1 Tor and Ext

Let K be a commutative ring and let A be a K-module. Consider a *free resolution* of A

$$0 \longrightarrow R \longrightarrow F \longrightarrow A \longrightarrow 0\,. \tag{1}$$

That is, a short exact sequence in which F and R are free K-modules and R stands for the relations, so that $A \cong F/R$. This does not always exist because R need not be a free K-module in general. We shall assume throughout this section that every K-module under consideration does have such a free resolution. This happens in several interesting cases: when K is a PID, and when the K-modules are finitely generated and K is a local ring or a polynomial ring over a PID. The main case to keep in mind is when $K = \mathbb{Z}$ and the modules are abelian groups.

1.1 Tor

The *torsion functor* is $\mathrm{Tor}^K(\ ,\ )$, and $\mathrm{Tor}^K(\ , B)$ measures the failure of $\otimes_K B$ to be exact. Thus for the free resolution (1) of A, we define

$$\mathrm{Tor}^K(A, B) = \mathrm{Ker}(R \otimes_K B \to F \otimes_K B)\,.$$

Ex on the torsion functor Tor

1. $\mathrm{Tor}^K(A, B) = 0$ if either A or B is free.
2. $\mathrm{Tor}^{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_{(p,q)}$ where (p, q) denotes the greatest common divisor of p and q.
3. More generally, $\mathrm{Tor}^K(K/pK, B) \cong \{b \in B \mid pb = 0\}$ for $0 \neq p \in K$.
4. $\mathrm{Tor}^K(A, B)$ is independent of the choice of the free resolution of A.
5. $\mathrm{Tor}^K(A, B) \cong \mathrm{Tor}^K(B, A)$.
6. Tor^K commutes with finite direct sums (biproducts):
$$\mathrm{Tor}^K(\oplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}^K(A_i, B)\,;$$
$$\mathrm{Tor}^K(A, \oplus_i B_i) \cong \bigoplus_i \mathrm{Tor}^K(A, B_i)\,.$$
7. More generally, this holds for arbitrary direct sums.

1.2 Ext

The *extension functor* is $\mathrm{Ext}_K(\ ,\ )$, and $\mathrm{Ext}_K(\ ,B)$ measures the failure of $\mathrm{Hom}_K(\ ,B)$ to be exact. Thus for the free resolution (1) of A, we define

$$\begin{aligned}\mathrm{Ext}_K(A,B) &= \mathrm{Coker}\left(\mathrm{Hom}_K(F,B) \to \mathrm{Hom}_K(R,B)\right) \\ &\cong \mathrm{Hom}_K(R,B)/\,\mathrm{Im}\,\mathrm{Hom}_K(F,B)\,.\end{aligned}$$

Ex on the extension functor Ext

1. $\mathrm{Ext}_K(A,B) = 0$ if A is free or if B is divisible. [We recall that B is a *divisible* K-module if and only if $kB = B$ for every $0 \neq k \in K$.]

2. The field of rationals is divisible as an abelian group, as is
$$\mathbb{Z}_{p^\infty} = \bigcup_{n=1}^{\infty} \mathbb{Z}_{p^n}$$
where p is any prime. B is a divisible abelian group if it is a direct sum of various $\mathbb{Z}_{p^\infty}$ and copies of the rationals $\mathbb{Q}$. It is also true that $\mathbb{R}$ and $\mathbb{C}$ are divisible abelian groups.

3. $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_q) \cong \mathbb{Z}_{(p,q)}$.

4. More generally, $\mathrm{Ext}_K(K/pK, B) \cong B/pB$ for $0 \neq p \in K$.

5. $\mathrm{Ext}_K(A,B)$ is independent of the choice of the free resolution of A.

6. Ext_K commutes with finite direct sums (biproducts):
$$\mathrm{Ext}_K(\oplus_i A_i, B) \cong \bigoplus_i \mathrm{Ext}_K(A_i, B)\,;$$
$$\mathrm{Ext}_K(A, \oplus_i B_i) \cong \bigoplus_i \mathrm{Ext}_K(A, B_i)\,.$$

7. More generally, for arbitrary collections,
$$\mathrm{Ext}_K(\oplus_i A_i, B) \cong \prod_i \mathrm{Ext}_K(A_i, B)\,;$$
$$\mathrm{Ext}_K(A, \Pi_i B_i) \cong \prod_i \mathrm{Ext}_K(A, B_i)\,.$$

8. $\mathrm{Ext}_K(A,B)$ is the K-module of all equivalence classes of *extensions of A by B*; that is, of all classes of short exact sequences
$$0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$$
with addition derived from direct sum (biproduct), a suitable scalar multiplication by elements of K, and the equivalence relation induced from the short 5-lemma.

2 A universal completion

Let M be any monoid and FM be the free group on the set M. Let N be the normal subgroup of FM generated by xyz^{-1} for $xy = z \in M$. Define $UM = FM/N$ and $(u : M \to UM) = (M \hookrightarrow FM \twoheadrightarrow UM)$, where $\hookrightarrow$ denotes the inclusion or insertion of the generators and $\twoheadrightarrow$ denotes the standard projection onto the quotient group.

Ex on this universal completion

1. Show that (u, UM) is universal for monoid morphisms $M \to G$ for groups G, so U is a functor and u corresponds to a natural transformation.

2. If M is abelian, so is UM and we may consider only abelian G.

3. A *semiring* satisfies all the axioms of a ring except that subtraction (existence of additive inverses) is not necessarily defined. Extend the construction of (u, UM) to semirings M, and show that the result is universal for semiring morphisms $M \to R$ for rings R.

4. Show that $U\,\mathbb{N} \cong \mathbb{Z}$ as rings.

5. Show that the category $\mathbf{Mod}_K$ of modules over a commutative ring K has the structure of a commutative semiring under $\oplus_K$ and $\otimes_K$, if equality in the axioms of a semiring is interpreted as isomorphism of K-modules.

6. Apply the functor U to the semiring $\mathbf{Mod}_K$ and discuss the resulting category.