EXAMPLES OF SECTIONAL CURVATURE
WITH PRESCRIBED SYMMETRY
ON 3–MANIFOLDS

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Abstract  In a previous paper, we determined the possible pointwise symmetry groups of sectional curvature considered as a rational function. We determined the naturally reductive homogeneous spaces with constant symmetry, and gave general descriptions of some examples of them. Here, we exhibit explicit forms of the metric tensors on some of these examples. We also give some inhomogeneous examples utilizing warped products, and begin the study of how the symmetry type can vary on a connected space.


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1 Introduction

Let $M$ be a smooth 3-manifold and $g$ a pseudoriemannian metric tensor on $M$. Let $G_2(M)$ denote the Grassmannian bundle with fibers $G_2(T_xM)$, the space of (2-dimensional) planes in the tangent space $T_xM$ at a point $x \in M$. Observe that each $G_2(T_xM)$ may be regarded as a (real) algebraic variety, diffeomorphic to the (real) projective plane $\mathbb{P}^2$. As in [3, 5], we shall regard the sectional curvature $K_x$ at each point $x \in M$ as a rational mapping of algebraic varieties $G_2(T_xM) \to \mathbb{R}$, or a rational function for short. The group of all automorphisms of $G_2(T_xM)$ is isomorphic to $PGL_3 \equiv PGL_3(\mathbb{R})$, the group of projective automorphisms of $\mathbb{P}^2$.

In [5] we determined the possible symmetry groups of $K$ at $x$; i.e., the largest subgroup of $PGL_3$ which leaves $K_x$ invariant as a rational function. We shall refer to any one of these as a sectional curvature symmetry, or SCS for short. We determined the existence of naturally reductive homogeneous spaces with constant SCS, and gave general descriptions of some examples of them.

In this paper, we exhibit explicit forms of the metric tensors on some of these examples (§4). We also give some inhomogeneous examples utilizing warped products (§3), and begin the study of how the SCS and CF-type can vary on a connected space (§5).

Our Lorentzian metric tensors will have signature $+--$. When necessary, we distinguish among the possible orderings $+ --$, $- + -$, $- - +$. (To convert to the other signature convention $++-$, see [9, p.92].) Thus a vector $v$ is timelike if $g(v, v) > 0$, lightlike or null if $g(v, v) = 0$, spacelike if $g(v, v) < 0$, and causal if $g(v, v) \geq 0$.

When convenient, we regard the Riemann tensor $R_{ijkl}$ as a quadratic form on $\Lambda^2 TM$; cf. [3, 5]. In local coordinates,

$$R = \begin{bmatrix}
R_{1212} & R_{1213} & R_{1223} \\
R_{1213} & R_{1313} & R_{1323} \\
R_{1223} & R_{1323} & R_{2323}
\end{bmatrix}.$$  

Then the sectional curvature appears as a rational function on $G_2(M)$ in the form of a quotient of two quadratic functions:

$$K = \frac{R}{\Lambda^2 g}.$$  

Also recall that the associated tensor $R_{ij}^{kl}$ represents the curvature operator $\hat{R} : \Lambda^2 TM \to \Lambda^2 TM$ in local coordinates. Note that if $R$ and $\hat{R}$ are written as matrices with respect to the same local coordinates, then $R = (\Lambda^2 g) \hat{R}$.

We denote the Lorentz group in $(n = p + q)$ dimensions of signature $(p, q)$ by $O_q^p = O_q^p(\mathbb{R})$, thus the (usual) orthogonal group by $O_n = O_n(\mathbb{R})$. 

1
Projectivization of any group of linear transformations is indicated by a prefixed $P$; for example $PGL_3 = GL_3 / \{ aI : 0 \neq a \in \mathbb{R} \} \cong SL_3$.

Some of the results in Section 3 were presented by Parker at the Bolyai Colloquium on Differential Geometry in 1984. Again, Parker thanks Cordero and the Departamento at Santiago for their extraordinary hospitality during his visits.

2 Preliminary Recollections

For the convenience of the reader, we state some of the main results of [5]. Recall that $\Lambda^2 diag [1, -1, -1] = diag [-1, -1, 1]$. This is Theorem 2.2 of [5]:

**Theorem 2.1** At each point $x$ of a Lorentzian 3-manifold $(M, g)$, there exists a choice of $g$-orthonormal coordinates with respect to which the Riemann tensor $R_x$ on $\Lambda^2 T_x M$ takes on exactly one of these canonical forms:

- **CF1** $\text{diag} [b, c, a]$;
- **CF2** \[
\begin{bmatrix}
  b & 0 & 0 \\
  0 & -\lambda & F \\
  0 & F & \lambda \\
\end{bmatrix}, \quad F \neq 0;
\]
- **CF3** \[
\begin{bmatrix}
  b & 0 & 0 \\
  0 & -\lambda \pm \frac{1}{2} & \pm \frac{1}{2} \\
  0 & \pm \frac{1}{2} & \lambda \pm \frac{1}{2} \\
\end{bmatrix};
\]
- **CF4** \[
\begin{bmatrix}
  -\lambda & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
  -\frac{1}{\sqrt{2}} & -\lambda & 0 \\
  -\frac{1}{\sqrt{2}} & 0 & \lambda \\
\end{bmatrix}.
\]

We note that these forms can also be characterized in terms of eigenvectors of $\bar{R}$: timelike, spacelike, double null, and triple null, respectively; compare [8, §4.3].

We also give Table 1 of [5]. Recall the group $HT$ of horocyclic translations (called “null rotations” in relativity because there is a fixed null direction). The identity component of this group consists of the matrices

$$
\begin{bmatrix}
  1 & -t & t \\
  t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\
  t & -\frac{t^2}{2} & 1 + \frac{t^2}{2}
\end{bmatrix}, \quad t \in \mathbb{R},
$$

Each component of $O^2_3(- - +)$ contains one component of $HT$.

Finally, we state Theorem 3.1 of [5].
Table 1: Lorentzian SCS

<table>
<thead>
<tr>
<th>Canonical form of $R_x$</th>
<th>Symmetry group of $K_x = R_x / \wedge^2 g_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF1 : diag $[b, c, a]$</td>
<td>$\begin{cases} A = -b = -c &amp; PGL_3 \ b = c \neq -a &amp; PO_2 \ a = -b \neq -c \ a = -c \neq -b \end{cases}$ PO$^1_1$</td>
</tr>
<tr>
<td>CF2</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>CF3</td>
<td>$b \neq -\lambda \mathbb{Z}_2$</td>
</tr>
<tr>
<td>CF4</td>
<td>$b = -\lambda$ PHT</td>
</tr>
</tbody>
</table>

**Theorem 2.2** If $M = G$ is an irreducible, naturally reductive, Lorentzian homogeneous space of dimension 3, then either $M$ is flat or of constant positive curvature. In the former case, $M$ is Minkowskian 3-space or one of its quotients by a discrete group of translations. In the latter, $M$ is $SO^2_1(\mathbb{R})$ or one of its coverings or quotients by a discrete subgroup.

**3 Some Warped Products**

To begin, we consider Lorentzian warped products of the form $M = \Lambda \times_r \Sigma$. Here, $\Lambda$ is either the circle $S^1$ or an open interval $I \subset \mathbb{R}$ with coordinate $t$, $r$ is a positive, smooth function on $\Lambda$ (the warping function), $\Sigma$ is a surface with line element $d\sigma^2$, and the line element on $M$ is $ds^2 = dt^2 - r^2 d\sigma^2$.

We refer to [9, pp. 204–211] for the geometry of warped products. Using the formulas there, we find $\text{spec} \bar{R} = \{\lambda_1, \lambda_2\}$ with $\lambda_i$ of multiplicity $i$, $\lambda_1 = \dot{r}/r$, and $\lambda_2 = (\dot{r}/r)^2 - k/r^2$, where the dot denotes $d/dt$ and $k$ is the intrinsic (Gaussian or sectional) curvature of $\Sigma$. Thus, in appropriate local orthonormal coordinates [5], $\bar{R} = \text{diag} [\lambda_2, \lambda_2, \lambda_1]$. Roughly, one may say that $K(\text{plane} \parallel \Sigma) = \lambda_1$ and $K(\text{plane} \perp \Sigma) = \lambda_2$. Consulting Table 1 from [5], we have

**Proposition 3.1** If $\lambda_1 = \lambda_2$, then $M$ has a constant SCS of $PGL_3$, hence constant curvature. If $\lambda_1 \neq \lambda_2$, then $M$ has a constant SCS of $PO_2$.

If we regard $\lambda_1$ and $k$ as given, then we find

$$\log r = \pm \int \sqrt{\lambda_1 + k/r^2} dt$$
subject to the consistency condition $\lambda_1 \geq -k/r^2$. If we regard $\lambda_2$ as given, then it follows from ODE theory (e.g., [4]) that a suitable $r$ exists for any continuous $\lambda_2$. With a smooth metric, our $\lambda_2$ is in fact smooth. Therefore,

**Theorem 3.2** All possible $\bar{R}$ of the form $\text{diag} [\lambda_2, \lambda_2, \lambda_1]$ can be obtained from Lorentzian warped products $\Lambda \times_r \Sigma$.

It remains to determine which of them have constant curvature. This happens if and only if

$$\frac{\ddot{r}}{r} = \left(\frac{\dot{r}}{r}\right)^2 - \frac{k}{r^2},$$

whence $k$ must be constant. Solving the ODE, we obtain the solution implicitly as

$$\log r = \pm \int \sqrt{c_1 + 2k/r} \, dt + c_2$$

for appropriate constants of integration $c_1$ and $c_2$. When $k = 0$, we can write $r$ explicitly as

$$r(t) = r(0)e^{ct}$$

for an appropriate constant $c$. If $c \neq 0$, $r$ cannot be periodic. Thus the only such examples with $\Lambda = S^1$ and a flat $\Sigma$ are the flat Lorentzian warped products $S^1 \times_r \Sigma$ with constant $r$.

Reflecting back on [5] and anticipating the next section, we observe that some warped products are homogeneous spaces. For example, if $\Sigma$ is a closed, orientable surface with a Riemannian metric of constant curvature and the warping function $r$ is constant, then $\Lambda \times_r \Sigma$ is $G$-homogeneous for $G = \mathbb{R} \times SO_3$, $\mathbb{R} \times \mathbb{R}^2$, or $\mathbb{R} \times SL_2$, respectively, when $\Sigma$ has genus $g = 0, 1, \text{or } \geq 2$, respectively. In light of the present work, it would be of interest to have effective criteria for determining when a warped product is homogeneous.

In this spirit, we make a simple observation. Assume that a connected, Lorentzian 3-manifold $M$ has a constant SCS of $PO_2$. Then $\bar{R}$ has a timelike eigenvector at every point. Equivalently, there is a distinguished spacelike plane at every point. We obtain a splitting $TM = L \oplus P$ into a timelike line bundle $L$ and this spacelike plane bundle $P$. When $P$ is integrable, the constant SCS of $PO_2$ implies that the leaves are totally umbilic.

**Proposition 3.3** If $P$ is integrable, then $M$ is locally an umbilic product.

This follows from the obvious Lorentzian version of a theorem of Bishop [2]. (See also [7].) As we shall make no use of this result here, we omit the proof. For the convenience of the reader, however, we recall that an umbilic product is a generalization of a warped product in which the warping function may depend on both factors.

We continue with warped products $M = \Sigma \times_r \Lambda$, where $r$ is now a smooth, positive function on $\Sigma$, $ds^2 = r^2 \, dt^2 - d\sigma^2$, and we keep the other notations
from the first part of this section. Consulting [9] again, we find in suitable local coordinates \((t = x_1, x_2, x_3)\)

\[
\begin{align*}
R_{2323} &= k, \\
R_{1212} &= -H^r(\partial_2, \partial_2)/r, \\
R_{1313} &= -H^r(\partial_3, \partial_3)/r, \\
R_{1213} &= -H^r(\partial_2, \partial_3)/r, \\
R_{1223} &= R_{1323} = 0,
\end{align*}
\]

where \(H^r\) denotes the Hessian of \(r\) on \(\Sigma\) and \(\partial_i = \partial/\partial x_i\). By means of an appropriate change of local coordinates on \(\Sigma\), we may assume that \(R_{1213} = 0\). Then we obtain \(R\) in CF1.

We have been able to solve these equations explicitly only when \(k = 0\). Even in that case, however, we obtain examples of all SCS in CF1. Indeed, in this case the system becomes

\[
\begin{align*}
\partial_2\partial_2 r &= f, \\
\partial_3\partial_3 r &= g, \\
\partial_2\partial_3 r &= 0.
\end{align*}
\]

It follows that \(f\) is a function of \(x_2\) only and \(g\) is a function of \(x_3\) only. Thus the system can be integrated directly for any such \(f\) and \(g\); in particular,

\[
\begin{align*}
\text{for } f = g = 0 & \text{ we obtain } PGL_3, \\
\text{for } f = g \neq 0 & \text{ we obtain } PO_2, \\
\text{for } \begin{cases} f \neq g = 0 \\ f = 0 \neq g \end{cases} & \text{ we obtain } PO^1_1, \\
\text{for } 0 \neq f \neq g \neq 0 & \text{ we obtain } \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\end{align*}
\]

Joint work in progress with Dean Allison will explore these warped products further, along with doubly warped products and umbilic products.

### 4 Some Homogeneous Examples

In this section we write out explicit details and metric tensors for some of the naturally reductive examples described in [5]. We begin by noting that the case of spaces of constant curvature is well known: cf. [10] and references there. Thus we omit explicit examples with constant SCS of \(PGL_3\). In order to simplify the notation, we shall write down our calculations using normalized versions in which the structure constants in [5] are taken as \(\pm 1\) whenever possible. We give only examples \(M = G/H\) with \(\dim G = 4\); among those Lie algebras we have been able to identify and integrate, no new features appear when \(\dim G = 5\).
For the SCS of $PO_2$, we take the structure equations of $g$ as
\[
[e_2, e_3] = e_4, \\
[e_3, e_4] = -e_2, \\
[e_4, e_2] = -e_3.
\]
The left-invariant 1-forms and the dual left-invariant vector fields are
\[
\omega_1 = dx_1, \\
\omega_2 = \cos x_4 \sinh x_3 dx_2 - \sin x_4 dx_3, \\
\omega_3 = \sin x_4 \sinh x_3 dx_2 + \cos x_4 dx_3, \\
\omega_4 = \cosh x_3 dx_2 + dx_4, \\
e_1 = \partial_1, \\
e_2 = \cos x_4 \text{csch} x_3 \partial_2 - \sin x_4 \partial_3 - \cos x_4 \coth x_3 \partial_4, \\
e_3 = \sin x_4 \text{csch} x_3 \partial_2 + \cos x_4 \partial_3 - \sin x_4 \coth x_3 \partial_4, \\
e_4 = \partial_4,
\]
where $\partial_i = \partial/\partial x_i$. The quotient $M = G/H$ is topologically $\mathbb{R}^3$ and given by \{$e_4 = 0$\} or \{$x_4 = \text{const.}$\}.

Taking $\varepsilon_4 = -1$ to get the signature $+--$, we find the line element
\[
d s^2 = dx_1^2 - \cosh 2x_3 dx_2^2 - dx_3^2
\]
with a curvature matrix of $\text{diag}[0, 0, 1]$. Regarding $x_1 \pmod{1}$ as a coordinate on $S^1$, we obtain a $G$-left invariant metric on $S^1 \times \Sigma$ where $\Sigma$ is a Riemannian open surface of constant curvature $-1$. Thus there are compact models $S^1 \times \Sigma_g$ where $\Sigma_g$ is a closed surface of genus $g \geq 2$.

For the SCS of $PHT$ in $\text{CF}3$ with $\nu = -\lambda = 0$, we take the structure equations of $g$ as
\[
[e_1, e_2] = 0, \\
[e_2, e_3] = ae_4, \\
[e_3, e_1] = 0, \\
[e_1, e_4] = 0, \\
[e_2, e_4] = -be_3, \\
[e_3, e_4] = be_1,
\]
where $a = c_{23}$ and $ab = \mp 1/2$. We obtain a semidirect product $g = g_1 \rtimes_\theta \langle e_2 \rangle$ with $g_1 = \langle e_3, e_4, e_1 \rangle$ isomorphic to the Heisenberg algebra and
\[
\theta(e_2) = \begin{bmatrix} 0 & -b & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Therefore, when \( ab = -1/2 \) we have

\[
\exp(-t\theta) = \begin{bmatrix}
\cosh \frac{1}{\sqrt{2}} t & b\sqrt{2} \sinh \frac{1}{\sqrt{2}} t & 0 \\
-a\sqrt{2} \sinh \frac{1}{\sqrt{2}} t & \cosh \frac{1}{\sqrt{2}} t & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

and when \( ab = 1/2 \) we have

\[
\exp(-t\theta) = \begin{bmatrix}
\cos \frac{1}{\sqrt{2}} t & b\sqrt{2} \sin \frac{1}{\sqrt{2}} t & 0 \\
-a\sqrt{2} \sin \frac{1}{\sqrt{2}} t & \cos \frac{1}{\sqrt{2}} t & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

We consider only the case \( ab = 1/2 \). Setting \( \xi = x_2/\sqrt{2} \) for convenience, the left-invariant 1-forms and vector fields are

\[
\omega_1 = dx_1 - bx_3 dx_4,
\]
\[
\omega_2 = dx_2,
\]
\[
\omega_3 = \cos \xi dx_3 + b\sqrt{2} \sin \xi dx_4,
\]
\[
\omega_4 = -a\sqrt{2} \sin \xi dx_3 + \cos \xi dx_4,
\]
\[
e_1 = \partial_1,
\]
\[
e_2 = \partial_2,
\]
\[
e_3 = \frac{x_3}{\sqrt{2}} \sin \xi \partial_1 + \cos \xi \partial_3 + a\sqrt{2} \sin \xi \partial_4,
\]
\[
e_4 = bx_3 \cos \xi \partial_1 - b\sqrt{2} \sin \xi \partial_3 + \cos \xi \partial_4.
\]

We perform the change of coordinates

\[
y_1 = x_1 + \frac{x_3^2}{2\sqrt{2}} \cot \xi,
\]
\[
y_2 = x_2,
\]
\[
y_3 = b\sqrt{2} x_4 \sin \xi + x_3 \cos \xi,
\]
\[
y_4 = x_4 \sec \xi,
\]
and obtain (interpreting \( \xi \) as \( y_2/\sqrt{2} \) now)

\[
\omega_1 = dy_1 + \left[ \frac{y_3^2 (\sqrt{2} - 4 \sin^2 \xi)}{4\sqrt{2} \sin^2 \xi \cos^2 \xi} + \frac{y_2^2 y_4^2}{2} \right] dy_2 - \frac{y_3}{\sqrt{2}} \sec \xi \csc \xi dy_3,
\]
\[
\omega_2 = dy_2,
\]
\[
\omega_3 = dy_3 + \left( \frac{y_3}{\sqrt{2}} \tan \xi - by_4 \right) dy_2,
\]
\[
\omega_4 = dy_4.
\]
\[ \omega_1 = dy_4 - ay_3 \tan^2 \xi \, dy_2 - a\sqrt{2} \tan \xi \, dy_3, \]
\[ e_1 = \partial_1, \]
\[ e_2 = -\left( \frac{y_3 \sec \xi \csc \xi - \frac{by_4}{\sqrt{2}}}{} \right)^2 \partial_1 + \partial_2 + \left( \frac{by_4 - y_3 \tan \xi}{\sqrt{2}} \right) \partial_3 + \frac{y_4}{\sqrt{2}} \tan \xi \partial_4, \]
\[ e_3 = \frac{y_3}{\sqrt{2}} \sec \xi \csc \xi \partial_1 + \partial_3 + a\sqrt{2} \tan \xi \partial_4, \]
\[ e_4 = \partial_4, \]

with \( \partial_i = \partial/\partial y_i \) now.

On the quotient \( G/H \), we find a family of left-invariant line elements in the parameter \( y_4 \). We have not written them out here as we found no reasonable simplifications of the coefficients. Recall that compact models are possible only if \( \exp(\theta) \) preserves a suitable lattice in \( G/H \). The case \( ab = -1/2 \) may be treated similarly.

5 Varying CF and SCS

Here we wish to consider how the canonical form type (CF1–CF4) and the symmetry group (SCS) can vary as one moves about in \( M \). We observe first that with a smooth metric tensor \( g \) on \( M \), the curvature tensor \( R \) will vary smoothly. Thus, a smooth curve in \( M \) lifts to a smooth curve in the bundle of possible curvature tensors. This bundle can be identified either as \( \text{End}_{SA}(\bigwedge^2 TM) \), the selfadjoint endomorphisms bundle, or (lowering indices) as \( Q(\bigwedge^2 TM) \), the bundle of quadratic forms; cf. [5].

To begin, we determine how the CF’s and SCS’s partition the space of possible curvature tensors at a fixed point \( p \in M \). For this purpose, it suffices to fix a normal coordinate chart at \( p \). This identifies \( T_pM = \mathbb{R}^3 \) and \( \bigwedge^2 T_pM = \bigwedge^2 \mathbb{R}^3 \cong \mathbb{R}^3 \). Thus we identify the space of possible curvature tensors at \( p \) with \( \text{Sym}_3 \), the symmetric \( 3 \times 3 \) matrices regarded as quadratic forms on \( \bigwedge^2 \mathbb{R}^3 \). We coordinatize \( \text{Sym}_3 \cong \mathbb{R}^6 \) via

\[
\begin{bmatrix}
B & D & E \\
D & C & F \\
E & F & A
\end{bmatrix} \mapsto (B, C, A, D, E, F); 
\]

see [5].

If the metric tensor \( g_p = \text{diag} [1, -1, -1] \), then \( \bigwedge^2 g_p = \text{diag} [-1, -1, 1] \). A change of normal coordinates at \( p \) acts on \( T_pM = \mathbb{R}^3 \) by an element of \( O^2_1(+-) \), the Lorentz group for \( g_p \). It is an easy exercise in linear algebra to verify that \( \bigwedge^2 O^2_1(+-) = SO^2_1(-+) \). Thus we consider the action of \( SO^2_1(-+) \) on \( \text{Sym}_3 \) where \( A \cdot R = A'R \) for \( A \in SO^2_1(-+) \) and \( R \in \text{Sym}_3 \).
Constant curvature $k$ at $p$ is characterized by $R_p = k \wedge^2 g_p$. Since $SO^2_1(-,+,\cdot)$ preserves $\wedge^2 g_p$, it also preserves $R_p$ in this case: each constant curvature $R_p$ is a fixed point. These comprise a line through the origin in $Sym_3$. This is the set where the SCS is $PGL_3$: $\{b = c = -A, d = e = f = 0\}$.

For the SCS of $PO_2$, there is the plane $\{b = c\}$ in $BCA$-space, containing the line $L$ of constant curvature $\{b = c = -A\}$. The $O_2$ subgroup of $SO^2_1$ fixes each point on this plane, and all other elements of $SO^2_1$ move every point of the two half-planes complementary to $L$. The two half-planes are not interchanged by the action. We obtain two closed orbits in $\mathbb{R}^6$, $PO_2 \cap$ and $PO_2 \cap$, each homeomorphic to $\mathbb{R}^4$. Figure 1 represents a plane perpendicular to $L$ in $BCA$-space. The view is from the third octant, toward the origin along $L$. In the figure, “dim $d$” means the full orbit in $\mathbb{R}^6$ has dimension $d$.

For the SCS of $PO^1_1$, there are two planes in $BCA$-space: $\{b = -A\}$ and $\{c = -A\}$. They intersect in $L$. Each point of each half-plane complementary to $L$ is fixed by the appropriate $O^1_1$ subgroup of $SO^2_1$. The two planes are not interchanged by the action of $SO^2_1$, but the half-planes are in pairs across the plane $\{b = c\}$; see Figure 1. We obtain two closed orbits in $\mathbb{R}^6$, $PO^1_1 \cap$ and $PO^1_1 \cap$, each homeomorphic to $S^1 \times \mathbb{R}^3$, which are cones over $\{b = c = -A\}$.

For the SCS of $Z_0 \oplus Z_2$, we have the complement of the union of the planes $\{b = c\} \cup \{b = -A\} \cup \{c = -A\}$ in $BCA$-space. The discrete subgroup of $SO^2_1$ which fixes each point in this set has two elements in each component of $SO^2_1$. The set consists of six connected components, pairs of which are interchanged across the plane $\{b = c\}$ by the $SO^2_1$-action. We obtain three open orbits in $\mathbb{R}^6$, each homeomorphic to $S^1 \times \mathbb{R}^5$, which are also cones over $\{b = c = -A\}$. They are pairwise separated in $BCA$-space by the preceding orbits, as indicated in Figure 1.

$CF_2$ has the SCS of $Z_0$ and $CF_3$ has the SCS of either $Z_2$ or $PHT$. Since $R_{1323} = f \neq 0$ in either, both consist of (affine) planes parallel to one seen previously. No part of the $CF_2$ or $CF_3$ orbits lies in $BCA$-space.

The planes of $CF_2$ are parallel to $\{c = -A\}$. The discrete subgroup of $SO^2_1$ which fixes each point of these planes has elements in both components of $SO^2_1$. The two connected components $\{f > 0\}$ and $\{f < 0\}$ are interchanged by the action. We obtain one open orbit in $\mathbb{R}^6$, homeomorphic to $S^1 \times \mathbb{R}^5$, which is a cone over $\{c = -A\}$. See Figure 2.

The two planes of $CF_3$ are also parallel to $\{c = -A\}$ and are translates of the two planes of $CF_2$ at $f = \pm 1/2$. The lines $\{b = -\lambda\}$ are the translates of the line of constant curvature $L$ by $\{0, \pm 1/2, \pm 1/2, \pm 1/2\}$ in $BCAF$-space, and lie inside the small bullets of Figure 2. Each point is fixed by the $HT$ subgroup of $SO^2_1$. The two lines are not interchanged by the action of $SO^2_1$. We obtain two closed orbits in $\mathbb{R}^6$, $PHT \cap$ and $PHT \cap$, each homeomorphic to $S^1 \times \mathbb{R}^3$, which are cones over $L$. Portions of them lie inside the sets $CF_3 \cap$ and $CF_3$
Figure 1: Contiguity Relations for CF1.

\[
\begin{align*}
PO_2 & \quad I \\
-A < c = b \\
dim 4
\end{align*}
\]

\[
\begin{align*}
PO_1 & \quad I \\
-A = b < c \\
dim 4
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad I \\
A < b < c \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad II \\
A = b < -c \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad III \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
PO_1 & \quad I \\
-A = c < b \\
dim 4
\end{align*}
\]

\[
\begin{align*}
PO_1 & \quad II \\
-A = c < b \\
dim 4
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad II \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad III \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad II \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad III \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad II \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
Z_2 \oplus Z_2 & \quad III \\
A < c < -b \\
dim 6
\end{align*}
\]

\[
\begin{align*}
PGL_3, \quad b = c = -a, \quad \text{dim} \ 1
\end{align*}
\]

Figure 2: Contiguity Relations for CF2 and CF3.

\[
\begin{align*}
CF_3 & \quad II \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
c < -a
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad II \\
c + a = 2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]

\[
\begin{align*}
CF_2 & \quad \text{dim} \ 6
\end{align*}
\]

\[
\begin{align*}
CF_1 & \quad Z_2 \oplus Z_2 \\
-A < c
\end{align*}
\]

\[
\begin{align*}
CF_3 & \quad I \\
c + a = -2F \\
dim 5
\end{align*}
\]
II of Figure 2, respectively.

The complement of these lines in the CF3 planes has the SCS of \( \mathbb{Z}_2 \). Again, the discrete subgroup of \( SO^2_1 \) which fixes each point has elements in both components of \( SO^2_1 \), and the translate at \( f = \pm 1/2 \) interchanges with a correspondent at \( f = \mp 1/2 \). Two translates at the same \( f \)-level, however, do not interchanged. We obtain two orbits in \( \mathbb{R}^6 \), \( \mathbb{Z}_2 \text{ I} \) and \( \mathbb{Z}_2 \text{ II} \), each of dimension 5, which are also cones over \( \{ c = -A \} \).

The unions \( \mathbb{Z}_2 \text{ I} \cup \text{ PHT I} \) and \( \mathbb{Z}_2 \text{ II} \cup \text{ PHT II} \) are closed sets in \( \mathbb{R}^6 \), each homeomorphic to \( S^3 \times \mathbb{R}^4 \). These are the sets CF3 I and CF3 II of Figure 2, which represents a plane parallel to \( \{ c = A \} \) in \( \text{CAF} \)-space. The view is from the fourth quadrant of the \( \text{CA} \)-plane toward the origin along the line \( \{ c = -A \} \). The lines have slopes of \( \pm 1/\sqrt{2} \), and the two bullets for CF3 have coordinates \( \pm (1/\sqrt{2}, 1/2) \). Considering the \( b \)-axis as perpendicular to the page, we have a representation of a translate of the 3-space with axes \( b, c = A \), and \( f \) along the \( b \)-axis in \( \text{BCAF} \)-space. The part of the \( b \)-axis with \( b > c \) is \( \text{PO}_1 \text{ I} \) and the part with \( b < c \) is \( \text{PO}_1 \text{ II} \). The \( SO^2_1 \)-orbits of \( \text{PO}_2 \) from CF1 are the portions of the elliptic cone \( \{(2b - c + A)^2 + 4f^2 = (c + A)^2\} \) above and below the CF1 regions, \( \text{PO}_2 \text{ I} \) to the right and \( \text{PO}_2 \text{ II} \) to the left.

Finally, CF4 is a line parallel to the line \( L \) in \( \text{BCA} \)-space. Since the SCS here is 1, we obtain one closed orbit of two connected components (CF4+ and CF4−) in \( \mathbb{R}^6 \), each homeomorphic to \( S^1 \times \mathbb{R}^3 \). Both components are cones over \( L \). No part of this orbit lies in \( \text{BCAF} \)-space.

In Figure 3, we see the three coordinate planes of \( \text{DEF} \)-space. Parts (a) and (b) have \( b = c = -A \). The orbits of the other canonical forms are contained in the axes; for example, rotations of CF1 are in the \( d \)-axis, and those CF2 with \( b = -A \) and certain boosts of CF3 are in the \( f \)-axis. Part (c) still has \( b + c + 2A = 0 \), but now we allow \( b \neq c \). Note that the four CF3 points lie at \( d = \pm 1 \), not at \( d = 0 \); their \( e \)- and \( f \)-coordinates are \( \pm 1/2 \sqrt{2} \). The CF4 lines can be either \( \text{CF4}^+ \) or \( \text{CF4}^- \), depending on the values of \( b \) and \( c \). The open regions are parts of the orbit of CF2, the others all being inside the axes. While rotations act naturally in \( \text{AEF} \)-space, their action in \( \text{DEF} \)-space is more complicated. Rotations acting here on CF4 sweep out the two branches of the quartic surface \( \nu^2(e^2 + f^2) = (e^2 - f^2)^2 \). We recall that a rotation acting through an angle \( \theta \) clockwise about the \( A \)-axis in \( \text{AEF} \)-space, acts through an angle \( 2\theta \) counterclockwise about the line \( \{ b = c, d = 0 \} \) in \( \text{BCD} \)-space.

Now, at each point \( x \) of our 3-manifold \( M \), the sectional curvature \( K \) has a certain symmetry and the curvature tensor \( R \) lies in a certain \( SO^2_1 \)-orbit in \( Q(\Lambda^2 TM) \). We shall call this latter the orbit type of \( R \) at \( x \). As a first simple result, we observe

**Theorem 5.1** The SCS is constant if and only if either the orbit type of \( R \) is constant or varies between CF2 and CF3 \( \mathbb{Z}_2 \) on connected \( M \).
Figure 3: Contiguity Relations for CF4.

(a) plane \( \{ f = 0 \} \)

\[
\begin{align*}
\text{CF4}^+ & \quad \text{CF4}^- \\
D = -E & \quad D = E
\end{align*}
\]

\[
\begin{align*}
\text{CF1} & \quad \text{CF1} \\
Z_2 \oplus Z_2 & \quad Z_2 \oplus Z_2
\end{align*}
\]

\[
\begin{align*}
\text{CF2} & \quad \text{CF2} \\
D = E & \quad D = -E
\end{align*}
\]

\[
\bullet \{ B = C = -A \}
\]

(b) plane \( \{ e = 0 \} \)

\[
\begin{align*}
\text{CF4}^- & \quad \text{CF4}^+ \\
D = -F & \quad D = F
\end{align*}
\]

\[
\begin{align*}
\text{CF1} & \quad \text{CF1} \\
Z_2 \oplus Z_2 & \quad Z_2 \oplus Z_2
\end{align*}
\]

\[
\begin{align*}
\text{CF2} & \quad \text{CF2} \\
D = F & \quad D = -F
\end{align*}
\]

\[
\bullet \{ B = C = -A \}
\]

(c) plane \( \{ d = 0 \} \)

\[
\begin{align*}
\text{CF4} & \quad \text{CF4} \\
E = F & \quad E = -F
\end{align*}
\]

\[
\begin{align*}
\text{CF3} \ I & \quad \text{CF3} \ I \\
at \ D = -1 & \quad at \ D = 1
\end{align*}
\]

\[
\begin{align*}
\text{CF3} \ II & \quad \text{CF3} \ II
\end{align*}
\]

\[
\begin{align*}
\text{CF4} & \quad \text{CF4} \\
E = F & \quad E = -F
\end{align*}
\]

\[
\bullet \{ B + c + 2A = 0, B \neq c \}
\]
One now can read information on how the SCS and orbit type can vary from Figures 1–3. As an example, we state

**Proposition 5.2** If a smooth curve connects a point with SCS of $PO_2$ to a point with SCS of $PO_1$, then there exists either a point of constant curvature or a relatively open set with SCS of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ along the curve.

On the other hand,

**Proposition 5.3** If a smooth curve connects a point with SCS of $PO_1$ to a point with SCS of $\mathbb{Z}_2$ and $R$ of CF3, then there need be no points on the curve with any other orbit type of $R$. However, there will be a last point with SCS of $PO_1$.

For warped products, we note

**Theorem 5.4** In a connected warped product $\Lambda \times_r \Sigma$, if there are no points of constant curvature, then $\lambda_2 < \lambda_1$ or $\lambda_1 < \lambda_2$ everywhere.

**Corollary 5.5** In a connected warped product $\Lambda \times_r \Sigma$, the open sets $\{\lambda_2 < \lambda_1\}$ and $\{\lambda_1 < \lambda_2\}$ are separated by a closed set of points of constant curvature.

**References**


