent bundle T^*M is the associated *phase* space. The special structure on T^*M that reflects this is a *symplectic* structure. In linear PDE, this same structure is the basis for propagation of singularities of solutions and a crucial ingredient in the theory of Fourier Integral Operators [20].

The main point for us is that this structure is naturally there: it is inherent merely in being a cotangent bundle.

5.1 Canonical forms on the cotangent bundle

Let M be a smooth n-manifold and $\pi : T^*M \to M$ the natural projection. Then $\pi_* : TT^*M \to TM$. Define the canonical 1-form θ on T^*M by

$$\theta(v) : T_vT^*M \to \mathbb{R} : z \mapsto -\langle v, \pi_* z\rangle.$$

Ex 5.1.1 Check that this is well defined.

The 1-form includes a sign choice for later convenience. The canonical 2-form is $\omega = d\theta$. It is occasionally useful to keep track of base points by writing (p, v) when $v \in T_p^*M$.

Recall that if (U, x) is a chart in M then a local coframe in $T^*M|_U$ is (dx^i). More generally, a collection $\{\xi_i\} \subseteq C^\infty(T^*M|_U)$ together with $\{x^i\}$ is a *canonical* coordinate system (x^i, ξ_i) on $T^*M|_U$ if and only if $\xi_i(v) = \langle v, \frac{\partial}{\partial x^i}\rangle$ for all $v \in T^*M|_U$. Thus $(T^*M|_U, x, \xi)$ is a special kind of chart in T^*M, a *canonical* chart. In these canonical coordinates,

$$\theta = -\sum_{i=1}^{n} \xi_i \, dx^i \quad \text{and} \quad \omega = \sum_{i=1}^{n} dx^i \wedge d\xi_i,$$

and the form of ω motivates the sign choice.

Ex 5.1.2 For any $\alpha \in \Omega^1 M$ we may consider

$$\begin{array}{ccc} T^*M & \xleftarrow{\alpha^*} & T^*T^*M \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_{T^*}} \\ M & \xrightarrow[\alpha]{} & T^*M \end{array}$$

Then $\alpha^*(-\theta) = \alpha$.

Observe that $d\omega = 0$ (one says ω is *closed*) and that for $v \in T^*M$, $\omega(v) : T_vT^*M \times T_vT^*M \to \mathbb{R}$ so that one can define a map $\omega^\flat(v) : T_vT^*M \to (T_vT^*M)^*$ by $z \mapsto \omega(v) \cdot (-, z)$. Note that $\omega(v)$ is an alternating bilinear functional on T_vT^*M and that $\omega^\flat(v)$ is an isomorphism: one says that ω is *nondegenerate*. To see that $\omega^\flat(v)$ is an isomorphism, assume $\omega(v) \cdot (u,\ ) = 0$ for all $u \in T_vT^*M$. Then in a canonical chart $(\pi^{-1}(U), x, \xi)$,

$$\begin{aligned} 0 = \omega(v) \cdot (u, z) &= \sum_{i=1}^{n} (dx^i \wedge d\xi_i)(v) \cdot (u, z) \\ &= \sum_{i=1}^{n} dx^i(v) \cdot u \, d\xi_i(v) \cdot z \; - \; dx^i(v) \cdot z \, d\xi_i(v) \cdot u \,, \end{aligned}$$

and since this holds for all u, running u over the dual basis to $\{dx^i(v), d\xi_i(v)\}$ one may conclude that $dx^i(v) \cdot u = d\xi_i(v) \cdot z = 0$ as desired.

Abstracting, a manifold P together with a nondegenerate closed 2-form σ is a *symplectic manifold* and such a σ is a *symplectic form*.

Ex 5.1.3 (T^*M, ω) is a symplectic manifold: ω provides the *canonical* symplectic structure on T^*M.

As usual, it helps to study the corresponding vector space theory first.

5.2 Symplectic vector spaces

{svs}

A *symplectic form* on a finite-dimensional $\mathbb{k}$-vector space E is an alternating nondegenerate bilinear functional σ on E. If σ is a symplectic form on E then (E, σ) is a *symplectic vector space*.

If (F, τ) is another symplectic vector space then $A \in L(E, F)$ is *symplectic* if and only if $A^*\tau = \sigma$; *i.e.*, if and only if $\sigma(e_1, e_2) = \tau(Ae_1, Ae_2)$ for all $e_1, e_2 \in E$. Observe that any symplectic map is injective. The set of symplectic maps $E \to E$, denoted by $Sp(E)$, is a subgroup of $GL(E)$ called the *Symplectic group*.

As in the case of the canonical 2-form, define $\sigma^\flat : E \to E^*$ by $e \mapsto \sigma(-, e)$, so $\sigma^\flat$ is an isomorphism. [The superscript $\flat$ is the musical "flat" and is pronounced that way.] Define $e \perp e_1$ if and only if $\sigma(e, e_1) = 0$. Alternating and symmetric bilinear forms are the only choices for $\perp$ to be a symmetric relation on E [1]. If L is a linear subspace of E, denoted $L \le E$, define $L^\perp := \{e \in E : e \perp l, \text{ for all } l \in L\}$, the (symplectic) *orthocomplement* of L in (E, σ).

Ex 5.2.1 The following conditions are satisfied by orthocomplements.

1. $L \le M$, $M^\perp \le L^\perp$
2. $(L^\perp)^\perp = L$
3. $(L \cap M)^\perp = L^\perp \oplus M^\perp$
4. $(L \oplus M)^\perp = L^\perp \cap M^\perp$
5. $\dim L^\perp = \dim E - \dim L$.

So far this looks like an inner product space, but since σ is alternating $\dim L = 1$ implies $L \le L^\perp$. Such a subspace is called *isotropic*. A maximal isotropic subspace is called *Lagrangian*.

Proposition 5.2.2 *If (E, σ) is a symplectic vector space and $L \le E$, then:* {svs1}

1. *L is Lagrangian if and only if $L = L^\perp$;*
2. $\dim E$ *is even;*
3. *if L is isotropic then L is Lagrangian if and only if* $\dim L = \frac{1}{2} \dim E$.

Proof: If $L \le L^\perp$ then there exists $e \in L^\perp - L$. Thus for all $l_1, l_2 \in L$, $\alpha_1, \alpha_2 \in \Bbbk$, $\sigma(l_1 + \alpha_1 e, l_2 + \alpha_2 e) = 0$ whence $L \oplus [\![e]\!]$ is isotropic so L is not Lagrangian. Clearly $L = L^\perp$ implies L is Lagrangian, so (1) is proved.

For (2), let $L \le E$ be Lagrangian so that $\dim L = \dim L^\perp = \dim E - \dim L$, hence $\dim E = 2 \cdot \dim L$. This also proves half of (3).

On the other hand, if L is isotropic and $\dim E = 2 \cdot \dim L$ then $\dim L = \dim L^\perp$ so $L = L^\perp$. □

The following theorem is the main structural description of symplectic vector spaces.

Theorem 5.2.3 *Let (E,σ) be a symplectic vector space. If L is a Lagrangian subspace, then there exists a Lagrangian subspace M such that $E = L \oplus M$. Moreover, $A : (E,\sigma) \to L \oplus L^*$ given by $(l,m) \mapsto (l, \sigma^\flat(m)|_L)$ is a symplectic isomorphism for the* canonical *symplectic structure* {svs2}

$$((a,a^*),(b,b^*)) \mapsto b^*(a) - a^*(b)$$

on $L \oplus L^*$.

Proof: Choose any isotropic subspace M_1 with $M_1 \cap L = \{0\}$ and assume $M_1 < M_1^\perp$. Then $M_1^\perp$ is not contained in $M_1 \oplus L$, for otherwise $M_1 \geq M_1^\perp \cap L^\perp = M_1^\perp \cap L$ whence $M_1^\perp \cap L = \{0\}$ so $\dim M_1^\perp \leq n = \frac{1}{2}\dim E$, which would be a contradiction. Thus there is an $e \in M_1^\perp - M_1 \oplus L$ such that $M_1 \oplus [\![e]\!]$ is isotropic and $(M \oplus [\![e]\!]) \cap L = \{0\}$. Iterating this procedure, one obtains the desired M.

Now observe that $\sigma(l+m, l_1+m_1) = \sigma(l,m_1) + \sigma(l_1,m) = \sigma^\flat(m_1)\cdot l - \sigma^\flat(m)\cdot l_1$ for $l, l_1 \in L$ and $m, m_1 \in M$, so A is symplectic and therefore an isomorphism. □

If L_1 and L_2 are n-dimensional $\mathbb{k}$-vector spaces, a linear map $A : L_1 \oplus L_1^* \to L_2 \oplus L_2^*$ is a (linear) *symplectomorphism* (or *canonical transformation* in classical mechanics) if and only if it is symplectic for the canonical symplectic structures. If $B \in L(L_1, L_2)$ is an isomorphism then $\widetilde{B} = B \oplus {}^tB^{-1}$ is the *induced symplectomorphism*. Thus every isomorphism $L \to \mathbb{k}^n$ induces a symplectomorphism $L \oplus L^* \to \mathbb{k}^n \oplus \mathbb{k}^{n*}$ with canonical structure, for some n.

If $A : E \to \mathbb{k}^n \oplus \mathbb{k}^{n*}$ is a symplectomorphism, then the inverse image of the standard basis is a *canonical basis* of (E,σ). Note that $\{e_i, f_i\}$ is a canonical basis if and only if $\sigma(e_i,e_j) = 0 = \sigma(f_i,f_j)$ and $\sigma(e_i,f_j) = \delta^i_j$, for all $1 \leq i,j \leq n$.

We will first study some properties and representations of symplectomorphisms. Most of the preceding carries over to the case of Banach spaces as will some of the later; *cf.* [53, 13].

{svs3} **Theorem 5.2.4** *Let (E,σ) be a symplectic $\mathbb{k}$-vector space. If $A : E \to E$ is a symplectic map and all the eigenvalues of A are in $\mathbb{k}$, then:*

1. *there exists a Lagrangian subspace L with $A(L) \leq L$;*

2. *on each canonical basis $\{e_i, f_i\}$ with $e_i \in L$, $1 \leq i \leq n$, A is of the form*

$$\begin{bmatrix} B & BS \\ 0 & {}^tB^{-1} \end{bmatrix};$$

3. each matrix of this form represents a symplectomorphism.

Proof: Choose an isotropic invariant subspace L_1 and assume $L_1 < L_1^\perp$. Since A is symplectic, $L_1^\perp$ is also invariant.

Define $A_1 : L^\perp - L_1 \to L_1^\perp - L_1$ by $e + L_1 \to Ae + L_1$. If λ is an eigenvalue of A_1 it is also an eigenvalue of A, hence in $\Bbbk$, so one can choose $f \in (L_1^\perp - L_1) - \{0\}$ with $A_1 f = \lambda f$. Now $f = e + L_1$ for some $e \in L_1^\perp - L_1$, so it follows that $[\![e + L_1]\!]$ is isotropic and invariant. By iterating one obtains the desired L proving (1).

For (2) and (3), compute on the required type of canonical basis ($\sigma(e_i, f_j)$ and $\sigma(f_i, f_j)$ in (2)). □

If $\{e_i\}$ is in addition a basis of L on which B assumes Jordan canonical form, one obtains a clear picture of the possibilities for the Jordan form of a symplectomorphism.

Corollary 5.2.5 *If $\Bbbk = \mathbb{R}$ then eigenvalues $\lambda, \overline{\lambda}, \lambda^{-1}$ and $\overline{\lambda}^{-1}$ of A all have the same multiplicity, and the multiplicities of ± 1 are even.*

Proof: Complexify and apply the theorem. From (2), λ and λ^{-1} have the same multiplicity, and since A is real so does $\overline{\lambda}$, hence also $\overline{\lambda}^{-1}$. If $\lambda = \lambda^{-1}$, (2) again yields even multiplicities. □

If $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$, then $Sp(E)$ is a closed subgroup of $GL(E)$. Recall that a *Lie group* is a smooth manifold with a group structure in which the group operation is smooth. It will be proved in Proposition 7.6.1 that the automorphism group of any bilinear form is a closed subgroup of $GL(E)$, hence a Lie group by Theorem 7.5.2.3; in particular, $Sp(E)$ is a Lie group.

Ex 5.2.6 The *winding line* on the torus $\mathbb{T}^2 = S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$ is a subgroup which is not closed. One may obtain it as the submanifold $(\mathbb{R}, i)$ where $i : \mathbb{R} \to T^2 : t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$ with α an irrational number.

The *Lie algebra* of a Lie group is the tangent fiber at the identity element. The Lie algebra of $Sp(E)$ can be concretely realized as the set of *all* linear maps $A : E \to E$ such that $\sigma(Ae_1, e_2) + \sigma(e_1, Ae_2) = 0$. Elements of the Lie algebra, denoted $\mathfrak{sp}(E)$, are *infinitesimal* (linear) symplectic transformations.

Theorem 5.2.7 *Let (E, σ) be a symplectic vector space over $\Bbbk$. If $A \in \mathfrak{sp}(E)$ and all the eigenvalues of A are in $\Bbbk$, then:* {lasp}

1. there is a Lagrangian subspace L with $A(L) \leq L$;

2. *on each canonical basis* $\{e_i, f_i\}$ *with* $e_i \in L$, $1 \le i \le n$, *A has a matrix of the form*
$$\begin{bmatrix} B & S \\ 0 & -{}^tB \end{bmatrix}$$
with S symmetric;

3. *every matrix of this form represents an element of* $\mathfrak{sp}(E)$.

Corollary 5.2.8 *If* $\Bbbk = \mathbb{R}$, *then the eigenvalues* $\lambda, -\lambda, \overline{\lambda}$ *and* $-\overline{\lambda}$ *all have the same multiplicity and the multiplicity of 0 is even.*

Proofs are omitted: they are similar to those for the case of $Sp(E)$ and in fact the calculations are easier. [If $\Bbbk \neq \mathbb{R}$ or $\mathbb{C}$, define $\mathfrak{sp}(E)$ by the formula.]

According to the structure theorem for symplectic vector spaces, the only invariant is dimension. Thus we shall work on $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and follow Arnol'd somewhat. Denote the usual inner product on $\mathbb{R}^{2n}$ by $\langle\, ,\rangle$ and write $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbb{R}^{2n}$ on the standard basis. Then $\langle (x_1, y_1), (x_2, y_2)\rangle = \sum_i (x_1^i x_2^i + y_1^i y_2^i)$. There is also the usual *complex structure* $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n} : (x, y) \mapsto (-y, x)$, corresponding to multiplication by $\sqrt{-1}$ if one considers $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Hence the canonical symplectic structure is given by $\sigma((x_1, y_1), (x_2, y_2)) = \langle J(x_1, y_1), (x_2, y_2)\rangle$.

The automorphisms of $\mathbb{R}^{2n}$ preserving each of these structures are $O(2n)$, the orthogonal group, $GL(n, \mathbb{C}) \hookrightarrow GL(2n)$, the complex general linear group, and $Sp(2n)$, the symplectic group, respectively. The embedding $GL(n, \mathbb{C}) \hookrightarrow GL(2n)$ is given by $A + iB \to \left[\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right]$.

Ex 5.2.9

$$O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(2n) = Sp(2n) \cap O(2n) = U(n)$$

[Hint: equality is easy; unitary follows from $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.]

5.3 Symplectic manifolds

{tdvf} A *time-dependent vector field* on M is a smooth map $V : \mathbb{R} \times M \to TM$ such that $V(t, \ ) = V(0, \ )$, $t \le 0$, $V(t, \ ) = V(1, \ )$, $t \ge 1$ and $\pi \circ V(t, \ ) = 1_M$ for all t. These have flows much as regular vector fields do; *viz.* [32, pp. 296ff], also [44].

Lemma 5.3.1 *If V is a time-dependent vector field with flow c and if* $\alpha \in \Omega^*$, *then* $\dfrac{d}{ds} c_s^* \alpha|_{s=t} = c_t^* \pounds_{V(t)} \alpha$.

Ex 5.3.2 Prove the Lemma by using the definition of $\pounds$ and the properties in Section 4.6.

Theorem 5.3.3 (Poincaré) *If α is a closed form on M ($d\alpha = 0$), then locally α is* exact *($\alpha = d\beta$).*

Proof: Since this is a local theorem it's actually a theorem about $\mathbb{R}^n$, so we work there. Let $c_t : \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto tx$ for $0 \leq t \leq 1$ and extend to $\mathbb{R}$ as constants at each end. Let $V(t)$ be a vector field with flow c_t, so V is a time-dependent vector field. Using the lemma,

$$\frac{d}{ds}\, c_s^*\alpha|_{s=t} = c_t^*\pounds_{V(t)}\alpha = c_t^* d(V(t) \lrcorner\, \alpha) = d(c_t^* V(t) \lrcorner\, \alpha).$$

Note that $c_0^*\alpha = 0$ and that $c_1^*\alpha = \alpha$. Thus

$$\begin{aligned} \alpha = c_1^*\alpha - c_0^*\alpha &= \int_0^1 \frac{d}{ds} c_s^*\alpha|_{s=t}\, dt \\ &= \int_0^1 d(c_t^* V(t) \lrcorner\, \alpha)\, dt \\ &= d\int_0^1 c_t^* V(t) \lrcorner\, \alpha\, dt \end{aligned}$$

since everything is C^∞. Therefore $d\alpha = 0$ implies $\alpha = d\beta$ on $\mathbb{R}^n$ so α is closed implies α is locally exact on M. □

This is the main result on the local structure of symplectic manifolds: the only local invariant is dimension. As can be seen, this is a dramatic contrast to the pseudoRiemannian case. Precisely,

Theorem 5.3.4 (Darboux) *If (M, σ) is a symplectic manifold then σ is locally constant. Thus M is even dimensional and there are local coordinates (x^i, ξ_i), $1 \leq i \leq n = \frac{1}{2}\dim M$, such that $\sigma = \sum_{i=1}^n dx^i \wedge d\xi_i$ (canonical coordinates).*

Proof: Since this is local we again work in $\mathbb{R}^m$ and without loss of generality near 0. Let $\sigma_t = \sigma(0) + t(\sigma - \sigma(0))$, $0 \leq t \leq 1$, and extend by constants to $\mathbb{R}$. It suffices to find a time-dependent vector field near 0 with flow C such that $C_t^*\sigma_t = \sigma(0)$, since this will provide a chart in which σ is constant. Since $GL(m)$ is open in $\mathfrak{gl}(m)$, $m = \dim M$, there is an open ball around 0 in which σ_t is nondegenerate. By the Poincaré theorem, $\sigma - \sigma(0) = d\alpha$ for some $\alpha \in \Omega^1$(this ball). Without loss of generality, $\alpha(0) = 0$. Define the time-dependent vector field V by $V(t) \lrcorner \sigma_t := -\alpha$, which is possible since σ_t is nondegenerate. Observe that V is smooth and that $V(t,0) = 0$, so that by restricting to a possibly smaller ball we can assume that the integral curves are defined for time at least 1. Now let c be the flow of V. Then

$$\begin{aligned}\frac{d}{ds}c_s^*\sigma_s|_{s=t} &= c_t^*\pounds_{V(t)}\sigma_t + c_t^*\frac{d}{ds}\sigma_s|_{s=t} \\ &= c_t^* d(V(t) \lrcorner \sigma_t) + c_t^*(\sigma - \sigma(0)) \\ &= -c_t^* d\alpha + c_t^*(\sigma - \sigma(0)) \\ &= 0\end{aligned}$$

whence $c_1^*\sigma_1 = c_0^*\sigma(0) = \sigma(0)$ and c_1 provides the desired chart. The other parts follow from the symplectic vector space theory. □

The proofs of the Poincaré and Darboux theorems given here are due to Weinstein [84] as inspired by Moser [58].

Definition 5.3.5 Let (M, σ) be a symplectic manifold and $L \leq M$ a submanifold. We say that L is a *Lagrangian* if and only if $\sigma|_L = 0$.

Equivalently, T_pL is a Lagrangian subspace of T_pM for every $p \in L$.

Example 5.3.6 The fibers of T^*M over M are Lagrangian submanifolds of (T^*M, ω).

Ex 5.3.7 For $\alpha \in \Omega^1 M$, $\alpha^*(d\theta) = d\alpha$. Thus if α is closed, $\operatorname{im}(\alpha)$ is a Lagrangian submanifold of T^*M.

Lagrangian submanifolds of cotangent bundles are the basis of Fourier Integral Operator (FIO) theory.

Naturally, maps that behave well with respect to the symplectic structure are of interest. Let (M, σ) and (N, τ) be symplectic manifolds. A *symplectic map* from M to N is a smooth map $f : M \to N$ such that $f^*\tau = \sigma$; *i.e.*, f_* is a linear symplectic map on each fiber of TM. Thus f is an immersion, hence a local diffeomorphism if $\dim M = \dim N$. If $N = T^*Z$ and $\tau = \omega$, the canonical 2-form, f is called a *canonical map* and thus is a canonical coordinatization when $Z = \mathbb{R}^n$ ($\dim M = 2n$). In mechanics, canonical maps are sometimes called contact maps, but these are more correctly the maps which preserve θ, the canonical 1-form.

Ex 5.3.8 If $f : M \to N$ is a local diffeomorphism then the induced map $\tilde{f} : T^*M \to T^*N : (x, \xi) \mapsto (f(x), {}^t(f_{*x})^{-1}(\xi))$ is a canonical map. Compare to symplectomorphism.

Let V be a smooth vector field on M and c its flow. V is *symplectic* (or an *infinitesimal symplectic transformation*) if and only if all the c_t are symplectic maps.

Theorem 5.3.9 *The following are equivalent:* {before}

1. *V is symplectic;*
2. *$\pounds_V \sigma = 0$;*
3. *$d(V \lrcorner \sigma) = 0$ so locally $V \lrcorner \sigma = df$ for some $f \in \mathfrak{F}$.*

Proof: Recalling the definition of $\pounds$, it is obvious that (1) implies (2). Since $d\sigma = 0, \pounds_V \sigma = d(V \lrcorner \sigma)$ so (2) and (3) are equivalent by the Poincaré theorem. □

Property (3) is of independent interest. Thus if $f \in \mathfrak{F}$, define the *Hamiltonian vector field* associated to f, H^f, by $df = H^f \lrcorner \sigma$. Every Hamiltonian vector field is (globally) symplectic and every symplectic vector field is locally Hamiltonian. The function f is sometimes called the *Hamiltonian* (or *energy*) *function* and is unique modulo constants.

Define the k^{th} de Rham cohomology group by

$$H^k_{dR}(M) := (\text{closed } k\text{-forms})/(\text{exact } k\text{-forms})\,,$$

where a k-form ω is *closed* if and only if $d\omega = 0$ and *exact* if and only if $\omega = d\alpha$ for some $(k-1)$-form α.

Theorem 5.3.10 (deRham) $H^k_{dR}(M) \cong \check{H}^k(M;\mathbb{R}) \cong H^k(M;\mathbb{R})$ *by an $\mathbb{R}$-algebra isomorphism taking the exterior product to the cup product.*

See [82] for an excellent proof. Note that $H^k_{dR}(M) = 0$ for $k > \dim M$. Also, a global Hamiltonian function can always be found if and only if $H^1_{dR}(M) = 0$; *i.e.*, this group measures the failure of every dynamical system on M to be Hamiltonian.

5.4 Some classical theorems

Hamiltonian vector fields are nice. To make this precise, we have a preliminary proposition and definition.

Proposition 5.4.1 *For (M, σ) a symplectic manifold, define the* canonical volume element *(or* phase volume*)*

$$\mu := \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \sigma \wedge \cdots \wedge \sigma$$

(n times). If (U, φ) is a canonical chart, then $\mu|_U = \varphi^*(dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n)$.

Proof: In $\mathbb{R}^{2n} = \{(x, y)\}$, we have

$$\left(\sum_1^n dx^i \wedge dy^i\right) \wedge \cdots \wedge \left(\sum_1^n dx^i \wedge dy^i\right)$$
$$= n!(-1)^{\frac{n(n-1)}{2}} \left[dx^1 \wedge \cdots \wedge dx^n\right] \wedge \left[dy^1 \wedge \cdots \wedge dy^n\right].$$ □

Theorem 5.4.2 (Liouville) *If c is the flow of a Hamiltonian vector field, then $c_t^* \mu = \mu$ for all t for which c_t is defined.*

Proof: $c_t^* \sigma = \sigma$ by Theorem 5.3.9 and pullbacks commute with exterior products. □

Thus if one considers c as the physical flow of a fluid, it represents an incompressible flow whenever the velocity field is Hamiltonian. In this case, for a velocity field H^f, one may regard f as measuring the total energy of the system.

Theorem 5.4.3 (Conservation of Energy) *If c is the flow of H^f, then $f \circ c_t = f$.*

Proof: Computations give us

$$\frac{d}{ds} f \circ c_s|_{s=t} = \langle df \circ c_t, H^f \circ c_t \rangle = \sigma \circ c_t \left(H^f \circ c_t, H^f \circ c_t\right) = 0.$$

Thus the integral curves of H^f are on the level surfaces of f. □

More generally, if $g \in \mathfrak{F}$ then

$$\begin{aligned} \frac{d}{ds} g \circ c_s|_{s=t} &= \langle dg \circ c_t, H^f \circ c_t \rangle \\ &= H^f(g) \circ c_t \\ &= \langle H^g \lrcorner\, \sigma, H^f \rangle \end{aligned}$$

$$= \sigma(H^g, H^f) \circ c_t \,.$$

Thus $\sigma(H^g, H^f) = H^f(g)$ measures the failure of the conservation along the flow of H^f; *i.e.*, it measures the failure of g to be H^f-invariant. $\{g, f\} := \sigma(H^g, H^f)$ is called the *Poisson bracket* of g and f.

Theorem 5.4.4 (Jacobi) *If (M, σ) and (N, τ) are $2n$-dimensional symplectic manifolds and $\varphi : M \to N$ is symplectic, then $\varphi^* H^f = H^{\varphi^* f}$ for every $f \in \mathfrak{F}(N)$. Conversely, if this condition holds on $\{f_j\}$ with $[\![d\varphi^* f_j(x)]\!] = T_x^*(M)$ for all $x \in M$, then φ is symplectic.*

Proof: Note that φ symplectic implies φ is a local diffeomorphism, so $\varphi^* H^f$ makes sense. Thus $\varphi^* H^f \lrcorner\, \varphi^* \tau = \varphi^* \left(H^f \lrcorner\, \tau\right) = \varphi^* df$, so $\varphi^* \tau = \sigma$ is equivalent to the condition $\varphi^* H^f = H^{\varphi^* f}$ for every $f \in \mathfrak{F}(N)$.

On the other hand, if the $d(\varphi^* f_j)(x)$ span $T_x^* M$ then φ_{*x} is injective. Also $[\![df_j(y)]\!] = T_y^* N$ so that $[\![\varphi^* H^{f_j}(x)]\!] = T_x M$ where $y = f(x)$ and $\varphi^* \tau = \sigma$ on them implies $\varphi^* \tau = \sigma$. □

Corollary 5.4.5 *The functions $x^i, \xi_i, 1 \le i \le n$, form a canonical coordinatization of (M, σ) for all $i, j, \{x^i, x^j\} = \{\xi_i, \xi_j\} = 0$ and $\{x^i, \xi_j\} = \delta^i_j$.*

Proof: True for standard coordinates in $\mathbb{R}^{2n}$ by calculation and thus in general by Jacobi's theorem. □

Note that in this case the $dx^i, d\xi_i$ comprise a local coframe. Thus the $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \xi_i}$ make up a local frame.

Ex 5.4.6 In this local frame

$$H^f = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x^i} \;-\; \frac{\partial f}{\partial x^i} \frac{\partial}{\partial \xi_i} \right]$$

and an integral curve $(x(t), \xi(t))$ satisfies *Hamilton's equations* (in fact, is characterized by them):

$$\dot{x}^i = \frac{\partial f}{\partial \xi_i}, \quad \dot{\xi}_i = \frac{\partial f}{\partial x^i},$$

where the superior dots denote time derivatives.

Theorem 5.4.7 *Let (M, σ) be a symplectic manifold. If $f, g \in \mathfrak{F}$, then $[H^f, H^g] = H^{\{g,f\}}$. Thus $\mathfrak{F}$ is a Lie algebra under the Poisson bracket.*

Proof: Let c be the flow of H^f. Since the c_t are symplectic,

$$\begin{aligned}
[H^f, H^g] &= \pounds_{H^f} H^g = \frac{d}{dt} c_t^* H^g|_{t=0} \\
&= \frac{d}{dt} H^{c_t^* g}|_{t=0} \\
&= H^{H^f(g)} = H^{\{g,f\}}.
\end{aligned}$$

Antisymmetry of the Poisson bracket follows from the definition and the Jacobi identity from the first part of this theorem. □

The following theorem is useful in applications, especially mechanics. Let U be a neighborhood of $x_0 \in M$. Functions $\{f_1, \dots, f_k\} \subseteq \mathfrak{F}$ are said to be *in involution* on U if and only if:

1. $df_1, \dots, df_k$ are linearly independent on U;

2. $\{f_i, f_j\} = 0$ for each $1 \le i, j \le k$.

Theorem 5.4.8 (Jacobi) *Let (M, σ) be a symplectic manifold, $x_0 \in M$ and U a neighborhood of x_0. If $\{f_1, \dots, f_k\}$ are in involution on U, then $k \le n = \frac{1}{2}\dim M$ and there exists a neighborhood $\widetilde{U}$ of x_0 and $f_{k+1}, \dots, f_n \in C^\infty(\widetilde{U})$ such that $\{f_1, \dots, f_n\}$ are in involution on $\widetilde{U}$.*

Proof: Since $\{f_1, \dots, f_k\}$ are in involution, $H^{f_1}, \dots, H^{f_k}$ are linearly independent on U by (1) of the definition and $S_x := [\![H^{f_1}(x), \dots, H^{f_k}(x)]\!] \le T_x M$ is isotropic (under $\sigma(x)$), hence $k \le n$.

Now assume $k < n$. Since $\{f, f\} = 0$, it suffices to solve

{diamond}
$$\{f, f_i\} = H^{f_i}(f) = 0, \quad 1 \le i \le k, \tag{5.4.1}$$

subject to $df_1, \dots, df_k, df$ linearly independent. By the preceding theorem the H^{f_i} commute, so *via* the Frobenius theorem we can choose local coordinates y^i with $H^{f_i} = \dfrac{\partial}{\partial y^i}$ and thus solve (5.4.1) near x_0. Moreover, we can prescribe f on any submanifold N with $T_{x_0}M = S_{x_0} \oplus T_{x_0}N$. Hence choose $u \in S_{x_0}^{\perp} \neg S_{x_0}$, N such that $u \in T_{x_0}N$, and then f such that $df(x_0) \cdot u \ne 0$. Then $df(x_0)$ is linearly independent of the $df_i(x_0)$, so it also is true in some neighborhood $\widetilde{U}$ of x_0. □

Ex 5.4.9 This theorem can be used to provide another proof of the Darboux theorem.

The existence of a set of functions in involution globally is quite restrictive.

{arnold} **Theorem 5.4.10 (Arnol'd)** *Let (M, σ) be a symplectic manifold of dimension $2n$. If $\{f_1, \ldots, f_n\}$ are in involution on M and each H^{f_i} is complete, then:*

1. *$f : M \to \mathbb{R}^n$ given by $x \mapsto (f_1(x), \ldots, f_n(x))$ is a fibration of M over an open set in $\mathbb{R}^n$;*

2. *each component Z of each fiber is diffeomorphic to a cylinder $T^k \times \mathbb{R}^{n-k}$, where $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ is the k-torus, and $\varphi : \mathbb{R}^n \to \mathrm{Diff}(Z)$ is given by $(t_1, \ldots, t_n) \mapsto c^n_{t_n} \circ \cdots \circ c^1_{t_1}$, where c^i is a transitive action of $\mathbb{R}^n$ on Z.*

Proof: Since the df_i are linearly independent, f is a submersion. Thus by the global implicit function theorem $F_e := \{x \in M : f(x) = e\}$ is a smooth embedded submanifold of $\dim n$ of M for each $e \in f(M)$. Observing that $H^{f_j}(f_i) = \{f_i, f_j\} = 0$, it follows that each F_e is invariant under each c^i. Also, since the H^{f_i} commute the c^i_t commute, so φ is an action of $\mathbb{R}^n$ on the components of F_e.

The H^{f_i} are linearly independent (the df_i are), so they frame the tangent space of F_e. Hence every point in the orbit $\varphi(\mathbb{R}^n)(z), z \in F_e$, has an open neighborhood (diffeomorphic to an open set in $\mathbb{R}^n$) that is contained in $\varphi(\mathbb{R}^n)(z)$ by the implicit function theorem applied to φ. Since orbits are either identical or disjoint, each component Z of F_e is an orbit and the action is transitive on Z. Thus $Z \cong \mathbb{R}^n/\mathbb{R}^n_z$ by the fundamental theorem on transitive actions. Using the global implicit function theorem it follows that $\mathbb{R}^n_z$ is discrete (0-dimensional), so $Z = T^k \times \mathbb{R}^{n-k}$ for some k. If $z(e) \in F_e$ depends smoothly on e, then the implicit function theorem with parameters shows that $\mathbb{R}^n_{z(e)}$ depends smoothly on e, whence the map f is a fibration. □

A curve $\gamma : \mathbb{R} \to M$ is called *quasiperiodic* if and only if its image is contained in a submanifold N with $\varphi : N \cong T^k$ for some k such that $\varphi \circ \gamma$ has constant velocity on T^k. Note that if $k = n$ in the preceding theorem, then each H^{f_i} has only quasiperiodic integral curves. It is a deep theorem (KAM) of Arnol'd-Kolmogorov and Moser that small perturbations of the Hamiltonian vector field preserve quasiperiodicity. These results are based on some number theory work of C. L. Siegal concerning "sufficiently irrational" numbers. The persistence (or stability) of quasiperiodic orbits is important in celestial mechanics where they form a set of positive measure in the space of all orbits.

Symplectic manifolds occur only in even dimensions. In odd dimensions, *contact* manifolds are the closest analogous objects. Around the middle of the twentieth century, symplectic theory began to branch out from its origins in classical mechanics, as exemplified by the KAM theorem.

Into the 1970s, symplectic manifolds and their geometry and topology continued to grow in interest. Weinstein [84, 85] provides a good survey of this era. In addition, important new applications in PDE theory and in geometric quantization appeared. The 1980s saw a large expansion of concepts, techniques, and results, many in what became a new branch called *symplectic topology.* In particular, the introduction of J-holomorphic (or pseudoholomorphic) curves ushered in a new era for symplectic theory. Another important application was to moment maps. Consolidation of the gains followed in the next decade, as in McDuff and Salamon [55].

6 The Tangent Bundle: Special Calculus

Comparing the bundle cocycles of TM and T^*M, we see that they are isomorphic vector bundles over M. But there is no *natural* one; a choice of one *imposes* some additional structure, albeit much less than that imposed by most other common structures (*e. g.*, a metric tensor).

While T^*M is the natural home of the Hamiltonian formulation of mechanics, TM is that of Lagrangian mechanics. But the former is *inherent* while the latter must be *imposed.*

What *is* inherently there on TM? There are: two vector-bundle structures π_T and π_* on TTM; a natural involution $\mathcal{I}$ interchanging them; the fixed (or invariant) subbundle $\mathscr{S}$ of the involution; and the sections of $\mathscr{S}$ over TM: the *second-order differential equations.*

There is also a question that is inherent: when is a (vector) subbundle E of TM *integrable*? By this is meant that there is a partition of M into equivalence classes, each of which is a submanifold, so that E is the tangent bundle to the partition. We begin with this question.

6.1 Foliations and the Frobenius Theorem

A simple generalization of a vector field is a *line field*: a line bundle $E \leq TM$. As we know, vector fields have *integral curves* with parameterizations. By contrast, *integrable* line fields have analogous unparametrized *integral paths*. These maximal (inextendible) integral paths of an integrable line field are 1-dimensional submanifolds that partition M.

More generally, a subbundle $E \leq TM$ of fiber dimension k is called a *k-plane field* (formerly, a *k-plane distribution*).

Definition 6.1.1 A submanifold $\iota : N \hookrightarrow M$ is an *integral submanifold* of E if and only if $\iota_* T_p N = E_{\iota p} \leq T_{\iota p} M$ for all $p \in N$. A k-plane subbundle {fol}

$E \leq TM$ is said to be *integrable* if and only if it has (maximal) integral submanifolds of dimension k that partition M. This defines a *foliation* $\mathscr{F}$ of M with *leaves* the maximal integral submanifolds.

Classical and traditional versions of such integrability vary; our version is the modern equivalent.

The theory of foliations is a major area of research in modern manifold theory, both smooth and topological. One may consult Tondeur [78] for a brief survey of the more geometric theory (with a 163-page bibliography), and Candel and Conlon [10, 11] for a more leisurely and thorough treatment, including deeper theory.

{folx1} **Example 6.1.2** The vertical bundle $\mathscr{V}$ of Definition 3.7.1 is always integrable. Its leaves are the fibers of TTM. The canonical horizontal bundle $\mathscr{H}$ of Example 3.7.14 is also integrable. Thus they provide transverse foliations of TTM.

Another classical and traditional property of subbundles E is necessary before the main theorem is enunciated.

{invol} **Definition 6.1.3** A subbundle $E \leq TM$ is said to be *involutive* if and only if $V, V' \in \Gamma E$ implies $[V, V']$ is also a section of E.

Ex 6.1.4 Equivalently, ΓE is a Lie subalgebra of $\mathfrak{X}M$.

Remark 6.1.5 One could define another sort of integral submanifold whose tangent spaces are proper subspaces of the fibers of E, $\iota_* T_p N \leq E_{\iota p}$. Then one might call such an E *partially integrable*, in contrast to involutive E being also called *completely integrable.*

{frob} **Theorem 6.1.6 (Frobenius)** *If E is a vector subbundle of TM, then E is integrable if and only if it is involutive.*

Proof: Let $V, V' \in \Gamma E$, $p \in M$, $\iota : N \hookrightarrow M$ an integral submanifold, and $\iota(q) = p$. Since E is integrable, then $\iota_* : TN \to E$ is an isomorphism on fibers. Then there exist vector fields $\overline{V}, \overline{V}' \in \mathfrak{X}N$ with $\iota_* \circ \overline{V} = V \circ \iota$ and $\iota_* \circ \overline{V}' = V' \circ \iota$, so that $\overline{V}$ and V (respectively, $\overline{V}'$ and V') are ι-related.

By Proposition 4.4.5 and Corollary 4.4.6, $\left[\overline{V}, \overline{V}'\right]$ and $[V, V']$ are also ι-related. Therefore

$$[V, V'](p) = \iota_* \left[\overline{V}, \overline{V}'\right](q) \in E_p\,.$$

As p was fixed but arbitrary, it follows that ΓE is a Lie subalgebra of $\mathfrak{X}M$.

Conversely, assume that E is involutive. Since integrability is a local condition, it suffices to consider a cubical chart (U, x) at $p \in M$. Let $h = n - k$ and decompose $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^h$ and $U = U_k \times U_h$ with k-dimensional slices. We need to show that

$$E|_U \cong TU_k \tag{6.1.1}$$ {int}

by inclusion, $U \times \mathbb{R}^k \hookrightarrow U \times \mathbb{R}^n$. Equivalently, we need to produce a smooth map

$$f : U_k \times U_h \to L(\mathbb{R}^k, \mathbb{R}^h)\,, \tag{6.1.2}$$ {vbi}

over 1_U and along $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Such an f characterizes $E \leq TM$ locally (over U).

Let $u, v : \mathbb{R}^n \to \mathbb{R}^n$ represent two vector fields on U with projections u_k, u_h, v_k, and v_h. Assume $u, v \in \operatorname{im}(f)$, so they represent local sections of E as $u = (u_k, f(u_k))$ and $v = (v_k, f(v_k))$. Then $[u, v] \in \operatorname{im}(f)$ implies that

$$Df(u) \cdot v_k - Df(v) \cdot u_k = 0. \tag{6.1.3}$$ {invl}

If for any $(x_0, y_0) \in U_k \times U_h$ there exist open neighborhoods V_k, V_h in U_k, U_h, respectively, and a unique $\alpha : V_k \times V_h \to U_h$ satisfying the initial value problem

$$\begin{aligned} D_x\alpha(x, y) &= f(x, \alpha(x, y)) \\ \alpha(x_0, y) &= y \end{aligned}$$

for $(x, y) \in V_k \times V_h$, then we claim that E is (locally) integrable. Indeed, letting $\alpha_y(x) = \alpha(x, y)$, then

$$D\alpha_y(x) = f(x, \alpha_y(x)) =: f \circ \varphi(x, y)$$

so that $D\varphi(x_0, y_0) = 1_{\mathbb{R}^n}$, whence φ is a local diffeomorphism; moreover,

$$\begin{aligned} D_x\varphi(x, y) \cdot (u, v) &= (u, D\alpha_y(x) \cdot u) \\ &= (u, f(x, \alpha_y(x)) \cdot u)\,. \end{aligned}$$

Integrability now follows from 6.1.1 and 6.1.2.

The existence and uniqueness of α follow from the FEUT *et al.* for a nonautonomous equation (time-dependent vector field) [32, Chapt. 15] in which $g(t, z, y) = f(x_0 + tz, y) \cdot z$, with $(z, y) \in \mathbb{R}^k \oplus \mathbb{R}^h$ here replacing the single space variable $x \in \mathbb{R}^n$ there. If β is the solution of the nonautonomous initial value problem with initial condition $\beta(0, z, y) = y$, then $\alpha(x, y) := \beta(1, x, y)$ exists, by shrinking V_k if necessary, and works, using 6.1.3.

Finally, maximal integral submanifolds (leaves) exist *via* an extension process analogous to that used for integral curves: extend the domain of the solution to the largest possible. □

The first part of the preceding proof is based on Warner's [82]; the second is adapted from Lang [45]: my f, α, and β coincide with his.

6.2 The double tangent bundle

Suppose E is a vector bundle of fiber dimension k over a manifold M. Scalar multiplication $m_a(v) = av$, $a \in \mathbb{R}$, induces a flow on E with fixed point set the 0-section. The flow field (or velocity field) is given by $A := v^j \bar{\partial}_j$, where $\bar{\partial}_j = \partial/\partial v^j$ and v^j are fiberwise coordinates on E, and is called the *Euler-Lagrange* vector field. We may also consider A for m_a with $a > 0$, and shall denote it the same way.

Ex 6.2.1 Both flows are global (the vector fields are complete), although the second is only on $E \neg 0$. [Hint: for $a > 0$ write $a = e^t$.]

The *sphere bundle* $SE := (E \neg 0)/m_{\exp t}$ has fibers diffeomorphic to S^{k-1}. A further quotient by the antipodal relation yields the *projective bundle* PE with fibers $\mathbb{R}P^{k-1}$.

Ex 6.2.2 Equivalently, the projective bundle can be constructed by taking m_a for $a \neq 0$.

A smooth function f on $E \neg 0$ is *homogeneous of degree* r if and only if $\pounds_A f = Af = rf$. Note that this agrees with the usual version of Euler's equation on $\mathbb{R}^k$. We shall denote this property by $hd(r)$. Similarly, a smooth form α is $hd(r)$ if and only if $\pounds_A \alpha = r\alpha$. By tradition, however, a vector field V is $hd(r)$ if and only if $\pounds_A V = (r-1)V$.

{hd1l} **Ex 6.2.3** Show that a smooth function is $hd(1)$ if and only if it is linear.

Ex 6.2.4 Recall the vertical bundle $\mathscr{V}E$ of Definition 3.7.1. Show that the Euler-Lagrange vector field A is *vertical*: it is a section of the vertical bundle $\mathscr{V}E$ over E.

We specialize (3.7.1) to $E = TM$.

{tpi2}
$$\begin{array}{ccc} TTM & \xrightarrow{\pi_* = T\pi} & TM \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ TM & \xrightarrow[\pi]{} & M \end{array} \tag{6.2.1}$$

This gives two vector bundle structures on TTM over TM, denoted here by π_T and π_*.

Recall from page 63 that we have put induced local coordinates (x, y) on TM such that x^i are local coordinates in the base manifold M and y^i is given by $y^i(u) = dx^i(u)\,\partial_i$, the ∂_i-component of $u \in TM$. We may also put induced coordinates (x, y, X, Y) on TTM. For $w \in TTM$, we let $X^i(w) = d(x^i \circ \pi)\cdot w\,\overline{\partial}_i$ and $Y^i(w) = dy^i(w)\,\widetilde{\partial}_i$, where $\overline{\partial}_i = \partial/\partial(x^i \circ \pi)$ and $\widetilde{\partial}_i = \partial/\partial y^i$.

We now examine the two vector bundle structures on $TTM \twoheadrightarrow TM$ in these *induced coordinates.* First, we may consider TTM as the tangent bundle of TM. The natural projection π_T is given by

$$\pi_T(x, y, X, Y) = (x, y).$$

Fiberwise addition in this bundle is given by

$$(x, y, X_1, Y_1) + (x, y, X_2, Y_2) = (x, y, X_1 + X_2, Y_1 + Y_2).$$

On the other hand, we may consider the induced tangent map π_* of the projection $\pi : TM \twoheadrightarrow M$. In our induced coordinates, $\pi_* : TTM \twoheadrightarrow TM$ is given by

$$\pi_*(x, y, X, Y) = (x, X)$$

and fiberwise addition is given by

$$(x, y_1, X, Y_1) \mathbin{+\!\!\!+} (x, y_2, X, Y_2) = (x, y_1 + y_2, X, Y_1 + Y_2).$$

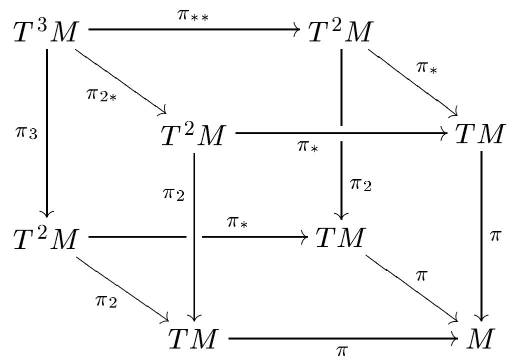
There is a natural involution $\mathcal{J} : TTM \to TTM$ that interchanges these two vector bundle structures.

$$\mathcal{J} : (x, y, X, Y) \mapsto (x, X, y, Y)$$

The vertical bundle $\mathscr{V} = \ker \pi_*$ is a vector subbundle with respect to both vector bundle structures on TTM. In induced local coordinates, elements of $\mathscr{V}$ look like $(x, y, 0, Y)$. The *vertical endomorphism* of TTM for π_T is given by $\mathcal{V}(x, y, X, Y) := (x, y, 0, X)$. It is nilpotent since $\mathcal{V}^2 = 0$.

Remarks 6.2.5 (Iterated Tangent Bundles) The tower of *iterated* (or sometimes *higher-order*) tangent bundles is defined by $T^rM := T(T^{r-1}M)$ for all $r \geq 1$, with $T^0M := M$. The manifold T^rM has dimension $2^r n$. Denote the natural projection by $\pi_r : T^rM \to T^{r-1}M$, so $\pi_2 = \pi_T$ and $\pi_1 = \pi$. There are r vector-bundle structures of T^rM over $T^{r-1}M$ and there are $\binom{r}{2}$ involutions (endomorphisms of order 2) of T^rM that interchange pairs of them. {itb}

The mapping diagram of the projections π_i and their derivatives (the functorial maps $T\pi_i = \pi_{i*}$) is the r-cube: T^rM is one vertex and r projections (natural and functorial) emanate from it with codomains $T^{r-1}M$. Here is the diagram for $r = 3$.



(6.2.2) {tpi3}

Numbering the structures 1 through r, we may denote the involution interchanging structures i and j in cycle notation as (ij). It is a standard algebra exercise to verify that the cycles (12), (23), … $(r-1\ r)$ generate the symmetric group S_r. It may be called the group of *quasi-automorphisms* of T^rM because these are isomorphisms of T^rM with itself with *different* vector-bundle structures. Denote this group by $\mathrm{qAut}(T^rM)$.

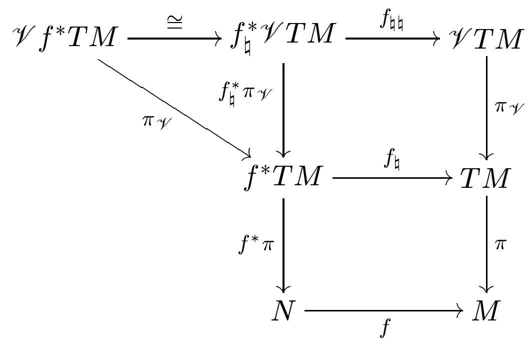
Ex 6.2.6 Verify all of the unobvious claims in the preceding Remarks.

Ex 6.2.7 Consider T^3M. Let structure 1 be that of π_3, 2 that of π_{2*}, and 3 that of π_{**}. Verify that $(12) : (x, y, X, Y, u, v, U, V) \mapsto (x, y, u, v, X, Y, U, V)$, $(13) : (x, y, X, Y, u, v, U, V) \mapsto (x, X, y, Y, u, U, v, V)$, and that these two involutions generate $\mathrm{qAut}(T^3M) = \{1, (12), (13), (23), (123), (321)\}$.

Ex 6.2.8 If appropriate for you, verify that the fixed-point set of $\mathrm{qAut}(T^rM)$ is isomorphic to $J^r_0(M) = J^r(\mathbb{R}_0, M)$ of dimension $(r+1)n$.

Continuing, from before the remarks, with $E = TM$ in Proposition 3.7.2, we obtain

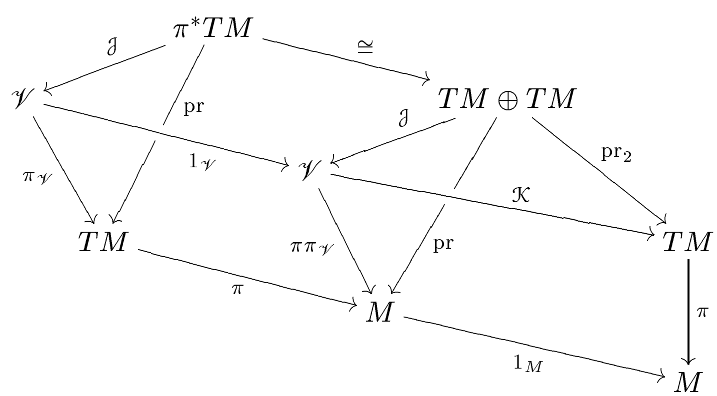
{vtmf} **Proposition 6.2.9** *If f^*TM is the pullback of TM along $f : N \to M$, then $\mathscr{V}f^*TM$ and $f_\natural^*\mathscr{V}TM$ are isomorphic, where $f_\natural$ is the pushforth of f,* *as shown.*



In summary, the vertical bundle is functorial. □

In this context, we usually abbreviate $\mathscr{V}TM$ as simply $\mathscr{V}$.

Summarizing the relationships of these bundles and their bundle morphisms, we have this diagram in which all bundle morphisms are isomorphisms of fibers.



(6.2.3) {jktm}

The two natural vector bundle morphisms $\mathcal{K} : \mathscr{V} \to TM$, respecting π_T, and $\mathcal{J} : (\pi^*TM \cong TM \oplus TM) \to \mathscr{V}$ are isomorphisms on fibers. These provide two extended versions of canonical parallel translation in a vector space.

Let E and F be vector bundles over M, and suppose that $u : E \to F$ is a smooth, fiber-preserving map over 1_M. In particular, u is not necessarily a morphism of vector bundles. We shall define a map $\mathcal{F}u : E \to \mathrm{Hom}(E, F)$ over M.

For all $p \in M$, recall that $T(E_p) \cong E_p \oplus E_p$ and $T(F_p) \cong F_p \oplus F_p$ naturally. We then define

$$\mathcal{F}u|_{E_p} := \left(u|_{E_p}\right)_* : E_p \oplus E_p \to F_p \oplus F_p : (v, w) \mapsto \left(u(v), \left(u|_{E_p}\right)_* w\right).$$

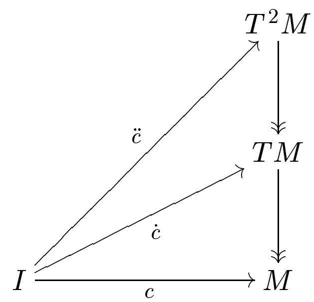
Applying pr_2 to both domain and codomain, we obtain (with some abuse of notation) $\mathfrak{F}u|_{E_p} : w \mapsto \left(u|_{E_p}\right)_* w$. This is actually just the second component of the induced tangent map $T\left(u|_{E_p}\right) = \left(u|_{E_p}\right)_*$.

Effectively, we have identified the vector spaces E_p and F_p with their own tangent spaces *via* canonical parallel translation in a vector space. In this bundle setting, it amounts to using the morphism $\mathfrak{J}$.

The resulting map $\mathfrak{F}u : E \to \mathrm{Hom}(E, F)$ is called the *vertical* (or *fiber*) *derivative* of u, or the *Legendre transform* of u.

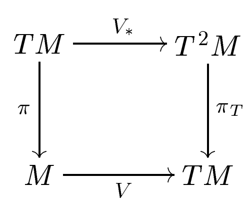
{ltf} **Example 6.2.10** Consider $f \in \mathfrak{F}E$ and regard $f : E \to M \times \mathbb{R}$ as fiber preserving. Then $\mathfrak{F}f : E \to \mathrm{Hom}(E, M \times \mathbb{R}) = E^*$ maps the vector bundle E to its dual E^*.

If $c : I \to M$ is a curve in M, then there are *natural lifts* $\dot{c}$ to TM and $\ddot{c}$ to T^2M as shown.



Here $\dot{c}(t) := (c(t), v)$ where v is the velocity vector of c at $c(t)$ as defined on page 62. It is also called the *velocity lift* of c. Then $\ddot{c}$ is the velocity lift of $\dot{c}$ or the *acceleration lift* of c.

Let V be a vector field on M. Then $V : M \to TM$, and we may consider V_* where



Letting V denote its own local representative, then

$$V_* : (x, y) \mapsto (x, V, y, y^j \partial_j V^i)$$

where $V^i = y^i(V)$ and $y^j \partial_j V^i = [\partial_j V^i] y^j$ is the second component of the induced tangent map.

{cvl} **Definition 6.2.11** The *canonical lift* V^{C} of $V \in \mathfrak{X}$ is the vector field $\mathcal{J}V_*$, locally given by $V^{\mathrm{C}}(x,y) = (x, y, V, y^j \partial_j V^i)$. The *vertical lift* V^{V} of V is the vector field $\mathcal{V}V^{\mathrm{C}}$, locally given by $V^{\mathrm{V}}(x,y) = (x, y, 0, V)$. Clearly V^{V} is a vertical vector field on TM.

The *natural* (as [89] said it should be called) or *canonical* or *complete* lift was first defined by Sasaki [69], extended to the full tensor algebra *etc.* by Yano and Kobayashi [89], developed at length by Yano and Ishihara [88], and extended to the full tower T^rM by Bucataru and Dahl [9].

Proposition 6.2.12 *If Φ is the local flow of $V \in \mathfrak{X}(M)$, then $\Phi^{\mathrm{C}} := \Phi_*$ is the (local) flow of $V^{\mathrm{C}} \in \mathfrak{X}(TM)$.* {clf}

Proof: We recall that $\dot{\Phi} = V$, so $\mathcal{J} \circ V_* = \mathcal{J} \circ (\dot{\Phi})_*$ and we must show that $\mathcal{J} \circ (\dot{\Phi})_* = (\Phi_*)^{\cdot}$. This can be done either with integral curves or with formulas in induced local coordinates. We leave the details as an exercise for the reader.

□

Corollary 6.2.13 *For $V \in \mathfrak{X}$, V^{C} is complete if and only if V is.* □ {ccl}

Ex 6.2.14 The vertical lift of a vector field is always complete.

Vector fields are $(1,0)$-tensors. To get a better sample of canonical and vertical lifts, we shall also consider $(0,0)$-tensors (functions), $(0,1)$-tensors (1-forms), and $(1,1)$-tensors (endomorphisms of TM).

For any $f \in \mathfrak{F}M$, define the vertical lift of f by $f^{\mathrm{V}} := f \circ \pi$. If $\omega \in \Omega^1 M$, then we define the vertical lift of ω by $\omega^{\mathrm{V}} := (\omega_i \circ \pi)\, \overline{dx}^i$, where $\overline{dx}^i$ are dual to the $\bar{\partial}_i$ defined above, and $\omega = \omega_i dx^i$. In induced local coordinates, the components of ω^{V} are $(x, y, \omega_i, 0)$. For any $F \in \mathfrak{T}^1_1 M$, define the vertical lift of F to be

$$F^{\mathrm{V}} := \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix} = F^i_j\, \widetilde{\partial}_i \otimes \overline{dx}^j \in \operatorname{End} T^2 M,$$

where the matrix representation of the fiber endomorphism of T^2M at $(x,y) \in TM$ is $2n \times 2n$ with $n \times n$ blocks.

Ex 6.2.15 If $V, W \in \mathfrak{X}$, $f \in \mathfrak{F}$, $\omega \in \Omega^1$, and $F \in \mathfrak{T}^1_1$, then the following identities hold for vertical lifts.

$$\begin{aligned} V^{\mathrm{V}} f^{\mathrm{V}} &= 0 \\ (V+W)^{\mathrm{V}} &= V^{\mathrm{V}} + W^{\mathrm{V}} \text{ and similarly for 1-forms} \end{aligned}$$

$$\begin{aligned} (fV)^{\mathrm{V}} &= f^{\mathrm{V}} V^{\mathrm{V}} \text{ and similarly for 1-forms} \\ \left[V^{\mathrm{V}}, W^{\mathrm{V}}\right] &= 0 \\ F^{\mathrm{V}} V^{\mathrm{V}} &= 0 \\ d(\omega^{\mathrm{V}}) &= (d\omega)^{\mathrm{V}} \end{aligned}$$

Next we define the canonical lift of each. If $f \in \mathfrak{F}$, define the canonical lift of f to be $f^{\mathrm{C}} := y^i \partial_i f$. If $\omega \in \Omega^1$, then its canonical lift is $\omega^{\mathrm{C}} := (y^j \partial_j \omega_i, \omega)$, or equivalently $\langle \omega^{\mathrm{C}}, V^{\mathrm{C}} \rangle = \langle \omega, V \rangle^{\mathrm{C}}$.

Ex 6.2.16 If $V, W \in \mathfrak{X}$, $f, g \in \mathfrak{F}$, and $\alpha, \omega \in \Omega^1$, then the following identities hold.

$$\begin{aligned} V^{\mathrm{V}} f^{\mathrm{C}} &= (Vf)^{\mathrm{V}} \\ \langle \omega^{\mathrm{V}}, V^{\mathrm{C}} \rangle &= \langle \omega, V \rangle^{\mathrm{V}} \\ (fg)^{\mathrm{C}} &= f^{\mathrm{C}} g^{\mathrm{V}} + f^{\mathrm{V}} g^{\mathrm{C}} \\ (V+W)^{\mathrm{C}} &= V^{\mathrm{C}} + W^{\mathrm{C}} \text{ and similarly for 1-forms} \\ (fV)^{\mathrm{C}} &= f^{\mathrm{C}} V^{\mathrm{V}} + f^{\mathrm{V}} V^{\mathrm{C}} \text{ and similarly for 1-forms} \\ (Vf)^{\mathrm{C}} &= V^{\mathrm{C}} f^{\mathrm{C}} \\ \left[V^{\mathrm{C}}, W^{\mathrm{C}}\right] &= [V, W]^{\mathrm{C}} \\ \left[V^{\mathrm{V}}, W^{\mathrm{C}}\right] &= [V, W]^{\mathrm{V}} \\ d\omega^{\mathrm{C}} &= (d\omega)^{\mathrm{C}} \\ (\alpha \wedge \omega)^{\mathrm{C}} &= \alpha^{\mathrm{C}} \wedge \omega^{\mathrm{V}} + \alpha^{\mathrm{V}} \wedge \omega^{\mathrm{C}}. \end{aligned}$$

For $F \in \mathfrak{T}^1_1$, define its canonical lift fiberwise by

$$F^{\mathrm{C}} := \begin{bmatrix} F & 0 \\ y^j \partial_j F^i_k & F \end{bmatrix} \in \operatorname{End} T^2 M.$$

Ex 6.2.17 For $F \in \mathfrak{T}^1_1$ and $V \in \mathfrak{X}$, we have

$$\begin{aligned} F^{\mathrm{C}}(V^{\mathrm{V}}) &= (FV)^{\mathrm{V}} = F^{\mathrm{V}}(V^{\mathrm{C}}) \text{ and} \\ F^{\mathrm{C}}(V^{\mathrm{C}}) &= (FV)^{\mathrm{C}}. \end{aligned}$$

Ex 6.2.18 Determine vertical and canonical lifts of a derivation $\mathcal{D} \in \operatorname{Der} \mathfrak{F}$.

Ex 6.2.19 For $F, G \in \mathfrak{T}^1_1 M = \operatorname{End} M$ we have $(F \circ G)^{\mathrm{C}} = F^{\mathrm{C}} \circ G^{\mathrm{C}}$. If $P \in \mathbb{R}[x]$, then $P(F)^{\mathrm{C}} = P(F^{\mathrm{C}})$.

Ex 6.2.20 Let $V, W \in \mathfrak{X}$, $f \in \mathfrak{F}$, and $\omega \in \Omega^1$. Then $\pounds_{V^{\mathrm{C}}} W^{\mathrm{C}} = (\pounds_V W)^{\mathrm{C}}$, $\pounds_{V^{\mathrm{C}}} \omega^{\mathrm{C}} = (\pounds_V \omega)^{\mathrm{C}}$, and $\pounds_{V^{\mathrm{C}}} f^{\mathrm{C}} = (\pounds_V f)^{\mathrm{C}}$. By induction, for any $K \in \mathfrak{T}M = \bigoplus \mathfrak{T}^r_s M$ we get $\pounds_{V^{\mathrm{C}}} K^{\mathrm{C}} = (\pounds_V K)^{\mathrm{C}}$.

6.3 SODEs

Let $\mathcal{I}$ be the canonical involution on T^2M, so it isomorphically exchanges the two vector bundle structures on T^2M. We denote the fixed set of $\mathcal{I}$ by $\mathscr{S}$ and observe that it is an *affine* subbundle of both π_T and π_*, but not a *vector* subbundle of either.

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\pi_*} & TM \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ TM & \xrightarrow[\pi]{} & M \end{array} \tag{6.3.1}$$ {tpi4}

Definition 6.3.1 A section S of T^2M over TM is a *second-order differential equation* (SODE) when $\mathcal{I}S = S$, or equivalently when $S \in \Gamma(\mathscr{S})$. The space of all SODEs is denoted by $\mathrm{DE}_2(M)$, those vanishing on the 0-section of TM by $\mathrm{QSpray}(M)$. Elements of the latter are called *quasisprays*, or *qsprays* for short. {sode}

Thus a SODE can be expressed locally as $S : (x, y) \mapsto (x, y, y, S(x, y))$ with a little abuse of notation.

Remark 6.3.2 Integral curves γ of a SODE are called *holonomic*; *i.e.,* a holonomic curve is the velocity lift of its projection to the base: $\gamma = \dot{c}$ for the curve $c = \pi \circ \gamma$ in M. As a further example, induced local coordinates are *holonomic coordinates*. One similarly defines holonomic local frames, *etc.* As a consequence of this behavior, SODEs are examples of *projectable* vector fields on TM.

Remark 6.3.3 If desired, one may work with jet spaces using $J^1(\mathbb{R}_0, M) \cong TM$ and $J^2(\mathbb{R}_0, M) \cong \mathscr{S}$, where the notation indicates jets with fixed source $0 \in \mathbb{R}$ and target any point in M. See [31, 56] for more on jets.

We observe that $\mathscr{S}$ is an affine subbundle of T^2M with translations $\mathscr{V}$. This allows us to regard $\mathrm{DE}_2(M)$ as an affine space with translations $\Gamma(\mathscr{V})$ and with $\mathrm{QSpray}(M)$ as a closed affine subspace, so that both are affine nuclear Fréchet spaces as in Example 3.3.4.

Let S be a SODE over M, p a point in M, and consider the value $S(0)$ for 0 in T_pM. Note that this is a vertical vector in T_0T_pM; in induced local coordinates it is $(x, 0, 0, S(0))$. Define a vertical vector field by

$$R(u) = \mathcal{J}_u \mathcal{K}(S(0)) \tag{6.3.2}$$ {vcf}

for each $u \in T_pM$ and for each $p \in M$. This looks more imposing than it is. One takes the vertical vector $S(0)$, regards it as an element of T_pM (applies $\mathcal{K}$), parallel translates it all over the fiber T_pM as a vertical vector (applies $\mathcal{J}_u$ for each $u \in T_pM$), and does this for each $p \in M$ to create the vertical vector field R on TM.

{vc} **Definition 6.3.4** Such an R is called *vertically constant* (VC) since it is constant along the fibers of TM in an obvious sense.

Clearly, $Q = S - R$ as a section of $\mathscr{S}$ is a quasispray on TM, using the structure of $\mathscr{S}$ as an affine bundle with translation vector bundle $\mathscr{V}$. This supports concentrating on quasisprays rather than general SODEs.

The next result is obvious from the definition of vertical lift.

Proposition 6.3.5 *A vector field on TM is VC if and only if it is a vertical lift of a vector field on M.* □

In mechanics, R as above represents an external force. For $R = W^{\mathrm{V}}$, W could be a wind blowing across M.

Ex 6.3.6 Using any three consecutive elements $T^rM, T^{r-1}M, T^{r-2}M$ from the tower of iterated tangent bundles, repeat the procedure of this section using only the two vector-bundle structures of π_r and $\pi_{(r-1)*}$. Then extend to the canonical and vertical lifts of SODEs and qsprays in this tower setting [9].

6.4 Lagrangian mechanics

The basic object of interest is a *Lagrangian function*; that is, a function $L \in \mathfrak{F}TM$. In classical mechanics, such an L is defined for any system as $L =$ (kinetic energy) − (potential energy). More generally, L is regarded as the (net) total energy of a *Lagrangian system*.

As in symplectic (Hamiltonian) mechanics, M is the configuration space of the system. Physicists want to obtain the so-called "equations of motion" of the system described by L. These describe how configurations change—move in M—as the system evolves. To get them, let c be a smooth curve in M, and consider variations of $\int_0^1 L(\dot{c}(t))dt$ over c. We then obtain the equations of motion by characterizing stationary (or extremizing) curves *via* standard methods of the calculus of variations. In particular, this yields the *Euler-Langrange equation*, which then directly yields the equations of motion.

There is another route to the same end.

{cl1dhj} **Example 6.4.1** The classical, local, 1-dimensional version of the *Hamilton-Jacobi problem* is stated as follows. Find the *principal function* S such that

$$\frac{\partial S}{\partial t} + H\left(x, \frac{\partial S}{\partial x}\right) = 0$$

where H is the Hamiltonian. Setting $W(x) = S(t,x) - tE$, where E is a constant, the *characteristic function* W satisfies

$$H\left(x, \frac{\partial W}{\partial x}\right) = E.$$

Both equations may be called *Hamilton-Jacobi*, but the second is the one that extends to manifolds and bundles more naturally.

There is a strong connection between this idea and the symplectic structure studied in Chapter 5. For any $L \in \mathfrak{F}TM$, define the *canonical 1-form* $\theta_L \in \Omega^1 TM$ by $\theta_L := dL \circ \mathcal{V}$. In induced local coordinates, θ_L is given by $\theta_L = (\widetilde{\partial}_i L)\overline{dx}^i$. Now define the *canonical 2-form* by $\omega_L := -d\theta_L$. There is also the *energy function* associated to L. It is given by $E_L := A(L) - L$, where A is the Euler-Liouville vector field, and $A(L)$ is called the *action*.

In this setting, the Euler-Lagrange equation is replaced by the *Hamilton-Jacobi equation*. Suppose there is a $V \in \mathfrak{X}M$ such that $V^*E_L = E$ is a constant, and $V^*\omega_L = 0$, where V^* is the pullback along $V : M \to TM$. The constant E is called the *energy*. In fact, $V^*\theta_L = df$ locally for some $f \in \mathfrak{F}M$; this is slightly stronger than $V^*\omega_L = 0$. If V is such a vector field with integral curve c, then it turns out that $(c, V \circ c)$ is a solution to the Euler-Lagrange equation.

Definition 6.4.2 A Lagrangian $L \in \mathfrak{F}TM$ is called *regular* if and only if ω_L {rlgr}
is nondegenerate (symplectic).

If L is regular, then there exists a unique $S_L \in \mathfrak{X}TM$ such that

$$S_L \lrcorner\, \omega_L = dE_L\,. \tag{6.4.1}$$ {lde}

This equation is called the *Lagrange (dynamical) equation*. The vector field S_L is called the *Lagrangian vector field* of the Lagrangian system.

Ex 6.4.3 S_L is a SODE.

Problem 6.4.4 The *generalized Lagrangian Hamilton-Jacobi problem* is de- {glhj}
scribed as follows. Given a regular Lagrangian L, find $V \in \mathfrak{X}M$ such that if c is an integral curve of V then $\dot{c}$ is an integral curve of S_L.

Note that $\operatorname{im}(V) \hookrightarrow TM$ is thus an S_L-invariant submanifold.

We can now restate equation (6.4.1) in terms of the calculus of variations interpretation mentioned in the introduction to this section. If c extremizes $\int_0^1 L(\dot{c}(t))dt$, then $\ddot{c} \lrcorner \omega_L = dE_L$. (The proof hinges on P4, p 87, which Besse calls the fundamental relation of the calculus of variations.) Since S_L is a SODE, if $\dot{c}$ is an integral curve then $\ddot{c} = S_L(\dot{c})$. For regular Lagrangians, this is the explicit relation between Euler-Lagrange and Hamilton-Jacobi.

{vrel} **Proposition 6.4.5** *A vector field V is a solution of Problem 6.4.4 if and only if V and S_L are V-related; namely $S_L \circ V = V_* \circ V$.*

Proof: If V is a solution, then every integral curve γ of V satisfies $S_L \circ \dot{\gamma} = (V \circ \gamma)^{\cdot} = V_* \circ \dot{\gamma} = V_* \circ V \circ \gamma$. On the other hand, $S_L \circ \dot{\gamma} = S_L \circ V \circ \gamma$. Since there is an integral curve through every point of M, then equivalently $V_* \circ V = S_L \circ V$. The converse is left to the reader. □

This equation for S_L provides a PDE for V, replacing the PDE for W in Example 6.4.1. We also have the following result.

{vinv} **Proposition 6.4.6** *A vector field V is a solution of Problem 6.4.4 if and only if $\operatorname{im}(V) \hookrightarrow TM$ is S_L-invariant;* i.e., *S_L is tangent to* $\operatorname{im}(V)$.

Proof: Set $v = V_p = V(p)$. By Proposition 6.4.5, $S_L(v) = T_pV(v)$. Conversely, $\operatorname{im}(V)$ being S_L-invariant means that $S_L(v) \in T_v \operatorname{im}(V)$. Hence there exists a u in T_pM such that $S_L(v) = T_pV(u)$. Then

$$v = \pi_* \circ S_L(v) = \pi_* \circ V_*(u) = (\pi \circ V)_*(u) = u$$

as $\pi \circ V = 1_M$ and $\pi_* \circ S_L = 1_{TM}$. Thus $S_L(v) = T_pV(v)$ for every $p \in M$. So $S_L \circ V = V_* \circ V$ and V is a solution of Problem 6.4.4 by the preceding Proposition. □

If V is a solution of the generalized Lagrangian Hamilton-Jacobi Problem 6.4.4, then the integral curves of V are π-projections of the integral curves of S_L lying in $\operatorname{im}(V)$. As we have not yet used that S_L is Lagrangian, all of these results actually hold for an arbitrary SODE S. Using now that S_L is Lagrangian, these results can be related to the Lagrangian energy function E_L as follows.

{ven} **Proposition 6.4.7** *A vector field V is a solution of Problem 6.4.4 if and only if $V \lrcorner V^*\omega_L = d(V^*E_L)$.*

Proof: Pulling back the Lagrange equation (6.4.1) along V yields

$$V^*(S_L \lrcorner\, \omega_L) = V^* dE_L = d(V^* E_L).$$

As V and S_L are V-related by Proposition 6.4.5, we get $V^*(S_L \lrcorner\, \omega_L) = V \lrcorner\, V^*\omega_L$, whence the desired equation.

Conversely, assume V satisfies the equation. Then

$$D_L := S_L \circ V - V_* \circ V : M \to T^2 M$$

is a vector field along V. Observe that D_L is vertical. Indeed, $\pi_* D_L = \pi_* S_L \circ V - \pi_* V_* \circ V = V - V = 0$. From equation (6.4.1) again, we get $V^*(S_L \lrcorner\, \omega_L) - V \lrcorner\, V^*\omega_L = 0$. Thus, for $p \in M$, $w \in T_p M$, and $v = V(p)$, we have

$$\begin{aligned} 0 &= [V^*(S_L \lrcorner\, \omega_L) - V \lrcorner\, V^*\omega_L]\, w \\ &= (\omega_L)_v(S_L(v), V_* w) - (V^*\omega_L)_p(v, w) \\ &= (\omega_L)_v(S_L(v), V_* w) - (\omega_L)_v(V_* v, V_* w) \\ &= (\omega_L)_v(D_L(p), V_* w). \end{aligned}$$

Moreover, for any vertical vector field W on TM, we have

$$(\omega_L)_v(D_L(p), W(v)) = (-d\theta_L)_v(D_L(p), W(v))$$

vanishes globally since $d\theta_L(W_1, W_2) = 0$ for all vertical vector fields W_1, W_2 on TM. As ω_L is nondegenerate, it follows that $D_L = 0$, whence the conclusion. □

The following *standard* Lagrangian Hamilton-Jacobi problem is usually simpler than the generalized version.

Problem 6.4.8 Given a regular Lagrangian $L \in \mathfrak{F}TM$, the *(standard) Lagrangian Hamilton-Jacobi problem* is to find solutions V to Problem 6.4.4 that additionally satisfy $V^*\omega_L = 0$. {slhj}

As $0 = V^*\omega_L = -V^* d\theta_L = -d(V^*\theta_L)$, every point in M has an open neighborhood U and a $W \in \mathfrak{F}U$ such that $V^*\theta_L = dW$ locally in U.

Corollary 6.4.9 *If V solves Problem 6.4.8, then $d(V^*E_L) = 0$.* □ {slhjc}

Observe that if V solves Problem 6.4.8, then $\mathrm{im}(V)$ is a Lagrangian submanifold of (TM, ω_L) contained in a level set of E_L. In fact, if $j : \mathrm{im}(V) \hookrightarrow TM$, then $j^*\omega_L = 0$ since $V^*\omega_L = 0$; *cf.* Example 6.4.1.

Combining previous results, we obtain one that also explains how and why the standard problem is usually simpler than the generalized one.

Theorem 6.4.10 *If $V \in \mathfrak{X}M$ satisfies $V^*\omega_L = 0$ for a symplectic ω_L, then the following are equivalent:* {lsbmfld}

1. *V solves Problem 6.4.8;*
2. *$d(V^*E_L) = 0$;*
3. *$\mathrm{im}(V) \hookrightarrow TM$ is an S_L-invariant ω_L-Lagrangian submanifold;*
4. *the integral curves of S_L with initial conditions in $\mathrm{im}(V)$ project to integral curves of V.*

□

For a Lagrangian L, consider the vertical derivative $\mathcal{F}L : TM \to T^*M$ as in Example 6.2.10. In mechanics, this is simply called *the Legendre transform.* It follows that L is regular if and only if $\mathcal{F}L$ is a local diffeomorphism of $TM \neg 0$ with $T^*M \neg 0$. Indeed, in induced local coordinates it maps $(x, y) \mapsto (x, \widetilde{\partial}_i L(y))$. Now the rank of $\mathcal{F}L$ is the rank of the matrix

$$\left[\frac{\partial^2 L}{\partial y^i \partial y^j}\right] = \left[\widetilde{\partial}_i \widetilde{\partial}_j L\right].$$

This matrix locally represents the *vertical* (or *fiber*) *Hessian* of L. In these induced local coordinates, recall that $\theta_L = \widetilde{\partial}_i L\, \overline{dx}^i$ (just after Example 6.4.1).

Ex 6.4.11 In these same induced local coordinates,

$$\omega_L = \widetilde{\partial}_i \widetilde{\partial}_j L\, \overline{dx}^i \wedge dy^j + \bar{\partial}_i \widetilde{\partial}_j L\, \overline{dx}^j \wedge \overline{dx}^i.$$

Thus ω_L is symplectic if and only if $\mathcal{F}L$ has maximal rank everywhere.

If the local diffeomorphism can be extended to a global diffeomorphism $TM \cong T^*M$, then the Lagrangian is called *hyperregular*. This allows for direct matching of Lagrangian mechanics on TM with Hamiltonian (symplectic) mechanics on T^*M. For example, the Lagrangian L corresponds to the Hamiltonian H as (net) total energy.

One of the most useful and essential ideas of Hamilton-Jacobi theory is that of a complete solution.

{glhjcs} **Definition 6.4.12** Consider a family $\{V_\lambda\}$ of solutions to Problem 6.4.4 depending on an additional parameter $\lambda \in \Lambda$ such that $\Phi : M \times \Lambda \to TM : (p, v) \mapsto V_\lambda(p)$ is a local diffeomorphism. We call $\{V_\lambda\}$ a *complete solution.*

Here Λ is a smooth n-manifold. It follows that a complete solution is a foliation of TM transverse to the fibers such that S_L is tangent to the leaves.

Remark 6.4.13 This can be further generalized to a bundle $E \to \Lambda$ with fiber dimension n.

If $\{V_\lambda\}$ is a complete solution and $(p_0, v_0) \in \operatorname{im} V$, then there exists a $\lambda_0 \in \Lambda$ such that $V_{\lambda_0}(p_0) = v_0$, and the velocity lift of the integral curve of V_{λ_0} through p_0 is the integral curve of S_L through (p_0, v_0). Whence the name "complete solution."

Furthermore, different transverse foliations (complete solutions) are merely different ways to smoothly collect all solutions such that they project to M coherently: integral curves of S_L in $\operatorname{im} V_\lambda$ project to integral curves of V_λ in M. The precise relation between S_L and complete solutions is described as follows.

1. If we have a family of n first integrals $\{f_1, \ldots, f_n\}$ of S_L such that $df_1 \circ \mathcal{V} \wedge \cdots \wedge df_n \circ \mathcal{V} \neq 0$, then $f_i = c_i$ define a transverse foliation locally, where the c_i are constants.

2. If $\Phi : M \times \Lambda \to TM$ is a complete solution, then the local surjections defining the foliation provide such families of first integrals locally.

3. If the foliation is ω_L-Lagrangian, then the first integrals are in involution locally; compare Theorem 5.4.10.

I used Besse [6] as a guide to the mathematical underpinnings and Cariñena *et al.* [12] as a guide to the overlying mechanics. The theory presented in this section has been further extended to certain (unconstrained) singular Lagrangians [12]: namely, those for which a SODE S_L exists globally on TM and ω_L, equivalently $\mathcal{F}L$, has constant rank.

7 *Lie Groups*

Lie started developing his theory of groups in the winter of 1873–74. One of his main incentives grew from Klein's remark comparing Lie's most recent results in solving differential equations admitting geometric symmetries to developments arising from Galois's work on algebraic equations. The previous geometric research, first by himself and then jointly with Klein, now spurred Lie to do for differential equations what Galois had done for algebraic equations: when there are enough symmetries of the right kind, they allow the equation to be solved explicitly just from the knowledge of their existence and structure as a group. Thus Lie devoted almost all of his efforts to his new theory of continuous transformation groups, as they came to be known. Lie's *finite* and *infinite* groups became our finite- and infinite-*dimensional* Lie groups, the spread of the concept of cardinality having changed the usage of these two words.

A common fable avers that when Klein and Lie realized the importance of transformation groups, they divided the world (without papal assistance) between them. Klein took discontinuous groups, as we might now say, and Lie took continuous groups. The former grew into automorphic functions and Langlands theory, the latter into symmetry and transformation group theory, and both diffused far beyond their purely mathematical origin into almost all corners of the sciences and other areas of applications. After remaining nearly separated for over a century, they were reunited recently in theoretical physics.

The original motivation from Galois theory was not seriously pursued until the middle of the twentieth century, when Kolchin almost single-handedly achieved that goal through his theory of differential algebra. Since then, it seems to have again fallen out of the main stream.

This chapter reflects the motivation for Lie theory in transformation groups and expounds the basics of Lie groups and algebras from that perspective. Of necessity, it thus follows Tondeur [77] to a considerable extent, more so initially than later when the differences become most apparent.

7.1 Lie groups

In order to obtain some simple statements about Lie groups, we replace continuous maps with smooth maps, and topological spaces with smooth manifolds, in many of the statements about topological groups.

{lgrp} **Definition 7.1.1** A *Lie group* G is a group with a differential structure such that $G \times G \to G$ given by $(g, h) \mapsto gh$ and $G \to G$ given by $g \mapsto g^{-1}$ are smooth.

Ex 7.1.2 Equivalently, $(g, h) \mapsto gh^{-1}$ is smooth.

{lgx1} **Example 7.1.3** The following are Lie groups:

1. any discrete topological group is a Lie group of dimension 0;
2. the additive group $\mathbb{R}^n$;
3. the unit circle $S^1 \subseteq \mathbb{C}$ with complex multiplication as the binary operation;
4. the n-dimensional torus $\mathbb{T}^n := S^1 \times \cdots \times S^1$ with n factors;
5. the (real) general linear group GL_n.

{lgx2} **Ex 7.1.4** If G is a Lie group and TG is the tangent bundle of G, then TG is a Lie group (because T preserves products); *cf.* Example 3.6.10.

{lgx3} **Ex 7.1.5** Prove that the product of Lie groups is a Lie group.

{lgmorph} **Definition 7.1.6** A *morphism* of Lie groups is a smooth morphism of groups.

Ex 7.1.7 Prove $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$.

{lgx4} **Ex 7.1.8** Let V be a finite-dimensional real vector space. Then V and $GL(V)$ are Lie groups.

{lgx5} **Example 7.1.9** Let G be a Lie group and TG the tangent bundle with the Lie group structure of Ex 7.1.4. Let T_1G be the tangent space at the identity element and $j : T_1G \hookrightarrow TG$ the natural inclusion. Since T_1G is a vector space and a Lie group as well (with vector space addition), j is a morphism of Lie groups. The natural projection $\pi : TG \to G$ is also a morphism of Lie groups. Now $0 \to T_1G \hookrightarrow TG \twoheadrightarrow G \to 1$ is a short exact sequence split by $s : G \to TG : g \mapsto (g, 0)$. This means that $TG \cong T_1G \ltimes_\theta G$ for some suitable $\theta : G \to \mathrm{Aut}_{\mathsf{Vec}}\, T_1G$.

Ex 7.1.10 Give θ explicitly. Apply to Example 3.6.10.

Ex 7.1.11 Example 7.1.9 poses a question: does it imply that TG is a trivial bundle?

Ex 7.1.12 Continuing from Ex 1.3.1.19, the identity component G_o of G is an open, closed, and normal subgroup.

Definition 7.1.13 A *G-manifold* M is a manifold and a G-set for a smooth action $\tau : G \times M \to M$. The pair (G, M) is called a *Lie transformation group* (or differential transformation group). {ltg}

In general, $\mathrm{Aut}_{\mathsf{Mfld}}\, M$ does not have any reasonable smooth structure making it an infinite-dimensional Lie group. One can still use this viewpoint for heuristics—but only with some care.

On the other hand, one may Kelleyfy $\mathrm{Aut}_{\mathsf{Mfld}}\, M$ with the Schwartz (weak C^∞) topology. Then $\tau : G \to k(\mathrm{Aut}_{\mathsf{Mfld}}\, M)$ is continuous in cgH as in Definition 1.3.1.3 and the comments after it.

Definition 7.1.14 Let M be a G-manifold, M' a G'-manifold, $f : M \to M'$ smooth, and $\rho \in \hom(G, G')$. Then f is *ρ-equivariant* if and only if the following diagram commutes. {leqv}

$$\begin{array}{ccc} G \times M & \xrightarrow{\ \tau\ } & M \\ {\scriptstyle \rho\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G' \times M' & \xrightarrow[\ \tau'\ ]{} & M' \end{array}$$

Example 7.1.15 An $\mathbb{R}$-manifold is called a 1-*parameter group (of transformations).* From the discussion in Section 4.4, p. 82, a 1-parameter group is a global flow. {1pargrp}

Example 7.1.16 A Lie group G acting on itself by left (right) translation or by conjugation makes G a G-manifold. {ltgx1}

Example 7.1.17 Let $V \in \mathsf{Vec}$ and $GL(V)$ be the general linear group of V. Then V and $GL(V)$ are both Lie groups. An action $\tau : G \to GL(V)$ is called a *linear representation* of G in $GL(V)$ (or on V). {ltgx2}

Example 7.1.18 Let M be a G-manifold. Then TM is a TG-manifold as T preserves products, hence a G-manifold as G can be regarded as a subgroup of TG from Example 7.1.9. {ltgx3}

Ex 7.1.19 Let G and G' be Lie groups and G' be a G-manifold for some $\tau : G \to \operatorname{Aut} G'$. Prove that the semidirect product $G' \ltimes_\tau G$ of Example 1.2.1.10 is a Lie group with the smooth (or differential or analytic) product structure on $G' \times G$. This generalizes Example 7.1.5 where τ was trivial. {ltgx4}

Ex 7.1.20 Ex 7.1.19 answers the question: it and Example 7.1.9 imply that TG is a trivial bundle. Thus all Lie groups are parallelizable, as in Definition 3.7.12.

{ltgx5} **Example 7.1.21** Let $V \in \mathsf{Vec}$. Then $GL(V)$ acts on V in the natural way, say by θ. The affine group of Example 1.2.1.13 $V \ltimes_\theta GL(V)$ is a Lie group by the preceding example.

{ltgx6} **Example 7.1.22** Let G be a Lie group and $0 \to T_1G \to TG \to G \to 1$ be a short exact sequence with splitting s as in Example 7.1.9. Observe that G acts on T_1G via $\tau_g = \kappa_{g*}|_{T_1G}$ with $T_1G \leq TG$ a normal subgroup and $\kappa_{g*} \in \operatorname{Inn} TG$ (see Example 1.2.1.10). This is a linear representation of G on T_1G called the *adjoint representation* and $TG \cong T_1G \ltimes_\tau G$.

7.2 Lie algebras

{sctla}

Recall the general definition of a Lie algebra, Definition 2.6.1.

Consider the $\mathbb{R}$-algebra $\mathfrak{F}$ of a manifold M, the commutator Lie algebra $\operatorname{End}(\mathfrak{F})$ of continuous linear endomorphisms and its Lie subalgebra $\operatorname{Der}(\mathfrak{F})$ of continuous linear derivations. It follows from Ex 4.6.7 and 4.6.9.1 that these constructions are functorial with respect to diffeomorphisms of M. Recall that $(\varphi_* V)_{\varphi(p)} = \varphi_* V_p$ and $(\varphi_* V)f = V(\varphi^* f)$ at $p \in M$ for a diffeomorphism φ, vector field V, and function f.

{Finv} **Theorem 7.2.1** *If M is a G-manifold and $\mathfrak{F}$ its function algebra, then* $\operatorname{End}(\mathfrak{F})$ *is a G-space* via τ_{g*} *which preserves the $\mathbb{R}$-algebra and commutator Lie algebra structures. Moreover, the invariant elements of* $\operatorname{End}(\mathfrak{F})$ *form a subalgebra and a Lie subalgebra, respectively.* □

{Feqv} **Theorem 7.2.2** *Let M, M' be G, G'-manifolds, respectively, $\rho : G \to G'$ a morphism, and $\varphi : M \to M'$ a ρ-equivariant diffeomorphism. Then φ_* :*

$\mathrm{End}(\mathfrak{F}) \to \mathrm{End}(\mathfrak{F}')$ *is a* ρ*-equivariant isomorphism. Moreover,* φ_* *maps* G*-invariant elements of* $\mathrm{End}(\mathfrak{F})$ *to* $\rho(G)$*-invariant elements of* $\mathrm{End}(\mathfrak{F}')$. □

{Xinv} **Corollary 7.2.3** *If* M *is a* G*-manifold and* $\mathfrak{X}$ *its Lie algebra of vector fields, then* $\mathfrak{X}$ *is a* G*-space* via τ_{g*} *which preserves the Lie algebra structure. Moreover, the invariant elements of* $\mathfrak{X}$ *form a Lie subalgebra.* □

Corollary 7.2.4 *Let* M, M' *be* G, G'*-manifolds, respectively,* $\rho : G \to G'$ *a morphism, and* $\varphi : M \to M'$ *a* ρ*-equivariant diffeomorphism. Then* $\varphi_* : \mathfrak{X} \to \mathfrak{X}'$ *is a* ρ*-equivariant isomorphism. Moreover,* φ_* *maps* G*-invariant elements of* $\mathfrak{X}$ *to* $\rho(G)$*-invariant elements of* $\mathfrak{X}'$. □ {Xeqv}

Now consider G acting on itself by left translation. Note that each left translation L_g is a diffeomorphism $G \to G$ and take $\rho = 1_G$; then apply the two preceding Corollaries.

Corollary 7.2.5 *Left-invariant vector fields on a Lie group* G *form a Lie algebra* $\mathfrak{g}$. □

Obviously the parallel result holds for right-invariant vector fields. In addition, one may replace "vector fields" with "derivations on $\mathfrak{F}(G)$."

Definition 7.2.6 Let G be a Lie group. The *Lie algebra* of G is the set of all left-invariant vector fields on G denoted by $\mathcal{L}G = \mathfrak{g}$. {lalg}

It is traditional to denote Lie groups by upper-case Latin letters and their Lie algebras by the corresponding lower-case Fraktur letters. Since the correspondence is functorial, it is convenient to have an explicit functor denoted by $\mathcal{L}$ as well.

Theorem 7.2.7 *Let* G *be a Lie group,* $\mathcal{L}G$ *its Lie algebra, and* $v \in T_1G$*. Then there exists a unique* $V \in \mathcal{L}G$ *such that* $V(1) = v$. {lvft1g}

Proof: Suppose that such a V exists. Then $V(g) = L_{g*}v$. This establishes uniqueness.

For existence we only need to show that V so defined is smooth. Let $\gamma : I \to G$ with $\dot\gamma(0) = v$. Then for $f \in \mathfrak{F}(G)$,

$$\left(V\left(L_g^* f\right)\right)(1) = \frac{d}{dt}(L_g^* f)\left(\gamma(t)\right)\Big|_{t=0} = \frac{d}{dt} f\left(g\gamma(t)\right)\Big|_{t=0}$$

which is smooth since f is. □

This defines a bijection $\mathcal{B} : T_1 G \to \mathcal{L}G = \mathfrak{g}$. It is easy to verify that it is linear, thus a vector space isomorphism. We now use it to transport the Lie algebra structure from $\mathcal{L}G$ to $T_1 G$. Thus from now on, we may regard the Lie algebra $\mathfrak{g}$ as whichever of $\mathcal{L}G$ or $T_1 G$ is more convenient at the moment. Note we also have shown $\dim G = \dim \mathfrak{g}$.

Ex 7.2.8 In fact, this isomorphism is natural.

{rla} **Remark 7.2.9** One may replace $\mathcal{L}G$ with $\mathcal{R}G$, the right-invariant vector fields, and obtain a quite similar isomorphism, also natural. It then follows that $\mathcal{L}G \cong \mathcal{R}G$ naturally.

{lar} **Example 7.2.10** Consider $\mathbb{R}$ as a Lie group. Then $T_0\mathbb{R} = \mathbb{R}$ so $\mathcal{L}\mathbb{R} = \mathbb{R}$ and $[x, y] = 0$ for all $x, y \in \mathbb{R}$. Hence $\mathbb{R}$ is an *abelian* Lie group and $\mathcal{L}\mathbb{R}$ is an *abelian* Lie algebra. Similarly, $\mathcal{L}\mathbb{T} = \mathbb{R}$ is an abelian Lie algebra and the torus $\mathbb{T}$ an abelian Lie group.

{tlav} **Example 7.2.11** Let $V \cong \mathbb{R}^n$ and regard $GL(V) \subseteq \mathrm{End}(V)$. Then

$$T_g GL(V) \cong \mathrm{End}(V)$$

for any $g \in GL(V)$. Multiplication in $GL(V)$ is the restriction of that in $\mathrm{End}(V)$. Then $\mathfrak{gl}(V) \cong \mathrm{End}(V)$, the latter with the commutator Lie algebra structure.

Ex 7.2.12 Verify the Lie bracket is correct.

{nfs3} **Example 7.2.13** Let G be a Lie group, $\mathfrak{F} = C^\infty(G)$, $\mathfrak{X} = \Gamma(TG)$, and $\mathfrak{g}$ the Lie algebra of G. Then $\mathfrak{X} \cong \mathfrak{F} \widehat{\otimes} \mathfrak{g}$ as nuclear Fréchet spaces.

In contrast to general endomorphisms of $\mathfrak{F}$, and vector fields in particular (see just after Definition 4.4.4), Lie group morphism-related left-invariant vector fields are quite well behaved.

{laeqv} **Lemma 7.2.14** *Let G and G' be Lie groups, $\varphi : G \to G'$ a morphism, and $V \in \mathcal{L}G$. Then there exists a unique $V' \in \mathcal{L}G'$ so that V and V' are φ-related.*

Proof: Take $\varphi_* : T_1 G \to T_1 G'$ and apply Theorem 7.2.7. Observe that φ-related at one point is equivalent to φ-related globally as V and V' are left-invariant. □

Note that the argument used only φ_{*1}. This suggests {locmor} **Definition 7.2.15** Let G and G' be Lie groups. A *local morphism* $G \to G'$ is a smooth map $\rho : U \to G'$, where U is an open neighborhood of the identity in G, such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ whenever both sides are well defined.

Any (global) morphism $G \to G'$ can be restricted to such a U. If we identify local morphisms that have the same (co)restrictions, then we may compose local morphisms to obtain the category $\mathsf{LGrp}_{\mathrm{loc}}$ with $\hom_{\mathrm{loc}}(G, G')$. It follows that G and G' are *locally isomorphic* if and only if there exist open neighborhoods U and U' of 1 and $1'$, respectively, and a diffeomorphism $\rho : U \to U'$, such that both ρ and ρ^{-1} are local morphisms.

Remark 7.2.16 An object of $\mathsf{LGrp}_{\mathrm{loc}}$ is actually the equivalence class of tails of the neighborhood filter of $1 \in G$: a *germ* of a Lie group. This should come as no surprise since 1) Lie groups are topological groups and Ex 1.3.1.18 applies; and 2) differentially, Lie groups are analytic. {lggerm}

Theorem 7.2.17 *Let G and G' be Lie groups and $\rho : U \to G'$ a local morphism. Then $\mathcal{L}\rho(V) := \rho_* V$ at $T_1 G$, then on all of G by left-invariance and Theorem 7.2.7, defines a Lie algebra morphism $\mathcal{L}\rho : \mathcal{L}G \to \mathcal{L}G'$ such that the diagram commutes.* {lglaloc}

$$\begin{array}{ccc} T_1 G & \xrightarrow{\rho_*} & T_1 G' \\ \downarrow & & \downarrow \\ \mathcal{L}G & \xrightarrow[\mathcal{L}\rho]{} & \mathcal{L}G' \end{array}$$

where the vertical arrows are the isomorphism $\mathcal{B}$ after Theorem 7.2.7. □

Corollary 7.2.18 *The maps $G \mapsto \mathcal{L}G$ and $\rho \mapsto \mathcal{L}\rho$ yield a (covariant) functor $\mathcal{L}$ from LGrp_{loc} to LAlg.* □ {lfct}

Also, $G \mapsto T_1 G$ and $\rho \mapsto \rho_{*1}$ is a functor. Theorem 7.2.17 implies that the isomorphism $\mathcal{B}$ after Theorem 7.2.7 is a natural isomorphism of these functors.

Corollary 7.2.19 *The Lie algebras of locally isomorphic Lie groups are isomorphic.* □ {isoloc}

Example 7.2.20 The projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a *local* isomorphism; therefore $\mathcal{L}\mathbb{R} \cong \mathcal{L}\mathbb{T}$, where we note that $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$. Conversely, if $\rho : \mathbb{T} \to \mathbb{R}$ is a morphism then $\rho(\mathbb{T})$ is compact. If there existed $t \in \mathbb{T}$ with $\rho(t) \neq 0$, then the Archimedean Principle would imply that $k\rho(t) \notin \rho(\mathbb{T})$ for some positive integer k. Therefore the only *global* morphism from $\mathbb{T} \to \mathbb{R}$ is the trivial one. {lfctx1}

Example 7.2.21 Let V be a finite-dimensional vector space with $\tau : G \to GL(V)$ a morphism, and recall that $\mathcal{L}GL(V) \cong \mathrm{End}(V) \cong \mathfrak{gl}(V)$. As τ is smooth, it induces $\mathcal{L}\tau : \mathfrak{g} \to \mathfrak{gl}(V)$. Thus a representation of a Lie group on V induces a representation of the Lie algebra in $\mathrm{End}(V) \cong \mathfrak{gl}(V)$. {lfctx2}

{tr} **Proposition 7.2.22** *The morphism* $\mathfrak{gl}(V) \to \mathbb{R}$ *induced by* $\det : GL(V) \to \mathbb{R}^* = \mathbb{R} - \{0\}$ *is the trace.*

Proof: Let $A \in \mathrm{End}(V)$ and γ a curve in $GL(V)$ with $\gamma(0) = 1$ and $\dot\gamma(0) = A$. Then

$$\det{}_* A = \frac{d}{dt}\big[\det\gamma(t)\big]_{t=0}.$$

Now, for any nondegenerate n-form ω on V and any $(v_1, \ldots, v_n) \in V^n$ we get

$$\omega(v_1, \ldots, v_n) \cdot \det{}_*(\gamma(t)) = \omega(\gamma(t)v_1, \ldots, \gamma(t)v_n)$$

whence, subscripting the argument of γ, we compute

$$\begin{aligned}
\omega(v_1, \ldots, v_n) \cdot \det{}_* A &= \frac{d}{dt}\big[\omega(\gamma_t v_1, \ldots, \gamma_t v_n)\big]_{t=0} \\
&= \sum_{i=1}^{n} \omega(\gamma_t v_1, \ldots, \gamma_t v_{i-1}, \dot\gamma_t v_i, \gamma_t v_{i+1}, \ldots, \gamma_t v_n)\big|_{t=0} \\
&= \sum_{i=1}^{n} \omega(v_1, \ldots, A v_i, \ldots, v_n) \\
&= \omega(v_1, \ldots, v_n) \cdot \mathrm{tr}\, A.
\end{aligned}$$

Thus we obtain $\mathcal{L}(\det)A = \mathrm{tr}\, A$. □

{trx} **Example 7.2.23** For $A, B \in \mathrm{End}(V)$, we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Note that $\mathrm{tr} : \mathrm{End}(V) \to \mathbb{R}$ is a Lie algebra morphism, thus linear. Therefore $\mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB - BA) = \mathrm{tr}[A, B] = [\mathrm{tr}\, A, \mathrm{tr}\, B] = 0$ and hence $tr(AB) = tr(BA)$.

{lap} **Definition 7.2.24** Let K be a commutative ring with identity (as usual) and L_1, L_2 be K-Lie algebras. Then $L_1 \oplus L_2$ with the componentwise bracket

$$[(V_1, V_2), (V_1', V_2')] = ([V_1, V_1'], [V_2, V_2'])$$

is the *product* K-Lie algebra.

{lapg} **Proposition 7.2.25** *Let* G, G' *be two Lie groups. Then* $\mathcal{L}(G \times G') \cong \mathcal{L}G \oplus \mathcal{L}G'$ *naturally.* □

{larn} **Example 7.2.26** We have $\mathcal{L}\mathbb{R}^n \cong \mathcal{L}\mathbb{R} \oplus \cdots \oplus \mathcal{L}\mathbb{R} \cong \mathbb{R}^n$ (n summands) with trivial bracket. Similarly, $\mathcal{L}\mathbb{T}^n \cong \mathbb{R}^n$. Note that these are connected abelian groups. Compare with what you now know about Example 3.6.10.

Let G be a Lie group, G^{op} the opposite (Lie) group, and $\mathcal{I} : G \to G^{op}$ defined by $g \mapsto g^{-1}$ (an anti-isomorphism; a cofunctor). Note $\mathcal{I}$ is a diffeomorphism.

Lemma 7.2.27 *We have* $\mathcal{I}_{*1} = -1_{T_1 G}$. {t1gop}

Proof: Consider $\varphi : G \to G$ by $g \mapsto gg^{-1}$. Then $\varphi_* = 0$ since φ is constant. But $\varphi_{*g} = (R_{g^{-1}})_{*g} + (L_g)_{*g^{-1}} \mathcal{I}_{*g}$, so

$$\begin{aligned} \mathcal{I}_{*g} &= -(L_g)^{-1}_{*g^{-1}} \circ (R_{g^{-1}})_{*g} \\ &= -(L_{g^{-1}})_{*1} \circ (R_{g^{-1}})_{*g} \,. \end{aligned}$$

Setting $g = 1$ the result follows. □

Definition 7.2.28 Let L be a K-Lie algebra. Define L^{op} by $[A, A']^{op} = [A', A]$. {laop}

Proposition 7.2.29 *Let G a Lie group and G^{op} the opposite group. Recall that we may regard $\mathcal{L}G = T_1 G$ and $\mathcal{L}(G^{op}) = T_1 G^{op}$. Then $\mathcal{L}(G^{op}) = (\mathcal{L}G)^{op}$.* {lagop}

Proof: Observe that $\mathcal{L}(\mathcal{I}) = \mathcal{I}_{*1}$ and for $V \in \mathcal{L}G$ we have $\mathcal{I}_* V = -V$ in $\mathcal{L}(G^{op})$. For $[V, V']$ we have $[V', V] = -[V, V']$ in $(\mathcal{L}G)^{op}$ and we also have $[-V, -V'] = -[V, V'] \in \mathcal{L}(G^{op})$ by linearity. So, $[V, V']_{(\mathcal{L}G)^{op}} = [V, V']_{\mathcal{L}(G^{op})}$. □

Corollary 7.2.30 *Let $\mathcal{R}G, \mathcal{L}G$ be the Lie algebras of right-/left-invariant vector fields, respectively. Using the natural isomorphism $\mathcal{B}$ after Theorem 7.2.7 and its dual, then $\mathcal{R}G \cong (\mathcal{L}G)^{op}$.* {rglgop}

Proof: Observe that left translations of G are right translations of G^{op} and *vice versa,* so that $\mathcal{L}G \cong \mathcal{R}G^{op}$ and $\mathcal{R}G \cong \mathcal{L}G^{op}$. By Proposition 7.2.29, $\mathcal{L}(G^{op}) = (\mathcal{L}G)^{op}$ whence $\mathcal{R}G \cong (\mathcal{L}G)^{op}$. □

Corollary 7.2.31 *If G is abelian, so is $\mathfrak{g}$.* □ {ala}

7.3 1-parameter subgroups

Recall from Example 7.1.15 that an $\mathbb{R}$-manifold is called a 1-parameter group of transformations. Also, from the discussion in Section 4.4, p. 82, a 1-parameter group is a global flow Φ with associated vector field $\dot{\Phi}$. This fundamental relationship is why vector fields were classically called *infinitesimal transformations*.

{1pgeqvvf} **Proposition 7.3.1** *Suppose Φ and Ψ are 1-parameter groups of M and N inducing vector fields V and U respectively, and let $\varphi : M \to N$ be a smooth map. If $\varphi \circ \Phi_t = \Psi_t \circ \varphi$ for all t, then V and U are φ-related; and conversely.*

Proof: First, differentiate with respect to t at $p \in M$:

$$\begin{aligned} \varphi_* V_{\Phi_t(p)} &= U_{\Psi_t(\varphi(p))}\,, \quad \text{or} \\ \varphi_* V_{\Phi_t(p)} &= U_{\varphi(\Phi_t(p))}\,; \end{aligned}$$

hence V and U are φ-related.

On the other hand, let $p \in M$ and define $\gamma_1(t) := \varphi\big(\Phi_t(p)\big)$ and $\gamma_2(t) := \Psi_t\big(\varphi(p)\big)$. Notice that $\gamma_1(0) = \gamma_2(0) = \varphi(p)$. Now, we have

$$\dot{\gamma}_1(t) = \varphi_* V_{\Phi_t(p)} = U_{\gamma_1(t)}$$

which follows from Corollary 4.4.6 and the assumption that V and U are φ-related. Moreover, we see that

$$\dot{\gamma}_2(t) = U_{\gamma_2(t)}$$

Thus γ_1 and γ_2 are solutions of the same initial value problem, hence must be equal: $\gamma_1 = \gamma_2$. □

{df1pg} **Corollary 7.3.2** *Let $\varphi : M \to M'$ be a diffeomorphism. If $V \in \mathfrak{X}(M)$ with local flow Φ_t, then $\varphi_* V \in \mathfrak{X}(M')$ has $\varphi \circ \Phi_t \circ \varphi^{-1}$ as its local flow.* □

This is immediate as V and $\varphi_* V$ are φ-related.

{aut1pg} **Corollary 7.3.3** *Let φ be a smooth automorphism of M, and $V \in \mathfrak{X}(M)$ with local flow Φ_t. Then $\varphi_* V = V$ if and only if $\varphi \circ \Phi_t = \Phi_t \circ \varphi$ for all t.* □

Let G be a Lie group, U be an open neighborhood of $1 \in G$, $\varphi : U \to \varphi(U)$ be a smooth local automorphism of G, and $V \in \mathfrak{X}(M)$. Then $\varphi_* V$ is a vector field on $\varphi(U)$, as defined in Section 4.6.

Now consider the local flow Φ_t of V as a local automorphism of M. As $\Phi_s \circ \Phi_t = \Phi_t \circ \Phi_s$ for all t, it follows from the preceding corollary that $\Phi_{s*} V = V$. This means that the velocity field $\dot{\Phi}$ of the local flow is invariant along the flow: Φ is a *stationary* local flow.

{lvf1pg} **Lemma 7.3.4** *Suppose G is a Lie group, and $V \in \mathcal{L}G$ with local flow Φ. Then $L_g \circ \Phi_t = \Phi_t \circ L_g$ for all $g \in G$ and all t for which Φ_t makes sense.* □

Theorem 7.3.5 *Every left-invariant vector field on a Lie group is complete.* {lvfc}

Proof: Let Ψ_t be the local 1-parameter group of $V \in \mathcal{L}G$. Suppose that $\Psi : I_\varepsilon \times U \to G$ for some $\varepsilon > 0$ and some open neighborhood U of $1 \in G$. Any extension $\widetilde{\Psi} : I_\varepsilon \times G \to G$ of Ψ must satisfy

$$\widetilde{\Psi}_t(g) = \big(\widetilde{\Psi}_t \circ L_g\big)(1) = \big(L_g \circ \widetilde{\Psi}_t\big)(1) = L_g\big(\widetilde{\Psi}_t(1)\big)$$

by the preceding lemma. Define $\widetilde{\Psi}$ by this formula, and the result follows from Lemma 4.4.10. □

Lemma 7.3.6 *For any group G, $\psi : G \to G$ is a right translation if and only if $L_g \circ \psi = \psi \circ L_g$ for all $g \in G$. Moreover, $\psi = R_{\psi(1)}$.* {rtch}

Proof: The condition is necessary by associativity. For the converse, suppose $L_g \circ \psi = \psi \circ L_g$ for all $g \in G$, and fix $g' \in G$. We have $g\psi(g') = \psi(gg')$, whence $g\psi(1) = \psi(g)$, or $\psi(g) = R_{\psi(1)}(g)$. □

Applying Lemma 7.3.4 yields

Proposition 7.3.7 *If G is a Lie group and $V \in \mathcal{L}G$ with local 1-parameter group Ψ_t, then $\Psi_t = R_{\Psi_t(1)}$.* □ {1pgrt}

Definition 7.3.8 A *1-parameter subgroup* of a Lie group G is a morphism $\gamma : \mathbb{R} \to G$ of Lie groups. {1ps}

We usually do not consider the trivial 1-parameter subgroup $1 \le G$ in the sequel.

The projection $\mathbb{R} \twoheadrightarrow \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ is a 1-parameter subgroup that is not injective. Also, if $V \in \mathfrak{X}(G)$ is complete with global flow Φ_t, set $\gamma(t) = \Phi_t(1)$. Then $\gamma : \mathbb{R} \to G$ and $\gamma(0) = \Phi_0(1) = 1$, but γ is not necessarily a 1-parameter subgroup.

Ex 7.3.9 Provide an explicit example.

Proposition 7.3.10 *Let G be a Lie group, $V \in \mathcal{L}G$ with global flow Φ, and $\gamma : \mathbb{R} \to G : t \mapsto \Phi_t(1)$. Then γ is a 1-parameter subgroup, $\Phi_t = R_{\gamma(t)}$, and Φ_t is (completely) characterized by γ.* {1pslvf}

Proof: Applying Lemma 7.3.4 to $\gamma(t_1 + t_2)$, we obtain

$$\begin{aligned}
\gamma(t_1 + t_2) &= \Phi_{t_1+t_2}(1) \\
&= \Phi_{t_1}\big(\Phi_{t_2}(1)\big) \\
&= \big(\Phi_{t_1} \circ L_{\Phi_{t_2}(1)}\big)(1) \\
&= \big(L_{\Phi_{t_2}(1)} \circ \Phi_{t_1}\big)(1) \\
&= \Phi_{t_2}(1)\Phi_{t_1}(1) \\
&= \gamma(t_2)\gamma(t_1)\,.
\end{aligned}$$

By the preceding proposition, this implies that $\Phi_t = R_{\gamma(t)}$. □

As noted in [77], the fact that $\Phi_t = R_{\gamma(t)}$ is often paraphrased in the literature by "the infinitesimal transformation generated by a left-invariant vector field is a right translation."

Given a Lie group G, we denote the set of all its 1-parameter subgroups by $\mathbb{L}G$. Our next endeavor is to show that $\mathbb{L}G \cong \mathcal{L}G$ as vector spaces. Thus we shall define a map $\mathscr{L} : \mathcal{L}G \to \mathbb{L}G$.

The fundamental existence and uniqueness theorem for differential equations in Section 4.4 immediately yields the following.

{1psde} **Lemma 7.3.11** *Let $V \in \mathcal{L}G$ and $\gamma = \mathscr{L}(V)$. Then γ is the unique solution of the initial value problem $\dot\gamma(t) = V_{\gamma(t)}$ with $\gamma(0) = 1$.* □

{1psltrt} **Lemma 7.3.12** *If $\gamma \in \mathbb{L}G$, then $\dot\gamma(t) = \big(L_{\gamma(t)}\big)_*\dot\gamma(0) = \big(R_{\gamma(t)}\big)_*\dot\gamma(0)$.*

Proof: Since γ is a group morphism and $\mathbb{R}$ is commutative, we have the following identity.

$$\gamma(t+s) = \gamma(t)\gamma(s) = \gamma(s)\gamma(t)$$

Differentiate to obtain

$$\begin{aligned}
\dot\gamma(t+s) &= \big(L_{\gamma(t)}\big)_{*\gamma(s)}\dot\gamma(s) \\
&= \big(R_{\gamma(t)}\big)_{*\gamma(s)}\dot\gamma(s)\,.
\end{aligned}$$

The result follows by evaluating at $s = 0$. □

{lvf1ps} **Theorem 7.3.13** *Let G be a Lie group. For any $V \in \mathcal{L}G$, define $\mathscr{L}(V) :=$ $\gamma \in \mathbb{L}G$ where $\dot\gamma(t) = V_{\gamma(t)}$ with initial condition $\gamma(0) = 1$. Then $\mathscr{L}$ is an isomorphism of vector spaces.*

Proof: Consider $\gamma \in \mathbb{L}G$. If $\gamma = \mathscr{L}(V)$ for some $V \in \mathcal{L}G$, then $V_1 = \dot\gamma(0)$. Therefore $\mathscr{L}$ is injective; or use Lemma 7.3.11.

If we define V to be the left-invariant vector field such that $V_1 = \dot\gamma(0)$, then $V_{\gamma(t)} = \big(L_{\gamma(t)}\big)_* \dot\gamma(0) = \dot\gamma(t)$ by the preceding lemma. Thus $\mathscr{L}$ is surjective; hence bijective. It follows readily that $\mathscr{L}$ is linear. □

Proposition 7.3.14 *If $V \in \mathcal{L}G$ and $\gamma = \mathscr{L}(V)$, then for any $g \in G$, $V_g =$* {1psrt}
$\frac{d}{dt}\big[g\gamma(t)\big]$.

Proof: Let Φ be the flow of V so that $\dot\Phi_t(g) = V_{\Phi_t(g)}$ for all $g \in G$. Then by Proposition 7.3.10,

$$\Phi_t(g) = R_{\gamma(t)}g = g\gamma(t), \text{ and } \dot\Phi_t(g) = \frac{d}{dt}\big[g\gamma(t)\big].$$

Evaluating at $t = 0$ yields $V_{\Phi_0(t)} = V_g = \frac{d}{dt}\big[g\gamma(t)\big]_{t=0}$. □

Proposition 7.3.15 *Let I be an open interval containing 0 and $\gamma : I \to G$ a* {1psloc}
local morphism. Then there exists a unique 1-parameter subgroup $\widetilde\gamma : \mathbb{R} \to G$ such that $\widetilde\gamma\big|_I = \gamma$.

Proof: Define $V \in \mathcal{L}G$ by $\dot\gamma(0) = V_1$ and $\widetilde\gamma = \mathscr{L}(V)$. Then $\dot\gamma(t) = \big(L_{\gamma(t)}\big)_*\dot\gamma(0)$, so that $V_{\gamma(t)} = \big(L_{\gamma(t)}\big)_*\dot\gamma(0) = \dot\gamma(t)$. But $\widetilde\gamma$ is also a solution of the same initial value problem. Thus, the result follows from the FEUT (p. 81f). □

As shown by Example 7.2.20 of the local isomorphism $\mathbb{T} \to \mathbb{R}$, there does not necessarily exist an extension of a local morphism $G \to G'$ to a (global) morphism $G \to G'$. As in the topological case, Theorem 1.3.1.20, it will turn out that G being 1-connected is sufficient.

Let $\rho : G \to G'$ be a morphism of Lie groups. For any $\gamma \in \mathbb{L}G$, the composition $\rho\gamma$ is in $\mathbb{L}G'$. If ρ is merely a local morphism, then $\rho\gamma$ is *a priori* also only defined locally. However, Proposition 7.3.15 tells us that $\rho\gamma$ can be extended uniquely to $\widetilde{\rho\gamma} : \mathbb{R} \to G'$. The map $\mathbb{L}(\rho) : \mathbb{L}G \to \mathbb{L}G'$ so defined is compatible with the map $\mathscr{L}$ of Theorem 7.3.13. This idea can be made more precise.

Theorem 7.3.16 *If G and G' are Lie groups, and $\rho : G \to G'$ is a morphism, then the following diagram commutes,* {LLnt}

$$\begin{array}{ccc} \mathcal{L}G & \xrightarrow{\mathcal{L}(\rho)} & \mathcal{L}G' \\ \big\downarrow{\scriptstyle \mathscr{L}} & & \big\downarrow{\scriptstyle \mathscr{L}'} \\ \mathbb{L}G & \xrightarrow[\mathbb{L}(\rho)]{} & \mathbb{L}G' \end{array}$$

where $\mathbb{L}(\rho)$ is left composition by ρ.

Proof: If $\gamma \in \mathbb{L}G$, then $\frac{d}{dt}\big[\rho \circ \gamma\big]_{t=0} = \rho_* \dot\gamma(0)$. Thus the following diagram commutes

$$\begin{array}{ccc} T_1G & \xrightarrow{\rho_*} & T_1G' \\ \big\downarrow{\scriptstyle \mathcal{B}} & & \big\downarrow{\scriptstyle \mathcal{B}'} \\ \mathcal{L}G & \xrightarrow[\mathcal{L}(\rho)]{} & \mathcal{L}G' \\ \big\downarrow{\scriptstyle \mathscr{L}} & & \big\downarrow{\scriptstyle \mathscr{L}'} \\ \mathbb{L}G & \xrightarrow[\mathbb{L}(\rho)]{} & \mathbb{L}G' \end{array}$$

where $\mathcal{B}$ and $\mathcal{B}'$ are the isomorphisms after Theorem 7.2.7. □

Introducing the forgetful functor $\mathcal{U} : \mathsf{LAlg} \to \mathsf{Set}$, Theorem 7.3.16 states that $\mathscr{L} : \mathcal{U} \circ \mathcal{L} \mapsto \mathbb{L}$ is a natural isomorphism of sets. We shall see later that $\mathscr{L}$ extends to a functor on LAlg with a suitable interpretation.

7.4 Exponential map

{exp} **Definition 7.4.1** The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by

$$\exp V = \gamma(1)$$

where $V \in \mathcal{L}G$ and $\gamma = \mathscr{L}(V)$, or by

$$\exp v = \gamma(1)$$

where $v \in T_1G$ and $v = V_1$. Note that $v = \dot\gamma(0)$.

For $\gamma \in \mathbb{L}G$ define $\beta(t) = \gamma(ts)$ for all t and fixed s. Then $\beta \in \mathbb{L}G$, and $\dot{\beta} = s\dot{\gamma}(0)$. Indeed, for $f \in \mathfrak{F}(G)$ we compute

$$\begin{aligned} \dot{\beta}(0)f &= \frac{d}{dt}\Big[f\big(\beta(t)\big)\Big]_{t=0} \\ &= \frac{d}{dt}\Big[f\big(\gamma(ts)\big)\Big]_{t=0} \\ &= s\dot{\gamma}(0)f\,. \end{aligned}$$

Therefore $\exp(tV) = \gamma(t)$ is smooth. Furthermore, since γ is a morphism, we have a formula that looks rather familiar:

$$\exp\big((t_1 + t_2)V\big) = \exp(t_1 V)\exp(t_2 V).$$

Example 7.4.2 Let $V \in \mathit{Vec}$ and $G = GL(V)$. By example 7.2.11, $\mathfrak{gl}(V) \cong$ {expx1}
$\mathrm{End}(V)$. For $A \in \mathrm{End}(V)$, let $\gamma = \mathscr{L}(A)$ and define

$$\beta(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

Differentiating with respect to t we obtain

$$\dot{\beta}(t) = \sum_{n=1}^{\infty} \frac{n}{n!}(tA)^{n-1}A = \beta(t)A \text{ and } \beta(0) = 1_V.$$

Therefore β and γ solve the same initial value problem, so $\beta = \gamma$. In particular, if $V = \mathbb{R}$ then $GL(V) = \mathbb{R}^*$ and $\mathcal{L}\mathbb{R}^* = \mathbb{R}$ so that $\exp : \mathbb{R} \to \mathbb{R}^*$ is the usual exponential function.

Proposition 7.4.3 *If $\rho : G \to G'$ is a local morphism of Lie groups, then the* {nexp}
following diagram commutes.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathcal{L}(\rho)} & \mathfrak{g}' \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp'} \\ G & \xrightarrow[\rho]{} & G' \end{array}$$

Proof: If $\gamma(t) = \exp(tv) \in \mathbb{L}G$, then $\exp(t\rho_* v) = \rho\exp(tv)$. When $t = 1$ we obtain $\exp(\rho_* v) = \rho\exp(v)$. □

This result tells us that the exponential map acts naturally on the category of germs of Lie groups. Using Proposition 7.2.22, we immediately obtain

$$\det(\exp A) = \exp(\operatorname{tr} A)$$

for all $A \in \mathfrak{gl}(V)$.

Example 7.4.4 Recall that $SL_2(\mathbb{R})$ is the group of 2×2 real matrices with determinant 1. For now, we assume that this group is connected. We will show that there exists an $A \in SL_2$ that is not a square. This will imply that $\exp : \mathfrak{sl}_2 \to SL_2$ is not a surjection. (Why?) {expx2}

Let $A \in SL_2$. Its characteristic polynomial is $\det(\lambda I - A) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A$. Since $A \in SL_2$, then $\det A = 1$ and the Caley-Hamilton theorem tells us that $A^2 - (\operatorname{tr} A)A + I = 0$. Applying the trace function tr we obtain

$$\operatorname{tr}(A^2) = (\operatorname{tr} A)^2 - 2 \geq -2 \, .$$

Now take $A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$. Clearly $A_0 \in SL_2$. However, $\operatorname{tr} A_0 < -2$ so A_0 is not a square.

Recall that sometimes one can identify $T_0 T_p M$ and $T_p M$ without any confusion occurring. The next result is one of these times.

{exp*0} **Proposition 7.4.5** *The tangent map* $T_0 \exp = \exp_{*0} : T_1 G \to T_1 G$ *induced from* $\exp : T_1 G \to G$ *is the identity map* $1_{T_1 G}$.

Proof: For $v \in T_1 G$ we compute

$$\begin{aligned} \exp_{*0} v &= \exp_{*tv} v \Big|_{t=0} = \exp_{*tv} \frac{d}{dt} [tv] \Big|_{t=0} \\ &= \frac{d}{dt} \big[\exp tv \big]_{t=0} = v \, . \end{aligned}$$

□

A straightforward application of the Inverse Function Theorem then yields

{expdfm} **Corollary 7.4.6** *There exist open neighborhoods* N_0 *in* $T_1 G$ *and* N_1 *in* G *such that* $\exp : N_0 \to N_1$ *is a diffeomorphism.* □

{log} **Definition 7.4.7** The inverse map $N_1 \to N_0$ is denoted by log, and implicitly defines a chart at $1 \in G$ called a *canonical chart*.

Suppose $\rho : G \to G'$ is a local morphism, and $(N_1, \log)$ is a canonical chart of G. Naturality of the exponential map implies that for all $g \in N_1$,

$$\rho(g) = \exp\big(\mathcal{L}\rho \circ \log g\big).$$

This means that $\rho|_{N_1}$ is determined by $\mathcal{L}\rho$. The following proposition is an application of this fact.

Proposition 7.4.8 *Let $\rho_1, \rho_2 : G \to G'$ be local morphisms of Lie groups. If $\mathcal{L}\rho_1 = \mathcal{L}\rho_2$ as morphisms from $\mathcal{L}G$ to $\mathcal{L}G'$, then there exists an open neighborhood U of $1 \in G$ with $\rho_1|_U = \rho_2|_U$.* {coin}

Proof: Take $U = N_1$ in a canonical chart. The formula then shows that $\rho_1 g = \rho_2 g$ for all $g \in U$. □

This corollary tells us that the functor $\mathcal{L} : \mathit{LGrp} \to \mathit{LAlg}$ is faithful on connected Lie groups (or germs of Lie groups) and their morphisms.

Corollary 7.4.9 *Let G be a connected Lie group with $\rho_1, \rho_2 : G \to G'$ Lie group morphisms. Then $\mathcal{L}\rho_1 = \mathcal{L}\rho_2$ implies that $\rho_1 = \rho_2$.* □ {ccoin}

Proposition 7.4.10 *Let G be a Lie group and $\gamma : \mathbb{R} \to G$ a continuous morphism (*i.e.*, a morphism of topological groups). Then there exists $v \in T_1 G$ such that $\gamma(t) = \exp tv$, whence $\gamma \in \mathbb{L}G$.* {cis1}

Proof: Let $(U, \log)$ be a canonical chart at $1 \in G$. We may assume without loss of generality that $U^2 \subseteq U$. If $g \in U$ then $g^2 \in U$, so that $\log g$ and $\log g^2$ are defined.

Consider the 1-parameter group $t \mapsto \exp\big(t \log g\big)$. When $t = 1$, we have $\exp\big(\log g\big) = g$. Now g^2 is on this 1-parameter subgroup, and $g^2 = \exp\big(2 \log g\big)$. On the other hand, $g^2 = \exp\big(\log g^2\big)$ as $g^2 \in U$, so we obtain $\log g^2 = 2 \log g$, or $g = \exp\big(\frac{1}{2} \log g^2\big)$. Thus g is uniquely determined by g^2.

Now consider the continuous morphism $\gamma : \mathbb{R} \to G$. There exists an $\varepsilon > 0$ such that $\gamma(t) \in U$ for all $|t| \le \varepsilon$. Rescaling if necessary, we may take $\varepsilon = 1$. Let $\gamma(1) = g \in U$; then $\big[\exp\big(\frac{1}{2} \log g\big)\big]^2 = g \in U$; uniquely so, by the preceding argument.

Thus $\gamma\big(\frac{1}{2}\big) = \exp\big(\frac{1}{2} \log g\big)$. Substituting $\log \gamma(1) = v$, this becomes

$$\log \gamma\left(\frac{1}{2}\right) = \frac{1}{2} v\,.$$

By iteration,

$$\log \gamma\left(\frac{1}{2^n}\right) = \frac{1}{2^n}\, v\,;$$

summing,

$$\log \gamma\left(\frac{p}{2^n}\right) = \frac{p}{2^n}\, v$$

for all $0 \le p \le 2^n$. This shows that $\log \gamma(r) = rv$ for all dyadic rational r in $(0,1)$; hence, by continuity, for all r in $(0,1)$. Therefore $\gamma(t) = \exp(tv)$. □

{t1gb} **Lemma 7.4.11** *For a Lie group G with $T_1G = S_1 \oplus S_2$, the map $\phi : S_1 \oplus S_2 \to G : (v_1, v_2) \mapsto (\exp v_1)(\exp v_2)$ is a local diffeomorphism at $0 \in T_1G$.*

Proof: We need only show that $\phi_{*0} : S_1 \oplus S_2 \to T_1G$ is an isomorphism. Now, $\phi = m \circ \big(\exp|_{S_1} \times \exp|_{S_2}\big)$, where m is the map defining the binary operation of the group G. Therefore

$$\begin{aligned} \phi_{*0}(v_1, v_2) &= m_{*(1,1)}\big(\exp_{*0} v_1, \exp_{*0} v_2\big) \\ &= \exp_{*0} v_1 + \exp_{*0} v_2 \\ &= v_1 + v_2 \end{aligned}$$

since $\exp_{*0} = 1_{T_1G}$ by Proposition 7.4.5. □

Lemma 7.4.12 *Let G and G' be Lie groups and $\rho : G \to G'$ a group morphism. If ρ is smooth at 1, then ρ is smooth everywhere.*

Proof: The result is clear since $\rho \circ L_g = L_{\rho(g)} \circ \rho$. □

{cis2} **Theorem 7.4.13** *If G and G' are Lie groups and $\rho : G \to G'$ is a continuous morphism, then ρ is smooth.*

Proof: Let $v \in T_1G$. The map $t \mapsto \rho\big(\exp tv\big)$ is continuous from $\mathbb{R}$ to G. Hence, there is a $v' \in T_1G$ such that $\rho\big(\exp tv\big) = \exp tv'$; explicitly $v' = \rho_* v$.

Now let $\{v_i\}$ be a basis of T_1G with corresponding $\{v_i'\} \subset T_1G'$ obtained by the previous construction. Then

$$\rho\Big(\prod_{i=1}^{n} \exp t_i v_i\Big) = \prod_{i=1}^{n} \exp t_i v_i'\,.$$

Thus $\phi : \mathbb{R}^n \to G : (t_i, \ldots, t_n) \mapsto \prod_{i=1}^n \exp t_i v_i$ is a local diffeomorphism at 0 by induction from Lemma 7.4.11. Therefore there exists a neighborhood V of $1 \in G$ such that every $g \in V$ may be written as $g = \prod_{i=1}^n \exp t_i v_i$, with each t_i a smooth function of g. Therefore ρ is smooth at 1, hence on G by the preceding lemma. □

As a corollary we obtain that the Lie algebra of a Lie group is completely determined by the underlying topological group.

Corollary 7.4.14 *If $G = G'$ as topological groups, then $G = G'$ as Lie groups.* {tla}

□

The complete explanation constitutes the solution of Hilbert's Fifth Problem [87].

7.5 Subgroups and subalgebras

Proposition 7.5.1 *Given a morphism $\rho : G \to G'$ of Lie groups, $\mathcal{L}\rho$ is injective (surjective) if and only if ρ_{*g} is injective (surjective) for all $g \in G$.* {l(i/s)}

Proof: Since ρ is a morphism, then $\rho(gh) = \rho(g)\rho(h)$. Thus when $h = 1$, we have $\rho_{*g} \circ (L_g)_* = (L_{\rho g})_* \circ \rho_*$ so $\rho_{*g} = (L_{\rho g})_* \circ \rho_* \circ (L_g)_*^{-1}$.

□

Definition 7.5.2 A subgroup H of a Lie group G is a *Lie subgroup* if and only if H is a Lie group and the inclusion $\iota : H \hookrightarrow G$ is smooth. {lsbgp}

In other words, ι is an injective immersion and H is a submanifold of G.

Example 7.5.3 Any submanifold H of G that is also a subgroup is a Lie subgroup. {lsbgpx1}

Example 7.5.4 A 1-parameter subgroup γ of a Lie group is a Lie subgroup if and only if γ is injective. {lsbgpx2}

Definition 7.5.5 Let H be a Lie subgroup of G. The monomorphism $\iota : H \hookrightarrow G$ induces a monomorphism of Lie algebras $\mathcal{L}\iota : \mathcal{L}H \hookrightarrow \mathcal{L}G$. We identify $\text{im}(\mathcal{L}\iota)$ with $\mathcal{L}H$, and say that $\mathcal{L}H$ is a *Lie subalgebra* of $\mathcal{L}G$. {lsbal}

Lemma 7.5.6 *If H is a Lie subgroup of G, then $\exp : T_1H \to H$ is the (co)restriction of $\exp : T_1G \to G$.* {expr}

Proof: This is an immediate consequence of the naturality of exp, Proposition 7.4.3. □

Proposition 7.5.7 *Let H_1 and H_2 be connected Lie subgroups of G. If $\mathcal{L}H_1 = \mathcal{L}H_2$, then $H_1 = H_2$.* {coinsg}

Proof: There exists an open neighborhood of $1 \in H_1$, and also one of $1 \in H_2$. Take a canonical chart and use the preceding lemma. $\square$

Lemma 7.5.8 *Let H be a Lie subgroup of G; then* {lalsg}

$$\mathcal{L}H = \{V \in \mathcal{L}G \mid t \mapsto \exp tV \text{ is continuous } \mathbb{R} \to H\}.$$

Proof: If $V \in \mathcal{L}H$, then $t \mapsto \exp tV$ is smooth $\mathbb{R} \to H$. Conversely, suppose $V \in \mathcal{L}G$ and $t \mapsto \exp tV$ is continuous $\mathbb{R} \to H \hookrightarrow G$. By Theorem 4.1.6 this map is smooth, so $V \in \mathcal{L}H$. $\square$

{tsg=} **Proposition 7.5.9** *Suppose H_1 and H_2 are Lie subgroups of G. If $H_1 = H_2$ as topological groups, then $H_1 = H_2$ as Lie groups.* $\square$

This also follows from Corollary 7.4.14.

{lsalsg} **Theorem 7.5.10** *If G is a Lie group with Lie subgroup H, then $\mathcal{L}H \leq \mathcal{L}G$. Every Lie subalgebra of $\mathcal{L}G$ is the Lie algebra of a unique connected Lie subgroup of G.*

Proof: It remains only to show that for a given Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a connected Lie subgroup H of G with $\mathcal{L}H = \mathfrak{h}$.

Take S to be a complementary subspace to $\mathfrak{h}$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus S$. Following Definition 7.4.7, we may take a cubical canonical chart N_1 in G with corresponding N_0 in $\mathfrak{g}$ such that

$$N_1 = \exp(\mathfrak{h} \cap N_0) \times \exp(S \cap N_0).$$

Then the (co)restriction $\exp(\mathfrak{h} \cap N_0)$ is a germ of a connected Lie (sub)group H with Lie algebra (isomorphic to) $\mathfrak{h}$.

Indeed, a topological subgroup H exists by Ex 1.3.1.18 applied to the germ. The germ also provides a chart at 1, hence at every point of H *via* left (or right) translations. Let $\iota : H \hookrightarrow G$ denote the inclusion; ι is smooth by Theorem 7.4.13. Observe that $\mathcal{L}\iota$ is injective, and by construction $\mathcal{L}H = \mathfrak{h}$. Using Proposition 7.4.8 and Corollary 7.4.9, ι is an immersion so H is a submanifold and hence a (connected) Lie subgroup of G as desired. $\square$

{lociso1} **Lemma 7.5.11** *Suppose $\rho : G \to G'$ is a local morphism. If $\mathcal{L}\rho : \mathcal{L}G \to \mathcal{L}G'$ is an isomorphism, then ρ is a local isomorphism.*

Proof: Since $\mathcal{L}\rho$ is an isomorphism, therefore ρ is locally invertible near $1 \in G$ by the InFT. Denote the local inverse by $\mu : G' \to G$. Since ρ is a local morphism, then so is μ; hence an isomorphism. $\square$

{locmor1} **Theorem 7.5.12** *Let G and G' be Lie groups and $h : \mathcal{L}G \to \mathcal{L}G'$ a Lie algebra morphism. Then there exists a local morphism $\rho : G \to G'$ such that $\mathcal{L}\rho = h$.*

Proof: Let $\mathfrak{k} = \{(V, h(v)) \mid V \in \mathcal{L}G\}$. Then $\mathfrak{k}$ is a subalgebra of $\mathcal{L}G \oplus \mathcal{L}G'$ with the product Lie algebra structure from Definition 7.2.24. Let K denote a connected Lie subgroup of $G \times G'$ that has $\mathfrak{k}$ as its Lie algebra. If $p_1 : G \times G' \to G$ is the projection, consider the morphism $p_1|_K : K \to G$. Then the map $\mathcal{L}\lambda : \mathfrak{k} \to \mathcal{L}G : (V, h(V)) \mapsto V$ is an isomorphism. By Lemma 7.5.11 λ is a local isomorphism $K \to G$ with local inverse $\mu : G \to K$. Moreover, $\mathcal{L}\mu(V) = (V, h(V))$ for $V \in \mathcal{L}G$. Now, composing $\mu : G \to K \hookrightarrow G \times G'$ with the restriction of the projection $p_2 : G \times G' \to G'$, the result is a local morphism $\rho = p_2 \circ \mu : G \to G'$. By construction, $\mathcal{L}\rho = h$. □

Taken together with Corollary 7.4.9, this theorem implies that $\mathcal{L}$ is a *completely faithful* functor from LGrp_{loc} to LAlg. Considering also germs of morphisms, one has a category with germs of Lie groups as objects and germs of Lie group morphisms as arrows. Corollary 7.2.19 states that a local isomorphism of Lie groups induces an isomorphism of Lie algebras. Thus $\mathcal{L}$ is also a functor from this germy category to LAlg.

Theorem 7.5.13 *Lie groups G and G' are locally isomorphic if and only if $\mathcal{L}G \cong \mathcal{L}G'$.* {lociso2}

Proof: If $h : \mathcal{L}G \to \mathcal{L}G'$ is an isomorphism, then by Theorem 7.5.12 there exists a local morphism $\rho : G \to G'$ inducing h. Lemma 7.5.11 then implies that ρ is actually a local isomorphism. □

This theorem is our most important result about Lie groups and algebras so far. It tells us exactly what information we can hope to obtain from the Lie algebra of a Lie group.

Ex 7.5.14 A connected Lie group is abelian if and only if its Lie algebra is abelian. {ablg}

Another point to make precise is the relation between local and global morphisms. Let $\rho : G \to G'$ be a local morphism. If G is connected, then by Corollary 7.4.9 there exists at most one global extension. Theorem 1.3.1.20 together with the lemma for Theorem 7.4.13 yields

Proposition 7.5.15 *If G and G' are Lie groups, G is 1-connected, and $\rho : G \to G'$ is a local morphism, then there exists a unique global extension $\widetilde{\rho} : G \to G'$.* {uext} □

Note that Proposition 7.3.15 is a particular case of this result.

{la2lg} **Corollary 7.5.16** *Let G and G' be Lie groups, $h : \mathcal{L}G \to \mathcal{L}G'$ a morphism of Lie algebras, and G be 1-connected. Then there exists a unique global morphism $\rho : G \to G'$ with $\mathcal{L}\rho = h$. If G' is also 1-connected and h is an isomorphism, then ρ is an isomorphism.*

Proof: Theorem 7.5.12 gives a local morphism ρ inducing h. Since G is 1-connected, Proposition 7.5.15 gives the unique global extension of ρ. If G' is 1-connected and h is an isomorphism in addition, then h^{-1} is similarly induced by a global morphism $\mu : G' \to G$ with $\mathcal{L}(\mu\rho) = 1_{\mathcal{L}G}$, so $\mu\rho = 1_G$. Similarly, $\rho\mu = 1_{G'}$. □

If G is an abelian Lie group, then $\exp : \mathcal{L}G \to G$ is a morphism of Lie groups: it is induced by $1_{\mathcal{L}G} : \mathcal{L}G \to \mathcal{L}G$ using the result of Proposition 7.4.5. Applying Corollary 7.5.16, we obtain

{abexp} **Proposition 7.5.17** *If G is an abelian, 1-connected Lie group, then* $\exp$: *$\mathcal{L}G \to G$ is an isomorphism of Lie groups.* □

Observe that Theorem 1.3.2.2 and Ex 1.3.2.4 imply that there exists a *universal covering group* of every connected Lie group. More precisely, let G be a connected Lie group. Then there exists a 1-connected Lie group $\widetilde{G}$ and a global morphism and local isomorphism $p : \widetilde{G} \to G$ such that $(\widetilde{G}, p)$ is a covering Lie group of G. It has the universal property that given any 1-connected Lie group H and morphism $\rho : H \to G$ there exists a unique morphism $\widetilde{\rho} : H \to \widetilde{G}$ with $p \circ \widetilde{\rho} = \rho$.

{abucg} **Ex 7.5.18** If G is abelian, then $(\mathcal{L}G, \exp)$ is its universal covering group.

Continue with the same G, let G' be any Lie group, and $\rho : G \to G'$ a local morphism. Then the local morphism $\rho \circ p : \widetilde{G} \to G'$ has by Proposition 7.5.15 a unique extension to a global morphism $\varphi : \widetilde{G} \to G'$. If G' is connected and if $(\widetilde{G}', p')$ is its universal covering group, then there exists a unique morphism $\widetilde{\varphi} : \widetilde{G} \to \widetilde{G}'$ with $p' \circ \widetilde{\varphi} = \varphi$.

{covmap} **Proposition 7.5.19** *Let G and H be connected Lie groups, and $\varphi : G \to H$ a morphism of Lie groups. Then φ is a covering map if and only if $\mathcal{L}\varphi : \mathcal{L}G \to \mathcal{L}H$ is an isomorphism.* □

In particular, if $\rho : G \to G'$ is a local isomorphism, then $\widetilde{\varphi} : \widetilde{G} \to \widetilde{G}'$ is a local isomorphism. By Corollary 7.5.16, $\widetilde{\varphi}$ is a global isomorphism. So a local isomorphism of connected Lie groups induces a global isomorphism of universal covering groups. This means that to every class of locally isomorphic Lie groups there corresponds a unique (up to isomorphism, of course) 1-connected Lie group that is the universal covering group of each member of the class. Moreover, each member is obtained from the universal covering group modulo a discrete (the fibers of a covering are discrete) normal subgroup. By Theorem 7.5.13, there is an injective map of classes of locally isomorphic Lie groups to isomorphism classes of $\mathbb{R}$-Lie algebras.

Now we need a theorem for whose proof we refer to [36, p. 199] or [80], for example.

Theorem 7.5.20 (Ado) *Every finite-dimensional real Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_n$ for some n.* {ado}

The connected subgroup G of GL_n corresponding to this Lie subalgebra by Theorem 7.5.10 is a Lie group with Lie algebra $\mathfrak{g}$. Note that this also implies that every Lie group is locally isomorphic to a Lie subgroup of some GL_n.

Combined with Ado's Theorem, the map of classes of locally isomorphic Lie groups to isomorphism classes of $\mathbb{R}$-Lie algebras is in fact bijective. This decomposes the classification of connected Lie groups into two broad steps. First, find all (isomorphism classes of) $\mathbb{R}$-Lie algebras. Second, find all discrete normal subgroups of a 1-connected Lie group.

7.5.1 discrete and open subgroups

Given a subgroup H of a Lie group G, it is natural to ask if it is possible to make H into a *Lie* subgroup of G. Proposition 7.5.9 says that given a topology on H, there exists at most one structure on H making it a Lie subgroup. However, a particular topology on H will not necessarily admit such a structure.

Example 7.5.1.1 Consider the rational numbers as a subgroup of the reals, $\mathbb{Q} \hookrightarrow \mathbb{R}$. In the induced topology it is not possible to make $\mathbb{Q}$ into a Lie subgroup of $\mathbb{R}$. However, by giving $\mathbb{Q}$ the discrete topology it can be made into a 0-dimensional Lie subgroup of $\mathbb{R}$. In general, any subgroup H of a Lie group G can be made into a 0-dimensional Lie subgroup by endowing it with the discrete topology, and there may be no other way to make H a Lie subgroup.

Definition 7.5.1.2 A *discrete subgroup* H of a topological group G is a subgroup of G which is also a discrete subspace in the induced topology. {dsg}

The previous example shows that $\mathbb{Q}$ is *not* a discrete subgroup of $\mathbb{R}$. Notice also that since G is assumed to be Hausdorff, every discrete subgroup is closed.

{kdnsg} **Proposition 7.5.1.3** *If G and G' are topological groups and $\rho : G \to G'$ is both a topological group morphism and a local isomorphism, then* $\ker(\rho)$ *is a discrete normal subgroup of G.*

Proof: There exist open neighborhoods N of $1 \in G$ and N' of $1 \in G'$ such that $\rho|_N : N \to N'$ is a homeomorphism. Therefore $\ker(\rho) \cap N = \{1\}$ and 1 is an isolated point in $\ker(\rho)$. This implies that every point of $\ker(\rho)$ is isolated since translations are homeomorphisms. □

{abkexp} **Corollary 7.5.1.4** *If G is an abelian Lie group, then* $\ker(\exp)$ *is a discrete subgroup of the additive group $\mathcal{L}G$.* □

For the next lemma, we need a preparatory result [24, Thm. 14.5]. (For finitely-generated abelian groups, this is an exercise in [51].)

Theorem 7.5.1.5 *Any subgroup of a free abelian group is free.*

Recall that an abelian group is the same thing as a $\mathbb{Z}$-module. Now we can prove the following.

{dsgRn} **Lemma 7.5.1.6** *Let $V \cong \mathbb{R}^n$ and D be a discrete additive subgroup of V with* $\dim(\operatorname{span} D) = p$ *for some $0 \leq p \leq n$. Then there exist p linearly independent vectors v_i in V which generate D.*

Proof: We shall use the Change of Rings universal process [51, p. 332] and the bijective correspondence of free generators (p. 333) for $\mathbb{Z} \hookrightarrow \mathbb{R}$.

First note that $\operatorname{span}(D) \cong D \otimes \mathbb{R}$ and that $\operatorname{rk}_{\mathbb{Z}} D = \dim_{\mathbb{R}} D \otimes \mathbb{R} = p$. Next, $D \leq \mathbb{Z}^n$ so by the theorem $D \cong \mathbb{Z}^p$. Using the correspondence of free generators, any p free generators $v_i \in D$ over $\mathbb{Z}$ correspond bijectively to the p linearly independent vectors $v_i \in D \otimes \mathbb{R}$ over $\mathbb{R}$. Choosing a basis of $D \otimes \mathbb{R} \leq \mathbb{R}^n$, we obtain a set of free generators of D as desired. □

{ablgi} **Lemma 7.5.1.7** *If G is a connected abelian Lie group with* $\exp : \mathcal{L}G \to G$, *then* $\mathcal{L}G/\ker(\exp) \cong G$ *as Lie groups.*

Proof: The morphism $\exp : \mathcal{L}G \to G$ is surjective by Proposition 7.4.5, so it induces $\varphi : \mathcal{L}G/\ker(\exp) \to G$ a group isomorphism. It remains to show that this isomorphism is a diffeomorphism.

By Corollary 7.5.1.4, $\ker(\exp)$ is a discrete subgroup of $\mathcal{L}G$. This implies that there is an open neighborhood U of $0 \in \mathcal{L}G/\ker(\exp)$ such that $\varphi|_U = \exp : U \to G$. By taking a possibly smaller open neighborhood $N_0 \subseteq U$ and employing Corollary 7.4.6, we obtain that $\varphi : N_0 \to N_1$ is a diffeomorphism near $0 \in \mathcal{L}G/\ker(\exp)$. Finally, G connected implies that φ is a global diffeomorphism. □

Theorem 7.5.1.8 *If G is an n-dimensional connected abelian Lie group, then there exists $p \in \mathbb{N}$, $0 \leq p \leq n$, such that $G \cong \mathbb{R}^{n-p} \times \mathbb{T}^p$.* {ablgc}

Proof: Lemma 7.5.1.6 says that $\ker(\exp) \cong \mathbb{Z}^p$ for some $0 \leq p \leq n$. Since $\mathcal{L}G \cong \mathbb{R}^n$, Lemma 7.5.1.7 implies that $G \cong \mathbb{R}^n/\mathbb{Z}^p \cong \mathbb{R}^{n-p} \times \mathbb{R}^p/\mathbb{Z}^p$. □

Proposition 7.5.1.9 *If H is a discrete normal subgroup of a connected topological group G, then $H \leq Z(G)$.* {dnsgc}

Proof: Let $h \in H$. The map $G \to H : g \mapsto ghg^{-1}$ is continuous, so its image is connected. Since H is discrete, this image must be a single point. Taking $g = 1$, it must be h. □

Let G be a Lie group and H an open subgroup and submanifold. Since H is open, therefore $\mathcal{L}H = \mathcal{L}G$. Thus H contains the connected component of the identity, G_o. Since $\mathfrak{h} = \mathfrak{g}$, then H_o and G_o actually coincide.

Example 7.5.1.10 Suppose V is a finite-dimensional vector space (over $\mathbb{R}$, as usual). The determinant map $\det : GL(V) \to \mathbb{R}^*$ is polynomial, so it is a morphism of Lie groups. Since $\mathbb{R}^*$ is disconnected so is $GL(V)$. Choosing an orientation on V, any two positive bases can be smoothly transformed into each other by automorphisms of V. Therefore $\det^{-1}\mathbb{R}^+ = GL(V)_o$. {ccgl}

Suppose we take an open subgroup $H \leq G$ with cosets G/H. Since all cosets are open in G, therefore the topology on G/H is discrete, and G/H is a 0-dimensional manifold. In particular, if H is normal in G, then G/H is a 0-dimensional Lie group. Applied to G_o, this means that the *0th homotopy group* $\pi_0(G) := G/G_o$ is a 0-dimensional Lie group.

Example 7.5.1.11 When $G = GL(V)$ for some finite-dimensional vector space V, then $\pi_0 \cong \mathbb{Z}_2$ by the previous example. {pi0gl}

In general, any connected component of a Lie group G is diffeomorphic to G_o, so $G \cong G_o \times \pi_0$. However this isomorphism is not necessarily natural (in the formal sense). Any splitting s of the short exact sequence $1 \to G_o \to G \to \pi_0 \to 1$ induces an isomorphism $G \cong G_o \ltimes_\tau \pi_0$, where $\tau : \pi_0 \to \mathrm{Aut}(G_o) : g \mapsto \kappa_{sg}|_{G_o}$ for all $g \in \pi_0$. Recall that κ_{sg} is the morphism *conjugation by sg* introduced in Example 1.2.1.8. This means that G is topologically trivial but has a twisted algebraic structure, akin to that of the tangent bundle of a Lie group.

{glodd} **Example 7.5.1.12** Let $V \in \mathit{Vec}$ be odd-dimensional and consider $G = GL(V)$. Reflection of V through 0 together with the identity map 1_V provides an embedding of $\mathbb{Z}_2 \cong G/G_o$ in G.

In case G is abelian, then $\kappa_g = 1_G$ for all g. Thus any splitting s defines an isomorphism $G \cong G_o \times \pi_0$, where the right hand side is now a direct product.

{dtrns} **Example 7.5.1.13** Suppose $V \in \mathit{Vec}$, $U \leq V$, and $v \in V \neg U$. The union of U with all translates by integer multiples of v is a Lie group G in the relative topology. Moreover, $G_o = U$, $\pi_0 \cong \mathbb{Z}$, the short exact sequence $0 \to U \to G \to \mathbb{Z} \to 0$ splits, and $G \cong U \times \mathbb{Z}$. That is, G is a countably infinite disjoint union of copies of U, each separated from its neighbors precisely by v.

7.5.2 closed subgroups

We turn now to closed subgroups, beginning with two general lemmas.

{ucsgl1} **Lemma 7.5.2.1** *Let H be a subgroup of a Lie group G, $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$, and (N_0, N_1) a canonical chart. If $\exp(\mathfrak{h} \cap N_0) = H \cap N_1$, then H with the subspace topology is a Lie subgroup of G and $\mathcal{L}H = \mathfrak{h}$.*

Proof: If H is a subgroup of G with the subspace topology, then by Proposition 7.5.9 a differential structure making it a Lie subgroup is unique. Hence we need only produce such a differential structure. Consider $\exp : \mathfrak{h} \cap N_0 \to H \cap N_1$. Take the differential structure (maximal atlas) on H containing the charts

$$\{(H \cap hN_1, \log \circ L_{h^{-1}}) \mid h \in H\}.$$

Clearly, this suffices. □

{ucsgl2} **Lemma 7.5.2.2** *Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. If $A, B \in \mathfrak{g}$, then for all sufficiently small t we have the asymptotic condition*

$$(\exp tA)\,(\exp tB) = \exp t(A+B) + O(t^2)$$

as $t \to 0$.

This is the first term in the Baker-Campbell-Hausdorff formula [80].

Proof: For t near 0 there is a curve γ in $\mathfrak{g}$ such that $(\exp tA)(\exp tB) = \exp \gamma(t)$; indeed, $\dot{\gamma}(0) = A + B$. Taylor's expansion with integral remainder at $t = 0$ is given by $\gamma(t) = t(A+B) + O(t^2)$. The conclusion follows. □

Theorem 7.5.2.3 *Let H be a closed subgroup of the Lie group G. Then there is a unique Lie group structure on H making it a Lie subgroup with the subspace topology.* {ucsg}

Proof: First we note that if such a structure exists, then it is unique by Proposition 7.5.9.

Now, for H closed define

$$\mathfrak{h} = \{A \in \mathcal{L}G \mid \exp tA \in H \text{ for all } t \in \mathbb{R}\}.$$

Taking $A, B \in \mathfrak{h}$, Lemma 7.5.2.2 implies

$$\lim_{n\to\infty} \left[\exp\left(\frac{t}{n}A\right)\exp\left(\frac{t}{n}B\right)\right]^n = \exp t(A+B)\,.$$

Thus $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ and $\exp \mathfrak{h} \subseteq H$. Let S be complementary to $\mathfrak{h}$ (so that $\mathfrak{h} \oplus S = \mathfrak{g}$). Then $\phi : \mathfrak{g} \to G : A + B \mapsto \exp A \cdot \exp B$ for $A \in \mathfrak{h}$ and $B \in S$ is a diffeomorphism from some open neighborhood of $0 \in \mathfrak{g}$ to an open neighborhood of $1 \in G$ by Lemma 7.4.11.

It remains to show that, for a sufficiently small neighborhood U of $0 \in \mathfrak{g}$,

$$\begin{aligned} H \cap \phi(U) &= \phi(\mathfrak{h} \cap U), \text{ or} \\ H \cap \exp U &= \exp(\mathfrak{h} \cap U). \end{aligned}$$

This will suffice by Lemma 7.5.2.1 with $U = N_0$ and $\exp U = N_1$ forming a canonical chart.

Suppose not. Then there exists a sequence $B_n \to 0$ with $B_n \in S \neg \{0\}$ and $\exp B_n \in H$. Passing to a subsequence if necessary, we may assume that $t_n B_n \to B \in S \neg \{0\}$ for some sequence $t_n \in \mathbb{R}$. But this means that B is in $\mathfrak{h}$, a contradiction. □

Corollary 7.5.2.4 *A Lie subgroup is closed if and only if it is embedded.* {clsbgp}

Proof: The theorem proves the "only if" half. So let $\iota : H \hookrightarrow G$ be a closed Lie subgroup and let $h_i \to g \in G$. Since ι is an embedding, there exists a cubical chart (U, x) centered at $1 \in G$ such that $\iota(H) \cap U$ consists of the single slice S through 1. Choose open neighborhoods $V \subseteq W \subseteq U$ of 1, cubical with respect to x, so that $V^{-1}V \subseteq \overline{W} \subseteq U$. As $h_i \to g$, there exists N such that $h_i \in gV$ for $i \geq N$. Then it follows that $h_N^{-1}h_i \in \overline{W}$ and that $h_N^{-1}h_i \in \iota(H)$ for $i \geq N$. Thus $h_N^{-1}h_i \in S \cap \overline{W}$ and converges to $h_N^{-1}g$ which must also be in $S \cap \overline{W}$. Hence $h_N^{-1}g \in \iota(H)$ so $g \in \iota(H)$ and $\iota(H)$ is closed. □

{ucsgc} **Corollary 7.5.2.5** *If H is a closed subgroup of a Lie group G made into a Lie subgroup as in the theorem, then its Lie algebra is given by*

$$\mathfrak{h} = \{A \in \mathcal{L}G \mid \exp tA \in H \text{ for all } t \in \mathbb{R}\}.$$

□

{ker} **Proposition 7.5.2.6** *If $\rho : G \to H$ is a morphism of Lie groups, then* $\ker(\rho)$ *is a closed Lie subgroup of G and* $\mathcal{L}(\ker\rho) = \ker(\mathcal{L}\rho)$.

Proof: We know that $\ker(\rho)$ is closed and is therefore a Lie subgroup. By Corollary 7.5.2.5,

$$\mathcal{L}(\ker\rho) = \{A \in \mathcal{L}G \mid \rho(\exp tA) = 1 \in H \text{ for all } t \in \mathbb{R}\}.$$

Naturality of the exponential map (Prop. 7.4.3) implies that $\rho(\exp tA) = 1$ is equivalent to $\exp(\mathcal{L}\rho\, tA) = 1$ for all real t. This implies that $\mathcal{L}\rho(A) = 0$, whence $\mathcal{L}(\ker\rho) = \ker(\mathcal{L}\rho)$. □

{im} **Proposition 7.5.2.7** *Let G be a connected Lie group, and $\rho : G \to K$ a Lie group morphism. Then* $\mathrm{im}(\rho)$ *is a Lie subgroup of K and* $\mathcal{L}(\mathrm{im}\,\rho) = \mathrm{im}(\mathcal{L}\rho)$.

Proof: Let H be the connected subgroup of K with $\mathcal{L}H = \mathrm{im}(\mathcal{L}\rho)$. H is generated by elements $\exp(\mathcal{L}\rho\, A)$ with $A \in \mathcal{L}G$, and $\mathrm{im}(\rho)$ is generated by elements $\exp(\mathcal{L}\rho\, A)$ with $A \in \mathcal{L}G$. But $\rho(\exp A) = \exp(\mathcal{L}\rho\, A)$ by naturality. Therefore $\mathrm{im}(\rho) = H$ as both are connected. □

Now consider a sequence

$$G' \xrightarrow{\rho'} G \xrightarrow{\rho''} G''$$

of Lie groups, and the induced sequence

$$\mathfrak{g}' \xrightarrow{\mathcal{L}\rho'} \mathfrak{g} \xrightarrow{\mathcal{L}\rho''} \mathfrak{g}''$$

of Lie algebras.

{exact} **Proposition 7.5.2.8** *If G is connected, then exactness of the sequence of groups implies exactness of the sequence of algebras.*

Proof: If $\mathrm{im}(\rho') = \ker(\rho'')$, then $\mathrm{im}(\mathcal{L}\rho') = \mathcal{L}(\mathrm{im}\,\rho') = \mathcal{L}(\ker\rho'') = \ker(\mathcal{L}\rho'')$ by the preceding propositions. □

Example 7.5.2.9 If G is a connected Lie group, then the exact sequence $0 \to T_1G \to TG \to G \to 1$ induces the exact sequence $0 \to T_1G \to \mathcal{L}TG \to \mathfrak{g} \to 0$. Note the natural splitting $G \hookrightarrow TG$ defines a splitting $\mathfrak{g} \hookrightarrow \mathcal{L}TG$. {exactx1}

Ex 7.5.2.10 The converse of Proposition 7.5.2.8 fails even if all of the groups are connected.

Example 7.5.2.11 If $\rho : G \to H$ is a local isomorphism of Lie groups, then $0 \to \mathfrak{g} \xrightarrow{\mathcal{L}\rho} \mathfrak{h} \to 0$ is exact. However, $1 \to G \xrightarrow{\rho} H \to 1$ is not necessarily exact. {exactx2}

Proposition 7.5.2.12 *If $\rho : G \to H$ is a Lie group morphism and both G and H are connected, then ρ is surjective if and only if $\mathcal{L}\rho$ is surjective.* {surj}

Proof: If ρ is surjective, then $\mathcal{L}\rho(\mathfrak{g}) = \mathfrak{h}$ so that $\mathcal{L}\rho$ is surjective. If $\mathcal{L}\rho$ is surjective, then $\rho_{*g} : T_gG \to T_{\rho(g)}H$ is surjective for every $g \in G$ by Proposition 7.5.1. Therefore $\mathrm{im}(\rho)$ is an open (hence closed) subgroup of H. Since H is connected, then $\mathrm{im}(\rho) = H$. □

The hypothesis that H is connected cannot be omitted: $H_o \hookrightarrow H$ for H disconnected induces an isomorphism of Lie algebras.

Ex 7.5.2.13 The analogue for injections fails.

7.6 Classical matrix groups

Suppose V is a finite-dimensional (real) vector space with automorphism group $GL(V)$, and let $\beta : V \times V \to \mathbb{R}$ be a bilinear form. Consider the subgroup H of $GL(V)$ that leaves β invariant.

$$H = \{g \in GL(V) \mid \beta(gv, gw) = \beta(v, w) \text{ for all } v, w \in V\}$$

For fixed v and w, consider the composition

$$GL(V) \to GL(V) \times GL(V) \to V \times V \to \mathbb{R} :$$

$$g \mapsto (g, g) \mapsto (gv, gw) \mapsto \beta(gv, gw)\,.$$

Since β is bilinear it is continuous, so the entire composition is continuous. Thus $S(v, w) := \{g \in GL(V) \mid \beta(gv, gw) = \beta(v, w)\}$ is closed in $GL(V)$. Since H is the intersection over all pairs (v, w) from V of the sets $S(v, w)$, H is a closed subgroup of $GL(V)$. Now everything that we have proved about closed subgroups applies to H. In particular, H is a Lie subgroup of $GL(V)$.

Recall from Example 7.2.11 that we identify $\mathcal{L}GL(V)$ with $\mathrm{End}(V) = \mathfrak{gl}(V)$ *ad libitum.*

{bf} **Proposition 7.6.1** *If H is the Lie subgroup of $GL(V)$ defined above, its Lie algebra is given by*

$$\mathcal{L}H = \{A \in \mathrm{End}(V) \mid \beta(Av, w) + \beta(v, Aw) = 0 \text{ for all } v, w \in V\}.$$

□

Proof: If $A \in \mathcal{L}H$, then $\exp tA \in H$ for all real t and

$$\beta\left(\exp(tA)v, \exp(tA)w\right) = \beta(v, w)$$

for all v, w in V. Differentiating with respect to t and evaluating at $t = 0$, we obtain $\beta(Av, w) + \beta(v, Aw) = 0$.

On the other hand, suppose $A \in \mathrm{End}(V)$ satisfies the hypothesis. Denote by $A^\dagger$ the adjoint of A with respect to β. We claim that $A^\dagger = -A$, or equivalently that $(\exp tA)^\dagger = (\exp tA)^{-1}$ for all $t \in \mathbb{R}$. The latter of these statements follows directly from the power series representation for $\exp tA$ in Example 7.4.2.

□

The group H is called the *automorphism group* of β and denoted $\mathrm{Aut}(\beta)$. The Lie algebra $\mathcal{L}H$ is isomorphic to the *derivation algebra* of β, denoted $\mathrm{Der}(\beta)$; *cf.* Definition 2.6.3.

{bo} **Ex 7.6.2** Using a similar argument, show that the automorphism group of a binary operation $b : V \otimes V \to V$ is a closed Lie subgroup $\mathrm{Aut}(b)$ of $GL(V)$ with Lie algebra $\mathrm{Der}(b)$.

Suppose we interpret $\mathfrak{g}$ as a vector space with binary (bilinear) operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. To make the vector space $\mathfrak{g}$ into a Lie algebra we must require that $[\ ,\ ]$ is skew and satisfies the Jacobi identity of Definition 2.6.1. In such cases we call $[\ ,\ ]$ a *Lie product* on $\mathfrak{g}$. We write $GL(\mathfrak{g})$ to denote the vector space automorphisms of $\mathfrak{g}$, reserving $\mathrm{Aut}(\mathfrak{g})$ for the Lie algebra automorphisms of $\mathfrak{g}$. This convention will be followed hereinafter.

{autla} **Ex 7.6.3** For any Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$ with Lie algebra the derivations of the Lie product, $\mathfrak{aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

{autlg} **Ex 7.6.4** If G is a 1-connected Lie group and $\mathrm{Aut}(G)$ its group of Lie automorphisms, then the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}) : \varphi \mapsto \mathcal{L}\varphi$ is a group morphism. It is mono by Corollary 7.4.9, and epi by Corollary 7.5.16. Declaring this isomorphism to be smooth, $\mathrm{Aut}(G)$ becomes a Lie group. Since every Lie group is covered by a 1-connected group, this result extends to any Lie group: $\mathrm{Aut}(G)$ is a Lie group for every Lie group G.

If we assume that the bilinear form β is also nondegenerate and symmetric, {ndsbf} then H is called the *(pseudo-)orthogonal* group $O(\beta)$, or $O(V, \beta)$ if the vector space is not clear from context.

Example 7.6.5 Suppose V is a 3-dimensional vector space, β is the Eu- {dim3} clidean inner product, and $\{e_1, e_2, e_3\}$ is a positively oriented orthonormal basis for V. To make V a Lie algebra, define

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad \text{and} \quad [e_3, e_1] = e_2 .$$

Now consider $\mathrm{End}(V) = \mathfrak{gl}(V)$, the Lie algebra of $GL(V)$, and take $a \in V$. Suppose $A \in \mathrm{End}(V)$ is defined by $Av = [a, v]$ for $v \in V$. Then

$$\beta(Av, w) + \beta(v, Aw) = \beta([a, v], w) + \beta(v, [a, w]) = 0$$

by a direct calculation. Therefore $A \in \mathcal{L}O(\beta)$ and $\mathcal{J} : V \to \mathcal{L}O(\beta) : a \mapsto A$ is linear. Moreover, if $[a, v] = 0$ for all $v \in V$ then $a = 0$. Thus $\mathcal{J}$ is a monomorphism. Both V and $\mathcal{L}O(\beta)$ are of dimension 3, so $\mathcal{J}$ is a linear isomorphism.

We have left to verify that $\mathcal{J}$ is a Lie morphism. Let $A_i v = [a_i, v]$, $i = 1, 2$, for all $v \in V$. We compute

$$[A_1, A_2]v = A_1 A_2 v - A_2 A_1 v = [a_1, [a_2, v]] - [a_2, [a_1, v]] = [[a_1, a_2], v]$$

where the last equality follows from the Jacobi identity. Therefore $\mathcal{J}$ is an isomorphism of Lie algebras: $\mathcal{J} : V \cong \mathcal{L}O(\beta)$.

To see the geometric interpretation of $\mathcal{J}$, consider the 1-parameter subgroup γ of $O(\beta)$ generated by A such that $\dot{\gamma}(t) = A\gamma(t)$ by Lemma 7.3.11. Let $v \in V$ and $v(t) = \gamma(t)v$ so that $\dot{v}(t) = \dot{\gamma}(t)v = A\gamma(t)v = Av(t)$ can be rewritten as $\dot{v}(t) = [a, v(t)]$. Thus the 1-parameter subgroup γ of $O(\beta)$ generated by A is the 1-parameter group of rotations about the axis a.

Continue with the finite-dimensional vector space V and nondegenerate symmetric bilinear form β. If β is also positive definite, then by a similar argument as for $GL(V)$ (Example 7.5.1.10) it follows that the connected component of the identity is ker(det). We denote this (sub)group by $SO(\beta)$.

{defsbf} **Proposition 7.6.6** *If $V \in$ Vec, β a positive definite symmetric bilinear form, $O(\beta)$ its orthogonal group, and $SO(\beta)$ as above, then $O(\beta)$ and $SO(\beta)$ are compact.*

Proof: Since $SO(\beta)$ is a kernel, then it is an open subgroup of $O(\beta)$; hence also closed. Therefore it suffices to prove that $O(\beta)$ is compact. Now $O(\beta)$ is closed in $GL(V)$, which in turn is open in $\mathfrak{gl}(V)$, so $O(\beta)$ is closed in $\mathfrak{gl}(V)$. Let $|\cdot| : \mathfrak{gl}(V) \to \mathbb{R}$ be the norm with respect to β given by

$$|A| = \sup_{0 \neq v \in V} \frac{\beta(Av, Av)^{1/2}}{\beta(v,v)^{1/2}}.$$

Then for any $A \in O(\beta)$, $|A| = 1$ and $O(\beta)$ is bounded in $\mathfrak{gl}(V)$. Thus, by the Heine-Borel Theorem, it is compact. □

Ex 7.6.7 Repeat for β negative definite, *mutatis mutandis.*

Let $\sigma : V \times V \to R$ be a nondegenerate alternating bilinear form on a vector space V of even dimension (Prop. 5.2.2). The subgroup of $GL(V)$ leaving σ invariant is the symplectic group $Sp(\sigma)$. As there is essentially a unique such σ (Prop. 5.2.3), any two symplectic groups on V are isomorphic (Thm. 5.2.4). The Lie algebra of $Sp(\sigma)$ consists of the endomorphisms of V which are skewadjoint with respect to σ (Thm. 5.2.7).

Finally, consider the morphism $\det : GL(V) \to R^*$. Denote the kernel by $SL(V)$. This is the *special linear group*.

{lasl} **Proposition 7.6.8** *If V is a finite-dimensional vector space, then*

$$\mathfrak{sl}(V) = \{A \in \mathrm{End}(V) \mid \mathrm{tr}\, A = 0\}.$$

Proof: Proposition 7.2.22 implies $\mathcal{L}(\det) = \mathrm{tr}$, and Proposition 7.5.2.6 then yields $\mathcal{L}(\ker\det) = \mathcal{L}SL(V) = \ker(\mathrm{tr})$. □

We next turn our attention to some concrete examples of matrix groups, each of which is a closed subgroup of $GL_n(\mathbb{C})$. One example that we have already seen is the real general linear group $GL_n(\mathbb{R})$. If we replace $\mathbb{R}$ with $\mathbb{C}$ in our previous considerations, all of the results still hold. On the other hand, $GL_n(\mathbb{C})$ is itself a closed subgroup of $GL_{2n}(\mathbb{R})$ and its Lie algebra is a Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{R})$. This means that which field we choose to regard as primary is merely a matter of preference.

We have already seen the n-dimensional special linear group over $\mathbb{R}$ in Proposition 7.6.8. The same result continues to hold when $\mathbb{R}$ is replaced by $\mathbb{C}$. Letting $\Bbbk$ denote either field, $SL_n(\Bbbk)$ consists of all matrices in $GL_n(\Bbbk)$ of determinant 1 and is a closed subgroup of $GL_n(\Bbbk)$. It has dimension $n^2 - 1$ over $\Bbbk$.

In $\mathbb{C}^n$ we have a nondegenerate sesquilinear form, called the *standard Hermitian form*, which satisfies $\langle z, z\rangle = |z|^2$, $\langle z, w\rangle = \overline{\langle w, z\rangle}$, and a suitable triangle or Cauchy-Schwarz inequality. We then define the n-dimensional *unitary group* as the automorphism group of this form.

$$U_n := \{A \in GL_n(\mathbb{C}) \mid \langle Az, Aw\rangle = \langle z, w\rangle \text{ for all } z, w \in \mathbb{C}^n\}.$$

Alternatively, the unitary group can be described by the defining equation $A \in U_n$ if and only if $A^{-1} = {}^t\overline{A}$. The subgroup of U_n which consists of those matrices with determinant 1 is denoted by SU_n. Since these groups are only defined over the complex numbers, the field is usually not specified.

Ex 7.6.9 The determinant maps U_n to $S^1 \subseteq \mathbb{C}$.

The Lie algebra of U_n is given by

$$\mathcal{L}U_n = \mathfrak{u}_n = \{A \in \mathfrak{gl}_n(\mathbb{C}) \mid \overline{A} + A^T = 0\}.$$

The Lie algebra $\mathfrak{su}_n$ has the additional condition that $\operatorname{tr} A = 0$. As real manifolds, $\dim U_n = n^2$ and $\dim SU_n = n^2 - 1$.

Now consider the standard nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on $\mathbb{C}^n$. This form differs from the standard Hermitian form in that $\langle z, w\rangle = \langle w, z\rangle$ for all $z, w \in \mathbb{C}^n$. We define the *orthogonal group* $O_n(\mathbb{C})$ to be the automorphism group of this form, and $SO_n(\mathbb{C}) \leq O_n(\mathbb{C})$ to be the closed subgroup consisting of those matrices with determinant 1.

Ex 7.6.10 What is the image of the determinant map applied to $O_n(\mathbb{C})$?

For the following considerations we let $\Bbbk$ denote either $\mathbb{C}$ or $\mathbb{R}$, as the results hold for either field. Recall that any $A \in GL_n(\Bbbk)$ may be used to change the basis of a symmetric bilinear form β by $\beta \mapsto A\beta A^T$. By this process, we may diagonalize β. In particular, any nondegenerate symmetric bilinear form may be represented uniquely by $\eta = \operatorname{diag}\{+1, \ldots, +1, -1, \ldots, -1\}$ in an *orthonormal* basis with p entries of $+1$, q entries of -1, and $p+q=n$. This is related to the Principal Axis Theorem of Linear Algebra.

The automorphism group of η is a matrix Lie subgroup of $GL_n(\Bbbk)$ called the *(pseudo-)orthogonal group*. It is denoted by $\mathrm{Aut}(\eta) = O(\eta) = O_p^q(\Bbbk)$ or $O_n(\Bbbk) = O_n^0(\Bbbk)$ if $q = 0$. There is an obvious isomorphism $O_p^q(\Bbbk) \cong O_q^p(\Bbbk)$, so we may as well assume $q \geq p$ when convenient.

Since $\eta = \eta^{-1} = \eta^T$, applying the change of basis process to η yields $A\eta A^T = \eta$ or $A^{-1} = \eta A^T \eta$. From Proposition 7.6.1, we obtain the defining equation for the Lie algebra $\mathfrak{o}_p^q(\Bbbk)$ is $A\eta + \eta A^T = 0$. A typical element of $\mathfrak{o}_p^q(\Bbbk)$ looks like

$$\begin{bmatrix} \mathfrak{o}_p & \mathfrak{m} \\ \mathfrak{m}^T & \mathfrak{o}_q \end{bmatrix}$$

where $\mathfrak{m}$ is an arbitrary $p \times q$ matrix. The comments preceding Proposition 7.6.1 show that $O_p^q(\Bbbk)$ and $SO_p^q(\Bbbk)$ are closed Lie subgroups of $GL_n(\Bbbk)$. The Proposition itself proves that $\mathfrak{o}_p^q(\Bbbk) = \mathfrak{so}_p^q(\Bbbk)$ is a Lie subalgebra of $\mathfrak{gl}_n(\Bbbk)$, and the matrix form shows that the dimension is $n(n-1)/2$ over $\Bbbk$.

We studied another matrix Lie group in Section 5.2 called the *symplectic group*. This group is only defined on even-dimensional spaces $\Bbbk^{2n}$. Recall that

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

is the usual complex structure (multiplication by $\sqrt{-1}$) on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. On the standard basis, the usual symplectic structure σ is given by $\langle Jx, y\rangle$. The symplectic group $Sp_{2n}(\Bbbk)$ is the automorphism group of σ.

It is also possible to define another type of symplectic group, called the *unitary* or *quaternionic* symplectic group.

{usmpl} **Ex 7.6.11** The following constructions are equivalent.

$$\begin{aligned} Sp_{2n} &= Sp_{2n}(\mathbb{C}) \cap U_{2n} \\ &= Sp_{2n}(\mathbb{C}) \cap SU_{2n} \\ &= Sp_{2n}(\mathbb{C}) \cap Sp_{4n}(\mathbb{R}). \end{aligned}$$

{smpl} **Ex 7.6.12** We have the following identifications.

$$\begin{aligned} U_{2n} &= Sp_{2n}(\mathbb{R}) \cap GL_{2n}(\mathbb{C}) \\ Sp_2(\Bbbk) &= SL_2(\Bbbk) \\ Sp_{2n}(\Bbbk) &\leq SL_{2n}(\Bbbk) \end{aligned}$$

The last one means that $Sp_{2n}(\Bbbk)$ is volume preserving.

There are other Lie groups which are not contained in $GL_n(\Bbbk)$ in general. Some are obtained as universal covering groups of the identity component of any of the preceding groups (or any other Lie group). One such class of Lie groups that are commonly found in mathematical physics are the *spin groups*. The Lie group $Spin_p^q(\Bbbk)$ is the universal covering group of the identity component of $O_p^q(\Bbbk)$.

Alternatively, write $\Bbbk^n = \Bbbk^p \oplus \Bbbk^q$. Over $\mathbb{R}$, each of the subspaces and the whole space may be oriented (positively or negatively); over $\mathbb{C}$ the orientation is part of the structure. Any two of the real orientations uniquely determines the third. Assume $\mathbb{R}^n$ is given the usual orientation. Then $O_n = O_n(\mathbb{R})$ consists of two connected components: SO_n (the identity component) which preserves the orientation, and another which reverses it.

Ex 7.6.13 Using the method of Example 7.5.1.10, show that $O_p^q(\mathbb{R})$ has four connected components. Alternatively, observe that $O_p(\mathbb{R}) \times O_q(\mathbb{R})$ is a (maximal) compact subgroup, and that $O_p^q(\mathbb{R})$ is homeomorphic to $O_p \times O_q \times \mathbb{R}^{pq}$. {opqcmps}

Let $O_p^q(\mathbb{R})^+$ denote the subgroup of the pseudo-orthogonal group that preserves the orientation of $\mathbb{R}^p$. Then $SO_p^q(\mathbb{R})^+$ is the identity component and $Spin_p^q(\mathbb{R})$ is its universal covering group. Similar considerations over $\mathbb{C}$ yield $Spin_p^q(\mathbb{C})$.

Ex 7.6.14 The following coincidental isomorphisms of real spin groups hold. {spin}

$Spin_1 \cong \mathbb{Z}_2 \cong O_1$	$Spin_2 \cong \mathbb{R}$	$Spin_3 \cong SU_2 \cong S^3$
$Spin_4 \cong S^3 \times S^3$	$Spin_5 \cong Sp_4$	$Spin_6 \cong SU_4$
	$Spin_8 \cong Spin_7 \times S^7$	
$Spin_1^1 \cong \mathbb{R}$	$Spin_1^2 \cong \widetilde{SL_2}(\mathbb{R})$	$Spin_1^3 \cong SL_2(\mathbb{C})$
$Spin_1^4 \cong SL_2(\mathbb{H}) \cong Sp_2^2$	$Spin_1^5 \cong SU_4^*$	$Spin_2^2 \cong \widetilde{SL_2}(\mathbb{R}) \times \widetilde{SL_2}(\mathbb{R})$
$Spin_2^3 \cong \widetilde{Sp_4}(\mathbb{R})$	$Spin_2^4 \cong SU_2^2$	$Spin_3^3 \cong \widetilde{SL_4}(\mathbb{R})$

7.7 Homogeneous manifolds

We prove the standard facts about homogeneous manifolds as coset spaces and consider some of the standard examples of them.

Theorem 7.7.1 *Let G be a Lie group, H a closed subgroup, and G/H the orbit space (coset space) obtained in Example 1.3.1.11 with the action of G on G/H from Theorem 1.2.3.13. Then there is a unique differential structure on G/H making it a G-manifold in the identification topology.* {coset}

Proof: Let S be a vector subspace of $\mathfrak{g}$ such that $\mathfrak{g} = S \oplus \mathfrak{h}$. Let $\pi : G \to G/H$ be the natural projection. We begin with a lemma.

{cosetl1} **Lemma 7.7.2** *There exists an open neighborhood U_S of $0 \in S$ such that $\exp|_{U_S} : U_S \to V := \exp(U_S)$ is a homeomorphism. Moreover, $\pi|_V : V \to N_1 := \pi \exp(U_S)$ is a homeomorphism onto an open neighborhood of $\pi(1) = [1] = H \in G/H$.*

Proof: The first part of the lemma follows from a slight refinement of Lemma 7.4.11.

Continuing with the refinement, we may assume that the image of $\exp : U_S \oplus U_{\mathfrak{h}} \to G$ is a product $\exp(U_S) \times \exp(U_{\mathfrak{h}})$ near $1 \in G$. This is effectively a cubical neighborhood. Since π is open, $N_1 = \pi \exp(U_S)$ is an open neighborhood of $H \in G/H$. □

To get the differential structure on G/H we declare π to be smooth and use left invariance. First,

$$(\exp|_{U_S})^{-1} \circ (\pi|_V)^{-1} : N_1 \to U_S \subseteq S$$

is a "chart" centered at $\pi(1) = H \in G/H$. Because G acts by homeomorphisms on G/H, we also obtain charts at every other point of G/H. Taking a maximal atlas containing all of these charts gives G/H a differential structure.

Finally, uniqueness of the differential structure on G/H follows from the usual swindle. If $(G/H)'$ is G/H with another such differential structure, then $1_{G/H}$ is smooth in both directions. Indeed, locally it may be represented as the composition of a local smooth slice σ (as defined below) followed by (a restriction of) π. Thus $1_{G/H}$ is a diffeomorphism. □

{slice} **Remark 7.7.3** The obvious map $\sigma : \pi \exp(U_S) \to U_S$ is called a local *section* or *slice*.

Ex 7.7.4 Using the existence of local slices everywhere, G is a fiber bundle over G/H with model fiber H.

{cosetc1} **Corollary 7.7.5** *A function f is in the algebra of smooth functions $\mathfrak{F}(G/H)$ if and only if $f \circ \pi \in \mathfrak{F}(G)$.*

Proof: Since π is smooth, we need only consider $f \circ \pi$ smooth. But f admits a local representation *via* local slices $f = (f \circ \pi) \circ \sigma$. □

{cosetsm} **Theorem 7.7.6** *Let M be a G-manifold for $\tau : G \to \operatorname{Aut} M$, select $x_o \in M$, let $H = G_{x_o}$ be the isotropy group at x_o, and consider the map $\varphi : G/H \to M : gH \mapsto \tau_g(x_o)$. If G/H has the differential structure defined above, then φ is smooth. If φ is a homeomorphism, then it is a diffeomorphism.*

Proof: Continue with the notation from the previous proof. Note that V is a submanifold of G, making $\iota : V \hookrightarrow G$ smooth. Denote by ψ the map $G \to M : g \mapsto \tau_g(x_o)$. Then $\varphi|_{N_1} = \psi \circ \iota \circ (\pi|_V)^{-1}$, and thus is smooth.

If φ is a homeomorphism, then φ will be a diffeomorphism provided φ is a local diffeomorphism. Since φ is equivariant, it suffices to prove this at x_o. The factorization $\varphi|_{N_1} = \psi \circ \iota \circ (\pi|_V)^{-1}$ shows that φ is an immersion, so we need only prove that $\psi_* : T_1 G \to T_{x_o} M$ is a surjection.

Clearly, $T_1 H \subseteq \ker \psi_*$. Consider $A \in \ker \psi_*$. To show that A is in $T_1 H$ it suffices to show that $\exp tA \in H$ for all real t. In turn, it suffices to show that $\dot\gamma = 0$ for $\gamma(t) = \psi(\exp tA)$. We compute

$$\begin{aligned}\dot\gamma(t) &= \psi_*\left(A_{\exp tA}\right)\\ &= \left(\tau_{\exp tA} \circ \psi \circ L_{\exp(-tA)}\right)_*\left(A_{\exp tA}\right)\\ &= \left(\tau_{*\exp tA} \circ \psi_{*1}\right) = 0\,.\end{aligned}$$

Therefore $\gamma(t) = x_o$ and $\exp tA \in H$ for all $t \in \mathbb{R}$. Then the ImFT implies that the ranks are correct. □

It follows that $\dim(G/H) = \dim G - \dim H$.

Remark 7.7.7 Another characterization of the differential structure on G/H is that it is the unique one making the standard action (p. 18) $G \times G/H \to G/H$ of Theorem 1.2.3.13 smooth. {cosetr1}

Theorem 7.7.8 *If G is a Lie group and H is a closed normal subgroup, then G/H is a Lie group, and $\mathcal{L}(G/H) \cong \mathfrak{g}/\mathfrak{h}$.* {norsg}

This means that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$; that is, $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.

Proof: Let $\sigma_1 : W_1 \to G$ and $\sigma_2 : W_2 \to G$ be local slices on open neighborhoods W_1 of $g_1 H$ and W_2 of $g_2 H$ respectively. Then the map $(g_1 H, g_2 H) \mapsto g_1 g_2^{-1} H$ is represented locally by $\pi \circ m \circ (\sigma_1 \times \sigma_2)$ where $m : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2^{-1}$. By construction, $\pi : G \to G/H$ is a Lie group morphism. By Proposition 7.5.2.6, $\ker \mathcal{L}\pi = \mathcal{L}(\ker \pi) = \mathfrak{h}$ so $\mathcal{L}\pi$ is an isomorphism $\mathfrak{g}/\mathfrak{h} \to \mathcal{L}(G/H)$. □

Ex 7.7.9 A subgroup is normal if and only if its subalgebra is an ideal.

Note that if H is not closed, then G/H is not Hausdorff.

Remark 7.7.10 A foliation of a manifold M is said to be *regular* if for every $p \in M$ there exists a cubical chart (U, φ) centered at p such that each leaf (coset) intersected with U is at most one slice. A theorem of Palais [64, p. 19] states that a foliation of M is regular if and only if it has a smooth quotient space. In particular, it does not require that the quotient be Hausdorff. This means that smoothness and Hausdorffness have essentially nothing to do with each other for coset spaces.

We now look at some examples of homogeneous spaces.

{sphere} **Example 7.7.11 (sphere)** Consider $S^n \subseteq \mathbb{R}^{n+1}$, the real n-sphere embedded in $n+1$-dimensional real space with usual basis $(e_1, \dots, e_{n+1})$. There is a natural action of O_{n+1} on S^n as Euclidean motions in $\mathbb{R}^{n+1}$. This action is clearly transitive. The isotropy group at the north pole e_{n+1} looks like this

$$\begin{bmatrix} O_n & 0 \\ 0 & 1 \end{bmatrix}$$

and is a closed subgroup. Therefore $S^n \cong O_{n+1}/O_n$ is a diffeomorphism. Similarly, $S^n \cong SO_{n+1}/SO_n$.

{spherec} **Example 7.7.12 (odd sphere)** Now consider $S^{2n-1} \subseteq \mathbb{C}^n$ with the usual (complex) basis $(e_1, \dots, e_n)$. There is the natural action of U_n on S^{2n-1} as Hermitian motions in $\mathbb{C}^n$. Again the action is transitive, and the isotropy group at e_n looks like this.

$$\begin{bmatrix} U_{n-1} & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore $S^{2n-1} \cong U_n/U_{n-1}$, and similarly $S^{2n-1} \cong SU_n/SU_{n-1}$.

Note that $SU_1 = \{1\}$ so that $SU_2 \cong S^3$, and S^3 is a Lie group thereby. Recall that S^1 is also a Lie group. No other nontrivial spheres are Lie groups [68].

{proj} **Example 7.7.13 (real projective space)** Recall that $\mathbb{P}^n \cong S^n/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting *via* the antipodal map. Thus S^n is a covering space of $\mathbb{P}^n$, making $\mathbb{P}^n$ a smooth manifold. Now SO_{n+1} acts transitively on S^n, hence on $\mathbb{P}^n$.

Recall that $\det : O_n \to O_1 = \{\pm 1\} \cong \mathbb{Z}_2$. Thus the isotropy group at $[e_{n+1}] \in \mathbb{P}^n$ looks like this

$$\begin{bmatrix} g & 0 \\ 0 & \det g \end{bmatrix}$$

where $g \in O_n$. Therefore $\mathbb{P}^n \cong SO_{n+1}/O_n$ when O_n is embedded in SO_{n+1} as indicated.

Example 7.7.14 (complex projective space) Complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$ will be expressed first as a quotient space of S^{2n-1}. In this case we consider complex lines through the origin. These intersect S^{2n-1} in copies of S^1. Consider the natural transitive action of SU_n on S^{2n-1} and the isotropy group at e_n looks like SU_{n-1}. If we embed U_{n-1} in SU_n as {projc}

$$\begin{bmatrix} g & 0 \\ 0 & \frac{1}{\det g} \end{bmatrix}$$

and recall that $\det g \in S^1$ for all $g \in U_{n-1}$, then this is the isotropy group of $[e_n] \in \mathbb{P}^{n-1}(\mathbb{C})$. Therefore we obtain the diffeomorphism $\mathbb{P}^{n-1}(\mathbb{C}) \cong SU_n/U_{n-1}$.

Example 7.7.15 (Stiefel manifold) Let $V_k(n)$ be the set of all k-frames in $\mathbb{R}^n$, $k = 1, \ldots, n$. Let GL_n act on $\mathbb{R}^n$ in the usual way and recall that this action of GL_n preserves linear independence. Now define an action $\tau : GL_n \times V_k(n) \to V_k(n)$ in the obvious way. If $v, w \in V_k(n)$, then there is a $g \in GL_n$ such that $\tau_g v = w$. Suppose H is the subgroup fixing v. Then elements of H look like this {stief}

$$\begin{bmatrix} I_k & A \\ 0 & B \end{bmatrix},$$

so H is closed and the map $gH \mapsto \tau_g v$ is a bijection $GL_n/H \to V_k(n)$. Now require this map to be a diffeomorphism. We call $V_k(n)$ the *Stiefel manifold* (or variety) of k-frames in $\mathbb{R}^n$. It has dimension kn.

Ex 7.7.16 Express the Stiefel manifold of orthonormal frames as a homogeneous space.

Example 7.7.17 (Grassmannian) Let $G_k(n)$ be the set of all k-planes in $\mathbb{R}^n$. We have an action $\tau : O_n \times G_k(n) \to G_k(n)$ that is transitive. For P the k-plane spanned by the first k elements of some ordered basis of $\mathbb{R}^n$, the isotropy group H at P looks like {grass}

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

for $A \in O_k$ and $B \in O_{n-k}$. Clearly H is closed. Then $\varphi : O_n/(O_k \times O_{n-k}) \to G_k(n)$ is a homeomorphism, thus also a diffeomorphism. Then $G_k(n)$ is a homogeneous space of dimension $k(n-k)$. We call $G_k(n)$ the *Grassmannian* or *Grassmann manifold* (or variety) of k-planes in $\mathbb{R}^n$.

Ex 7.7.18 Replace O_n by GL_n *et seq.* and obtain the same result.

{hconn1} **Proposition 7.7.19** *If H is a closed subgroup of G, and if H and G/H are connected, then so is G.*

Proof: For nonempty open sets, write $G = U \cup V$. Then $G/H = \pi(U) \cup \pi(V)$ is also a union of nonempty open sets. As G/H is connected, there exists a point (coset) gH in the intersection of $\pi(U)$ and $\pi(V)$. Combining these, we get $gH = (gH \cap U) \cup (gH \cap V)$ each open in gH, whence $(gH \cap U) \cup (gH \cap V)$ is not empty since gH is homeomorphic to H. Thus $U \cap V \neq \varnothing$ and G is connected. □

{hconn2} **Theorem 7.7.20** *Each of SO_n, SU_n, and U_n is connected and O_n has 2 components for $n \geq 1$.*

Proof: Observe that SO_1 and SU_1 are trivially connected, and $U_1 \cong \{x \in \mathbb{C} \mid |z| = 1\} \cong S^1 \cong \mathbb{T}^1$ is connected. Using the spheres as homogeneous spaces, the proof follows by "bootstrapping" on n. □

For the last example we look at a matrix Lie group that is not classical and its homogeneous manifold modulo a discrete subgroup.

{dcs} **Definition 7.7.21** Let $\mathfrak{g}$ be a Lie algebra and set $\mathfrak{g}_0 := \mathfrak{g}$. We define the *descending central series* of $\mathfrak{g}$ iteratively by $\mathfrak{g}_{k+1} := [\mathfrak{g}, \mathfrak{g}_k]$.

{nilp} **Definition 7.7.22** A Lie algebra $\mathfrak{n}$ is *nilpotent* if and only if $\mathfrak{n}_k = 0$ for some integer k. For k the smallest such integer, $\mathfrak{n}$ is called *k-step nilpotent*. It follows that the Lie algebra $\mathfrak{n}$ is abelian if and only if $k = 1$.

Ex 7.7.23 If $\mathfrak{n} \neq 0$ is nilpotent, then the center $\mathfrak{z}$ of $\mathfrak{n}$ is nontrivial: $\mathfrak{z} \neq 0$.

{nilpg} **Definition 7.7.24** A Lie group N is *nilpotent* if and only if its Lie algebra $\mathfrak{n}$ is nilpotent.

{expnilp} **Theorem 7.7.25** *If N is a connected nilpotent Lie group, then* $\exp : \mathfrak{n} \to N$ *is a covering map.*

In particular, if N is 1-connected then exp is a diffeomorphism.

Ex 7.7.26 Prove the theorem. If $\mathfrak{n} \neq 0$, then $\mathfrak{z} \neq 0$, $\mathfrak{n}/\mathfrak{z}$ is nilpotent, and $\dim(\mathfrak{n}/\mathfrak{z}) < \dim(\mathfrak{n})$. Now induct (or bootstrap) on $\dim \mathfrak{n}$.

Example 7.7.27 (Heisenberg group) We consider the subgroup of GL_3 consisting of all matrices of the form {h3}

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

This Lie group is called the *Heisenberg group* and usually is denoted by H_3. It is diffeomorphic to $\mathbb{R}^3$, 1-connected, and 2-step nilpotent; *i.e.*, H_3 is a *nilmanifold*.

Let Γ_3 be the subgroup of H_3 with $x, y, z \in \mathbb{Z}$. Then the quotient $\Gamma_3 \backslash H_3$ is a compact principal $\mathbb{T}^1$-bundle over $\mathbb{T}^2$. Such a space is also called a nilmanifold, sometimes an *infranilmanifold* since it is covered by a nilmanifold. The Heisenberg group has no compact subgroups.

Example 7.7.28 We may define a family of generalized Heisenberg groups, although we must be careful calling them that as this name has other meanings. Define H_{2n+1}, $n \geq 1$, to be the subgroup of GL_{n+2} comprised of matrices of the form

$$\begin{bmatrix} 1 & x_1 & \cdots & x_n & z \\ & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ & & & & 1 \end{bmatrix}$$

with zeros making up the missing entries. In analogy with the previous example, we may form the discrete subgroup $\Gamma_{2n+1} \leq H_{2n+1}$ such that $x_i, z, y_i \in \mathbb{Z}$. The quotient $\Gamma_{2n+1} \backslash H_{2n+1}$ is a compact principal $\mathbb{T}^1$-bundle over $\mathbb{T}^{2n}$. Again, H_{2n+1} has no compact subgroups.

Suppose $\mathfrak{g}$ is a Lie algebra of dimension n with basis $\{e_1, \ldots, e_n\}$ as a vector space. By (bi)linearity it suffices to specify all $[e_i, e_j]$.

Definition 7.7.29 The *structure equations* of $\mathfrak{g}$ are given by {se}

$$[e_i, e_j] = c_{ij}^k e_k$$

where the summation convention is obeyed, and we may as well assume $i < j$. The coefficients c_{ij}^k are called the *structure constants*.

Ex 7.7.30 Is c_{ij}^k a tensor? If so, which kind? If not, what is it?

{rs} **Definition 7.7.31** We say $\mathfrak{g}$ has a *rational structure* if and only if there exists a basis on which all c_{ij}^k are rational. Such a $\mathfrak{g}$ is simply said to be *rational Lie algebra.*

{nLgdsg} **Theorem 7.7.32** *A nilpotent Lie group N has a cocompact discrete subgroup if and only if $\mathfrak{n}$ is rational.*

7.8 Adjoint representations

Recall from Example 1.2.1.21 that a Lie group G acts on itself by conjugation, $\kappa : G \to \operatorname{Aut} G$. The functor $\mathcal{L}$ maps the G-group G to the G-algebra $\mathfrak{g}$ as in Proposition 1.2.1.14. Then the induced action on $\mathfrak{g}$ is given by

$$G \xrightarrow{\kappa} \operatorname{Aut} G \xrightarrow{\mathcal{L}} GL(\mathfrak{g})$$

where $GL(\mathfrak{g})$ is the general linear group of the vector space $\mathfrak{g}$.

{Ad} **Definition 7.8.1** The *adjoint representation* of G on $\mathfrak{g}$ is the one induced by conjugation: $\operatorname{Ad}_g = \mathcal{L}(\kappa_g) \in GL(\mathfrak{g})$. Alternatively, it is the induced tangent functor of conjugation at the identity: $\kappa_{g*} : T_1 G \to T_1 G$.

{ad} **Definition 7.8.2** The adjoint representation of $\mathfrak{g}$ on itself is given by the induced tangent map of the representation $\operatorname{Ad} : G \to GL(\mathfrak{g})$. Thus we define $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by

$$\operatorname{ad} = T \operatorname{Ad} = \mathcal{L} \operatorname{Ad} = \operatorname{Ad}_*$$

so that ad is a morphism of Lie algebras, $\mathfrak{gl}(\mathfrak{g})$ being the commutator Lie algebra of Example 2.6.2.

{adeqv} **Theorem 7.8.3** *If G and G' are Lie groups and $\rho : G \to G'$ is a morphism of Lie groups, then $\mathcal{L}\rho : \mathfrak{g} \to \mathfrak{g}'$ is ρ-equivariant with respect to the adjoint representations of G and G' on $\mathfrak{g}$ and $\mathfrak{g}'$ respectively.*

Proof: The diagram

$$\begin{array}{ccc} G & \xrightarrow{\rho} & G' \\ {\scriptstyle \kappa_g}\downarrow & & \downarrow{\scriptstyle \kappa_{\rho g}} \\ G & \xrightarrow[\rho]{} & G' \end{array}$$

commutes for all $g \in G$ by the second part of Example 1.2.1.25. The result follows from the functoriality of $\mathcal{L}$. □

Consider the natural isomorphism $\mathcal{B} : T_1G \to \mathcal{L}G$ that was introduced after Theorem 7.2.7. From Theorem 7.2.17 it follows that the effect of Ad_g on T_1G is given by $\kappa_{g*} : T_1G \to T_1G$. Thus the action of G on T_1G in Example 7.1.22 is just the adjoint representation.

Ex 7.8.4 Another description of the adjoint representation is given by

$$\mathrm{Ad}_g A = \left(R_{g^{-1}}\right)_* A.$$

This shows that the action of G on itself by right translation induces a right action of G on $\mathcal{L}G$, and that the adjoint representation describes the effect.

Now we present a plethora of commutative diagrams. Recall that $\mathrm{ad} = T(\mathrm{Ad})$. This gives us the following commutative diagram.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & \mathfrak{gl}(\mathfrak{g}) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\mathrm{Ad}]{} & GL(\mathfrak{g}) \end{array}$$

Next, using the definition of Ad *via* κ_g,

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_g} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\kappa_g]{} & G \end{array}$$

so $\exp\left(t\,\mathrm{Ad}_g A\right) = \kappa_g\left(\exp tA\right)$. If $G = GL(V)$, then we get

$$\begin{array}{ccc} \mathfrak{gl}(V) & \xrightarrow{\mathrm{ad}} & GL(\mathfrak{gl}(V)) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ GL(V) & \xrightarrow[\mathrm{Ad}]{} & \mathfrak{gl}(\mathfrak{gl}(V)) \end{array}$$

$$\begin{array}{ccc} \mathfrak{gl}(V) & \xrightarrow{\operatorname{Ad}_B} & \mathfrak{gl}(V) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ GL(V) & \xrightarrow[\kappa_B]{} & GL(V) \end{array}$$

so that $\operatorname{Ad}_B A = BAB^{-1}$. Indeed, we compute

$$\begin{aligned} \operatorname{Ad}_B A &= \frac{d}{dt}\Big|_{t=0} \kappa_B \exp tA \\ &= \frac{d}{dt}\Big|_{t=0} B e^{tA} B^{-1} \\ &= \frac{d}{dt}\Big|_{t=0} e^{tBAB^{-1}} = BAB^{-1}. \end{aligned}$$

The most interesting results occur when $V = \mathfrak{g}$.

{adrep} **Proposition 7.8.5** *If G is an arbitrary Lie group with $A, B \in \mathfrak{g}$, then*

$$\operatorname{ad}_A B = [A, B]\,.$$

Proof: We compute

$$\begin{aligned} \operatorname{ad}_A B &= \left(\frac{d}{dt}\Big|_{t=0} \operatorname{Ad}(\exp tA) \right) B \\ &= \frac{d}{dt}\Big|_{t=0} \operatorname{Ad}_{\exp tA} B \\ &= \frac{d}{dt}\Big|_{t=0} (\kappa_{\exp tA})_* B. \end{aligned}$$

Let Φ denote the flow of $A \in \mathcal{L}G$, and continue with the computation.

$$\begin{aligned} \operatorname{ad}_A B(1) &= \frac{d}{dt}\Big|_{t=0} \left(R_{\exp(-tA)}\right)_* (L_{\exp tA})_* B(1) \\ &= \frac{d}{dt}\Big|_{t=0} \left(R_{\exp(-tA)}\right)_* B|_{\exp tA} \\ &= \frac{d}{dt}\Big|_{t=0} \Phi_{-t*} B_{\Phi_t(1)} \\ &= \pounds_A B(1) = [A, B](1) \end{aligned}$$

The final line follows from Ex 4.6.8. Left-invariance then implies that the result holds everywhere. □

This shows that ad is a linear morphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

{alaad} **Remark 7.8.6** For an abstract Lie algebra (not necessarily of a Lie group), if we define ad by the formula $\mathrm{ad}_x\, y := [x, y]$ then it follows from the Jacobi identity, Definition 2.6.1 item 2, that ad is a Lie algebra morphism. Indeed, merely rewrite the Jacobi identity as $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$ whence $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$.

It turns out that the rewritten version is preferred when looking at related kinds of algebras, such as Poisson, Jordan, Dirac, or Frölicher-Nijenhuis.

Theorem 7.8.7 *If H is a connected Lie subgroup of a connected Lie group G, then $H \trianglelefteq G$ if and only if $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.* {norsg1}

Proof: Assume $\mathfrak{h}$ is an ideal, $A \in \mathfrak{g}$, $B \in \mathfrak{h}$, and let $g = \exp A$. Then

$$\begin{aligned} \kappa_g \exp B &= \exp(\mathrm{Ad}_g\, B) \\ &= \exp\left(\exp \mathrm{ad}_A\right) B \\ &= \exp\left[B + [A, B] + \frac{\mathrm{ad}_A^2}{2!} B + \cdots\right]. \end{aligned}$$

As $\mathfrak{h}$ is an ideal, the series converges in $\mathfrak{h}$ so that $\kappa_g \exp B \in H$. By Ex 1.3.1.18 and Corollary 7.4.6, H is generated by elements $\exp A$. Together with the previous sentence, this implies that H is normal.

Conversely, assume H is normal, let $s, t \in \mathbb{R}$, $A \in \mathfrak{g}$, $B \in \mathfrak{h}$, and put $g = \exp tA$. By the first part, $\kappa_g \exp sB = \exp\left(\mathrm{Ad}_g\, sB\right) = \exp s\left[\left(\exp \mathrm{ad}_{tA}\right) B\right]$. Since H is normal, $\kappa_g \exp sB \in H$. It follows that $\left(\exp \mathrm{ad}_{tA}\right) B \in H$ for all real t. Now

$$\begin{aligned} \left(\exp \mathrm{ad}_{tA}\right) B &= \exp\left(t\, \mathrm{ad}_A\right) B \\ &= B + t[A, B] + \frac{t^2}{2!}\left[A, [A, B]\right] + \cdots \end{aligned}$$

is a smooth curve $\gamma \in \mathfrak{h}$ with $\dot\gamma(0) = [A, B]$. Thus $[A, B] \in \mathfrak{h}$ and $\mathfrak{h}$ is an ideal in $\mathfrak{g}$. □

Definition 7.8.8 The *center* of $\mathfrak{g}$ is given by $\mathfrak{z}(\mathfrak{g}) = \{A \in \mathfrak{g} \mid [A, B] = 0 \text{ for all } B \in \mathfrak{g}\}$, and the center of G is $Z(G) = \{g \in G \mid gh = hg \text{ for all } g \in G\}$. {ctr}

Theorem 7.8.9 *If G is connected, then $Z(G) = \ker(\mathrm{Ad})$.* {ctrt1}

Proof: Let $g \in Z(G)$ and $A \in \mathfrak{g}$. Then $\exp tA = \kappa_g \exp tA = \exp(t \operatorname{Ad}_g A)$ for all $t \in \mathbb{R}$. Thus, $A = \operatorname{Ad}_g A$ so $g \in \ker(\operatorname{Ad})$.

Conversely, if $g \in \ker(\operatorname{Ad})$, then the calculation in the first part still holds, so g commutes with every element sufficiently near $1 \in G$. As G is connected, g commutes with every element of G, so $g \in Z(G)$. □

{ctrc1} **Corollary 7.8.10** *If G is connected, then $Z(G)$ is closed with Lie algebra $\mathfrak{z}(\mathfrak{g})$.* □

{ctrc2} **Corollary 7.8.11** *A connected Lie group G is abelian if and only if $\mathfrak{g}$ is.* □

{dim3x} **Example 7.8.12** We continue from Example 7.6.5. Observe that $O(\beta)$ acts in V by automorphisms of the Lie algebra structure. The isomorphism $\mathfrak{J}$ defines therefore a representation $\sigma : O(\beta) \to \mathcal{L}O(\beta)$. Now σ is defined by $\sigma_g A(v) = [ga, v]$ for $g \in O(\beta)$, $a \in V$, $A = \mathfrak{J}(a)$, and every $v \in V$. Observe that σ is the adjoint representation of $O(\beta)$ because $(\sigma_g A)v = [ga, v] = g[a, g^{-1}v] = gA(g^{-1}v) = \kappa_g Av$ for all $v \in V$, whence $\sigma_g A = \kappa_g A$.

{kf} **Definition 7.8.13** For $A, B \in \mathfrak{g}$, define $\beta(A, B) = \operatorname{tr}(\operatorname{ad}_A \circ \operatorname{ad}_B)$, the *Killing form* on $\mathfrak{g}$.

{kfx1} **Ex 7.8.14** The Killing form is a bi-invariant, symmetric bilinear form.

{kfx2} **Ex 7.8.15** A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\beta = 0$.

{ss} **Definition 7.8.16** We say that $\mathfrak{g}$ is *semisimple*(*simple*) if and only if β is nondegenerate (and $\mathfrak{g}$ has no proper ideals). If $\mathfrak{g} = \mathcal{L}G$, we also say that G is semisimple (simple).

There is a complete classification of simple Lie algebras over $\mathbb{C}$ and $\mathbb{R}$, and this yields a complete structure theory of semisimple Lie algebras and groups over both $\mathbb{C}$ and $\mathbb{R}$.

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Natural intraentry ordering is preferred; reverse ordering is deprecated.