Semidirect Products

Notations and terms not defined here are as in class, and usually as in MacLane and Birkhoff. **Ex** denotes either an example or an exercise. The choice is usually up to you the reader, depending on the amount of work you wish to do. Those which direct or expect you to verify something, however, should be done.

1 Exact sequences

We begin with the inevitable preliminaries.

Definition 1.1 *A* sequence *of groups and morphisms is given by a diagram*

$$\cdots \xrightarrow{f_{-1}} G_{-1} \xrightarrow{f_0} G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \tag{1.1}$$

which may be infinite in either or both directions. If it actually is infinite in either direction, it is said to be long.

As usual, the individual groups are frequently referred to as the *terms* of the sequence.

The most important sequences in algebra are those in which pairs of successive morphisms are nicely related.

Definition 1.2 *A sequence such as (1.1) is said to be* exact *if and only if*

$$\mathrm{im}(f_{n-1}) = \ker(f_n)$$

for every n corresponding to a term with arrows on both sides.

Ex

1. $1 \to G \xrightarrow{f} H$ is exact if and only if f is a monomorphism.
2. $G \xrightarrow{f} H \to 1$ is exact if and only if f is an epimorphism.

3. $1 \to G \xrightarrow{f} H \to 1$ is exact if and only if f is an isomorphism.

The most important exact sequences are either long or *short:* those consisting of exactly five terms with 1 on each end.

$$1 \longrightarrow N \xrightarrow{\ i\ } G \xrightarrow{\ j\ } H \longrightarrow 1 \tag{1.2}$$

These are generically described by the abbreviation SES.

Ex Here $N \trianglelefteq G$ and $H \cong G/N$.

Actually it's iN that's the normal subgroup of G, but as usual we try to suppress the insertion (or inclusion) map whenever possible. There are times and places when this may create more confusion than it provides simplification; let's hope this isn't one.

And, finally, some SES are more important than others.

Definition 1.3 *The* SES *(1.2) is said to (be a)* split *(*SES*) if and only if there exists a morphism $s : H \to G$ such that $js = 1_H$.*

Clearly these are nicer—they have an extra property. But *why* is this extra property worthwhile?

2 Semidirect products

Again, there's a preliminary definition, written suggestively.

Definition 2.1 *Let N and H be groups with a morphism $\theta : H \to \mathrm{Aut}(N) : h \mapsto \theta_h$. We define a* twisted *group structure on the* ***set*** *$N \times H$ by means of*

$$(n, h)(n', h') = (n\,\theta_h n', hh')\,.$$

The ***set*** *$N \times H$ together with this twisted multiplication is called the* semidirect *product (of N and H by θ) and denoted by*

$$N \ltimes_\theta H\,,$$

and we may suppress θ when it is clear from context or would otherwise not cause any confusion to do so.

One says that the group structure (multiplication) is twisted by θ. Then the usual product group is called the *direct* product and is either *untwisted* or *trivially twisted* by the trivial morphism which sends all of H to $1 \in \mathrm{Aut}(N)$. This is also known as the trivial representation of H on N. In general, θ is a representation or action of H on N, so H is thought of as a transformation group or group of operators on N.

Ex N is a normal subgroup of $N \ltimes_\theta H$, but H need not be. When is H normal?

Now, at last, we get to the point.

Theorem 2.2 *The* SES *(1.2) splits if and only if* $G \cong N \ltimes_\theta H$ *for some* $\theta : H \to \mathrm{Aut}(N)$.

Proof: Given a splitting $s : H \to G$, define $\theta_h(n) = s(h)\, n\, s(h)^{-1}$. Then the isomorphism is

$$G \to N \ltimes_\theta H : g \mapsto \left(g\, s\big(j(g)^{-1}\big)\, , j(g)\right)$$

with inverse $(n, h) \mapsto n\, s(h)$.

Conversely, given an isomorphism $f : G \to N \ltimes_\theta H$, define $s : H \to G : h \mapsto f^{-1}(1, h)$. □

Ex Fill in the details.

We have already seen an example of this.

Ex $E(2) \cong \mathbb{R}^2 \ltimes_\theta O(2)$ with θ the natural action of $O(2)$ on $\mathbb{R}^2$.

More generally this holds for $E(n)$, and a similar result holds for the affine group $A(n)$ with $O(n)$ replaced by $GL(n)$. Note that the essential difference between the affine and Euclidean groups (in fact, between the general linear and orthogonal groups) is the similarities in the former, although it also allows generalized similarities with different factors in each dimension. This follows from the *polar decomposition* of $GL(n)$: every matrix in $GL(n)$ can be written as a product SQ where S is positive-definite symmetric and Q is orthogonal.

Ex Explain how it follows.