FORTY TWO PROBLEMS OF FIRST DEGREE FROM DIOPHANTUS' ARITHMETICA

A Thesis by

Tinka Davis

Bachelor of Science, Sofia University, 1995

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Applied Mathematics.

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Phillip E. Parker, Committee Chair

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William H. Richardson, Committee Member

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Mara Alagic, Committee Member

DEDICATION

To my husband Brent Davis

The fear of the LORD is the beginning of wisdom,
and knowledge of the Holy One is understanding.
Proverbs 9:10

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ABSTRACT

This work brings to the audience Diophantus' problems of first degree in a literal word for word English translation from Ver Eecke's French translation of Arithmetica. In addition, these problems are accompanied by commentary in modern notation, as well as some modern and general solutions to appropriate problems.

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Chapter 1

INTRODUCTION

1.1 The works of Diophantus

Diophantus, often referred to as the *father of algebra*, is best known for his *Arithmetica*, a unique collection of 189 algebraic equations and their solutions. Little is known about the life of Diophantus and even the exact time when he lived is a matter of debate among historians. We know he must have lived after 150 BC since he quotes in his treatise *On Polygonal Numbers* the definition of polygonal number from the work of Hypsicles. On the other hand, Diophantus is quoted by Theon of Alexandria around 350 AD. This leaves us a span of about 500 years. If we consider a letter of Michael Psellus (1018 to ca.1090), a Byzantine Neo-Platonist and rhetorician, where he states that Anatolius, who became a bishop of Laodicea in 280 AD, dedicated a tract on Egyptian computation to his friend Diophantus, we may conclude that Diophantus and Anatolius are contemporaries, and perhaps Dionysius, to whom Arithmetica is dedicated, is indeed the same person who became a bishop of Alexandria around 248-265 AD. With all that said we are led to believe that Diophantus flourished in Alexandria around 250 AD or not much later. [6]

The details we have about Diophantus' personal life (and these may be totally fictitious) come from the Greek Anthology, compiled by Metrodorus around 500 AD. [9] This collection of numerical riddles contains one about Diophantus:

"*This tomb holds Diophantus. Ah, what a marvel! And the tomb tells scientifically the measure of his life. God vouchsafed that he should be a boy for the sixth part of his life; when a twelfth was added, his cheeks acquired a beard; He kindled for him the light of marriage after a seventh, and in the fifth year after his marriage He granted him a son, Alas! Late begotten and miserable child, when he had reached the measure of half his father's life,* *the chill grave took him. After consoling his grief by this science of numbers for four years, he reached the end of his life."*[1]

Based on this puzzle Diophantus married at the age of 26 and had a son who died at the age of 42, four years before Diophantus himself died at the age of 84.

1.1.1 Arithmetica

With respect to the title *Arithmetica* we must note that the meaning of arithmetica is slightly different from its modern equivalent. The Greeks distinguished between arithmetica and logistica, even though both dealt with numbers. According to Geminos [6] the object of arithmetica is the abstract properties of numbers, while logistica gives solutions to problems with specific numbers, in other words, logistica is the science of calculations. We see that in Diophantus the calculations have abstract nature and thus the distinction between logististica and arithmetica is not clear.

"*Knowing, my most esteemed friend Dionysius, that you are anxious to learn how to investigate problems in numbers, I have tried, beginning from the foundations on which the science is built up, to set forth to you the nature and power subsisting in numbers."* [15] Thus Diophantus begins his great work *Arithmetica*, the highest level of algebra in antiquity. And indeed, as promised, Diophantus starts "from the foundations by treating simple linear equation with one unknown (I:7-11,39[2]), determinate systems of first degree (I:1-6,12,13,15-21), and finally indeterminate equations of first degree (Lemmas to IV:34,35,36). This foundation, which consists of methods of solutions of problems of first degree, is the ground work for understanding and appreciating the culmination of Diophantus' work, i.e. his indeterminate equations of second and higher degree.

If we take a "bird's eye view" of *Arithmetica* [6], we see that Book I consists primarily of equations and system of equations of first degree.[3] There are a few other linear problems

[1] If x was the age of death, then $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x,\ x = 84$

[2] Book I, problems 7-11, 39

[3] In this section we discuss the extant Greek books. The Arabic books will be described in section 1.4.

scattered throughout the other books, [4] most of which are interpolated by later commentators and probably did not originally belong to Book I. Indeterminate analysis of second degree equations is the object of Books II and III, while Books IV and V treat indeterminate equations and systems of third and fourth degree.[5] The last extant Greek book, Book VI, consists of problems of constructing right-angled triangles with sides in rational numbers and satisfying various conditions. [6]

On the subject of the missing books of *Arithmetica*, we must note that of the 13 Greek books only 6 survive. Due to the freedom ancient scribes and commentators exercised to add, omit, and even change certain parts of the manuscript, we do not know for sure how the original books were divided by the author. Some manuscripts divide the six books into seven and others list the separate work *On Polygonal Numbers* as Book VII [6]. The numerous interpolated problems[6] in *Arithmetica* bear evidence of the extreme difficulty a math historian faces trying to give true credit to the original work.

In [6] Heath presents Nesselman's view that the missing books were originally placed between Books I and II, while Tannery and Heath share the opinion that it is the last and most difficult books that we lack. As time showed, neither of these views was correct. The discovery of the Arabic books in 1968 and their placement between Books III and IV [12] solved at least part of the puzzle, since they can only account for the four missing books; the last two are still a mystery.

Many scholars and great minds have dedicated their time and efforts to study *Arithmetica*: Xylender, Bachet, Fermat, and Euler, to name a few, but their focus has always been the indeterminate equations and systems of equations of higher degree (Books II-VI), which is not surprising, since these problems are without a doubt, the pinnacles of Greek Algebra. The problems in Book I, however, even though easier than the rest, are the true foundation and the proper doorway to a genuine appreciation and understanding of the remaining work.

[4] II:6, II:1-5, 7, 17,1 8, IV:33, 36, Lemmas to IV:34, 35, 36

[5] There are only two problems of 4th degree in the Greek books (V:18, 29).

[6] problems contributed voluntarily by later commentators and embedded in the manuscript as part of the book, such as II:1-7,17,18 and others

Repeatedly in Book I (I:1-3, 5, 6, 12, 13, 20) Diophantus asks us to divide a given number into two or three parts with certain conditions leading to solving linear equations or systems of linear equations, while in Books II-V he states similar problems leading to equations or systems of equations of higher degree: for example in II:8 and II:9 a given number must be divided into two squares; in II:14, 15, IV:24, 31 and V:10 a given number must be divided into two parts plus additional conditions involving squares; in II:18, III:20, 21, IV:25, 32 and V:11-13, 20 a given number must be divided into three parts; in V:14 a given number must be divided into 4 parts; and in IV:1 a given number must be divided into two cubes. There is no question that solving second and higher degree equations and systems of equations requires more rigor and manipulation than treating a similar problem of first degree, but the process of naming the unknown quantities, setting the equation or system of equations, and the general approach, is very similar and uniquely Diophantine.

The construction of each problem in *Arithmetica* follows this pattern:

(a) General problem (*To divide a given number into two numbers such that a given fraction of the first number exceeds a given fraction of the second number by a given number.*) Book I:6.

(b) Necessary condition[7] (*It is necessary, that the later given number be smaller than the number obtained when the greater given fraction is taken of the first given number.*) Book I:6.

(c) Specific values for the given number(s) (*Let it be required to divide 100 into two numbers such that the difference of one fourth of the first number and one sixth of the second number be 20 units.*) Book I:6.

d) Detailed solution in paragraph form followed by careful check that the found quantities indeed satisfy the given conditions (*Let one sixth of the second number be 1 arithmos, then this number will be 6 arithmoi. From then on, one forth of the first number will be 1*

[7] Not all problems have necessary condition. Some necessary conditions prevent having negative solutions since Diophantus avoided these as being absurd numbers, such as the condition in I:6; others are genuinely necessary for the proper solution of the equation, such as the condition in V:11.

arithmos plus 20 units; so the second number will be 4 arithmoi plus 80 units. Moreover, we want the numbers added together to produce 100 units. Now, these two numbers added together form 10 arithmoi plus 80 units, which equal to 100 units. Subtract like from like: it remains that 10 arithmoi are equal to 20 units, and the arithmos becomes 2 units. Returning to our conditions, it was proposed that one sixth of the second number be 1 arithmos, which is 2 units, then the second number will be 12 units. On the other hand, since one fourth of the first number is 1 arithmos plus 20 units, which is equal to 22 units, then the first number will be 88 units. From then on, it is established that one fourth of the first number exceeds one sixth of the second number by 20 units and that the sum of the required numbers is the given number.) Book I:6.

The object of Diophantus' problems is simply finding a positive rational solution, which in the case of indeterminate equations, is not sufficient. Therefore, Diophantus is often criticized for not accounting for all solutions of indeterminate problems and his unwillingness to accept non-positive numbers, as well as the lack of general methods. All that being true, we must understand that these expectations are unrealistic and premature considering the insignificant history of indeterminate analysis before Diophantus[8] and the general practice of working exclusively with positive numbers at his time.[9] If a person undertakes the task of studying a new field, such as the indeterminate analysis, isn't it natural to start with just finding a solution? Even that is sometimes hard enough. The next natural step is of course classification of problems and a general method for solving which will produce all solutions. This task Diophantus left to his successors.

Another obvious imperfection of *Arithmetica* is the lack of obvious organization and classification of problems, at least in the eyes of a person just getting acquainted with the book. Thus Burton [3] on p. 210 comments that *his methods varied from case to case, and*

[8] One of the few indeterminate problems before Diophantus, the so called Archimedean Cattle problem, was not solved until 20th century.

[9] The existence and validity of negative as well as positive roots was first affirmed by the Hindu mathematician Bhaskara (born 1114). Europeans have admited them only since the 16th or 17th century. [3]

there was not a trace in his work of a systematic theory. Each question required its own special technique, which would often not serve for the most closely related problems.

Although numerous examples in *Arithmetica* can disprove the above statement we will present here just one comparison: I:17, as printed in [3] and the problem proceeding it in Book I, namely I:16. For the purpose of clarity we will use modern notation for I:16 as taken from [6].

Book I:16. *To find three numbers such that the sums of pairs are given numbers.*

Let $(1) + (2) = 20$, $(2) + (3) = 30$, $(3) + (1) = 40$.

Let x *be the sum of the three. Therefore the numbers are* $x - 30$, $x - 40$, $x - 20$. *The sum* $x = 3x - 90$ *and* $x = 45$. *The numbers are* 15, 5, 25.

Book I:17. *Find four numbers such that when any three of them are added together, their sum is one of four given numbers. Say the given sums are 20, 22, 24, and 27. Let* x *be the sum of all four numbers. Then the numbers are just* $x - 20$, $x - 22$, $x - 24$, *and* $x - 27$. *(For instance, if* $(1) + (2) + (3) = 20$, *then when* (4) *is added to both side of this equation,* $x = (1) + (2) + (3) + (4) = 20 + (4)$ *or* $(4) = x - 20$.*) It follows that*

$$x = (x - 20) + (x - 22) + (x - 24) + (x - 27)$$

and so $3x = 93$, *or* $x = 31$. *The required numbers are therefore* 11, 9, 7, *and* 4.

We see that both problems are very similar: I:16 asks to find three numbers, while I:17 asks for four numbers; as well as their method of solution, in I:16 the sum of the three required numbers is x and in I:17 the sum of the four required numbers is x. Setting and solving the equation in each case is almost identical, with the exception that there is one more term in I:17.

It is important to remark that the problems of first degree, once set up as equations, are quite easy to solve; it is the process of defining the variable(s) and describing the relationships between the quantities that is challenging. Thus, Diophantus, allowing himself to use only one variable for the unknown, is very careful in making these choices. There are several occasions where he defines in the same problem a second unknown with the same name *arithmos* but each time he works both parts so independently that there is not much room for confusion[10]. In dozens of *Arithmetica's* problems where solving by using systems of equations is a natural approach for the modern math student, Diophantus succeeds in finding solutions with more primitive and yet elegant methods. Despite the fact that Diophantus uses only one variable for the unknown and works under severe notational handicap having to state in paragraph form some rather long and complicated mechanical procedures, his work represents the highest level of algebra in the Greek mathematics. This is to say that lack of tools is never a good excuse for ignorance.

On the other hand, setting his problems using modern methods, such as systems of two or more equations, is elementary, but arriving at the solution(s) this way is sometimes bothersome and discouraging (See problem I:25 in section 3.4). As any true mathematician will agree obtaining solution(s) with a quick and elegant approach is a matter of mathematical virtue, and this is exactly what Diophantus reveals: a surprisingly slick and unusual method.

In conclusion, although Diophantus' book, with no table of contents, references or overt structure, is not even publishable based on our modern standards, to the real Diophantine follower that will present no difficulty. He will judge the book not by its *cover*, but by the *content* of its character. Each problem has its own value and even a kind of repetitive order. The one hungry to feed his mind with the unsurpassed wisdom of the Greeks will receive his reward.

1.1.2 On polygonal numbers

In addition to Arithmetica, Diophantus' name is associated with the following treatises: *On Polygonal Numbers, The Porisms* (or *Porismata*), *Teaching of the Elements of Arithmetic, and Moriastica.* [4, 6, 11] Of these, only a fragment of *On Polygonal numbers* survives, from which we detect little originality, but mainly a review of topics long known to the earlier Greeks.

After introducing polygonal numbers as numbers greater than or equal to 3 and

[10] See I:20 in section 3.2

having as many angles as they have units,[11] Diophantus proceeds with propositions regarding numbers in arithmetic progression. The first proposition states that:

If there are three numbers with common difference[12], *then 8 times the product of the greatest and middle plus the square of the least is equal to a square, the side of which is the sum of the greatest and twice the middle number.*[13]

Faithful to the Greek tradition, Diophantus applies deductive geometric proofs throughout the treatise. For the three numbers with a common difference, i.e., in arithmetic progression, he chooses three segments such that their lengths represent numbers in arithmetic sequence.

The second problem in the treatise is notable because it gives a formula for a general term in arithmetic sequence:

If there are any numbers, as many as we please, in arithmetic progression, the difference between the greatest and the least is equal to the common difference multiplied by the number of terms less one.

In other words, if in arithmetic progression a_1 is the first term, a_n the greatest term,[14] and d the common difference, then $a_n - a_1 = d(n-1)$ or $a_n = a_1 + d(n-1)$.

Another familiar formula is found in the third proposition:

If there are as many numbers as we please in arithmetic progression, then the sum of the greatest and the least multiplied by the number of terms is equal to twice the sum of the terms. In other words, if a_1 is the first term in arithmetic sequence, a_n the last (or, greatest in this case, since negative numbers are excluded), and d the common difference, then the sum of the terms is given by the formula:

$$S = \frac{(a_1 + a_{n)}n}{2}$$

[11] A polygonal number is a number represented by dots in a shape of a polygon, for example 3, 6, 10, 15,... are triangular numbers; 4, 9, 16,... are square numbers, and so on.

[12] in arithmetic progression

[13] We use Heath's translation of On Polygonal Numbers with the only modification of substituting modern notation with equivalent English words.

[14] Diophantus does not consider negative terms in the sequence, therefore the last term is usually the greatest.

The fourth proposition is rather lengthy and the proof difficult to follow because of the heavy geometrical representation; however, it is worth mentioning since it gives another formula for arithmetic series. It simply states, in modern notation, that if the first term is 1, the common difference d, the number of terms n, and the sum of all terms S, then:

$$8Sd + (d-2)^2 = [d(2n-1)+2]^2,$$

or

$$S = \frac{[2+(n-1)d]n}{2}.$$

It is easy to see that the above formula can be directly obtained from the previous two, which give expressions for the general term and the sum of all terms. However, Diophantus tackles this problem with heavy deductive reasoning about length segments, which is so bothersome to follow that Heath in [6] is forced to provide an algebraic equivalent parallel to the original proof.

The fifth proposition summarizes the previous results as stating that *In arithmetic progression beginning from 1, the sum of the terms is polygonal; for it has as many angles as the common difference increased by 2, and its side is the number of the terms set out including 1.*

Immediately following the proof of the above statement the manuscript reads:

And thus it is demonstrated what is stated by Hypsicles in his definition, namely, that, If there are as many numbers as we please beginning from 1 and increasing by the same common difference, then, when the common difference is 1, the sum of all the terms is a triangular number; when 2, a square; when 3, a pentagonal number [and so on]. And the number of the angles is called after the number exceeding the common difference by 2, and the side after the number of terms including 1.

We shall now state the above proposition in modern notation. For an arithmetic sequence

$$1, 1+d, 1+2d, ..., 1+(n-1)d$$

the sum of the terms, given by the formula

$$S = \frac{[2 + (n-1)d]n}{2},$$

is the nth polygonal number which has $(d + 2)$ angles.

For example, if the common difference $d = 1$ then $S = \frac{n(n+1)}{2}$ is a triangular number; if $d = 2$ then $S = n^2$ is obviously a square number, and so on.

The last portion of the manuscript includes some **Rules for Practical Use**, such as: *To find the number from the side* and *To find the side from the number*. The first rule is actually a formula for the nth a-gonal number in terms of the number of angles and the side:

$$P = \frac{[2 + (2n-1)(a-2)]^2 - (a-4)^2}{8(a-2)}$$

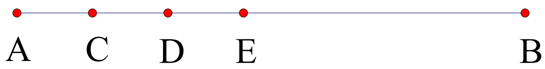
For example, if $a = 3$, that is for triangular numbers, $P = \frac{(2n+1)^2 - 1}{8}$ will give all triangular numbers: 1, 3, 6, ...

The very last proposition,[15] *Given a number, to find in how many ways it can be polygonal*, does not include the entire proof and the fragment abruptly ends leaving us with a question of whether Diophantus indeed had a complete proof. The reader may consider the suggested restoration of the proof by Wertheim, included in [6].

It is interesting to note that in this work Diophantus uses the phrase *as many numbers as we please* when referring to sequences with arbitrarily many terms. Take for example the second proposition: *If there are any numbers, as many as we please, in arithmetic progression, the difference between the greatest and the least is equal to the common difference multiplied by the number of terms less one.* In modern notation, the statement reads that $a_n - a_1 = (n-1)d$, where $a_1, a_2, ...a_n$ are in arithmetic progression with common difference d. Here, instead of working with numbers as in his *Arithmetica*, Diophantus chooses four segments, AB, BC, BD and BE, such that their lengths have the same common difference,

[15] last in the survived manuscripts, not last in the actual work

i.e., in arithmetic progression.



$AC = CD = DE$, therefore

$AE = AC$x(number of terms AC, CD, DE) $= AC$ x (number of terms in the series -1). Since there could be *as many numbers as we please*, i.e. the number of terms in the sequence can be very large, Diophantus cannot make all geometric representations and constructions necessary to carry on the proof, so he performs them for 3 segments ($n = 3$) and concludes the premises are true for all n, thus using the notion of mathematical induction without the induction step. We do not know whether Diophantus wrote *On Polygonal Numbers* before or after *Arithmetica*; however, based on his advanced performance with algebraic equations in Arithmetica and the old geometrical approach to arithmetic problems in *On Polygonal Numbers*, we may speculate that the treatise *On Polygonal Numbers* is an earlier work.

1.1.3 The Porisms

No manuscript with a title *The Porisms*[16] exists and no mentioning of such work from later commentators is evident; however, allusions in Arithmetica lead scholars to believe that Diophantus is also responsible for a collection of propositions called *The Porisms*. Three times in *Arithmetica*[17] he quotes certain propositions having to do with properties of numbers and their divisibility in certain number of squares.

The question of whether *The Porisms* is an independent work or part of *Arithmetica* is discussed in full detail by Heath in [6]. There we see Nesselman's view that *The Porisms* might have been part of *Arithmetica* and placed somewhere between Books I and II, thus *The Porisms* could account for at least some of the missing books of *Arithmetica*. On the other hand, if indeed *The Porisms* were part of *Arithmetica*, then it will be hard to explain why they were lost while other additions, such as the corollaries following problems I:34 and

[16] Porisma means corollary to a proposition; a kind of proposition that is intermediate between a theorem and a problem. [14]

[17] V:3, 5, 16

I:38 with the title *porisma*; and also the lemmas before IV:34, 35, 36; V:7, 8; VI:12, 15 remained untouched.

Nesselman's view is strongly opposed by Hultsch and Heath [6] with the sound argument that when a writer makes a remark such as, *We have it in the Porisms that...*, he refers to an independent source which is not part of the current work. In other words, one does not usually give a reference if that reference is part of the same work. If *The Porisms* were part of *Arithmetica*, then they will be referred to as certain propositions from a particular book or even vaguely such as *as it was already proven*, or *as it will be shown later*. That, and also the fact that Diophantus not only uses a proper name for his reference *The Porisms*, but even quotes the entire proposition in three separate occasions, lead us to conclude that *The Porisms* was an independent work and not part of *Arithmetica*. It is also plausible that the main purpose for the existence of *The Porisms* is to provide the reader with the necessary prerequisites and ground work for the intense algebraic content of *Arithmetica*.

As was already mentioned, Diophantus explicitly quotes three propositions from *The Porisms* in *Arithmetica*. In V:3 he states: *We have it in The Porisms that 'If each of two numbers and their product when severally added to the same given number produce squares, the squares with which they are so connected are squares of two consecutive numbers.'*[18] Translated in modern notation this means: If $x + a = m^2, y + a = n^2$, and if $xy + a = p^2$, then m and n are consecutive numbers. It is interesting to note that Euler wrote a paper discussing a general method of finding such porisms with the sole purpose of when some conditions are satisfied, the remaining are simultaneously satisfied. The above Porism from V:3 is a particular case in his paper.[6]

The text with Porism 3 (V:16) is difficult to read because of manuscript damage and therefore we rely on Tannery to supply the missing three words: *The difference of any two cubes is also the sum of two cubes.*[6] In other words, given two cubes, to find two other

[18] The text is taken from [6].

cubes whose sum is equal to the difference of the given cubes. In modern notation this can be written as $a^3 - b^3 = x^3 + y^3$.

It was pointed out earlier that Diophantus simply states a proposition from *The Porisms* in the midst of solving a problem in *Arithmetica*, and since the original collection of these porisms has not survived we have no indication of the nature of the porisms' proofs and whether other similar propositions were considered by him or not. For example, similarly to Porism 3, we may ask whether the sum of two cubes can be written as a difference of two other cubes, that is, given two cubes a^3 and b^3, can we find two other cubes such that $a^3 + b^3 = x^3 - y^3$. And similarly, another variation, given two cubes a^3 and b^3, to find two other cubes such that $a^3 - b^3 = x^3 - y^3$. This subject of transformation of sums and differences of cubes is further developed by Vièta, Bachet, and Fermat.[6]

In addition to the three explicit mentions of *The Porisms* in Book V, it is evident through the work of *Arithemtica* that Diophantus indirectly uses and applies other propositions not specifically called porisms even though they have the same or similar character. Heath summarizes these in two classes. The first class of propositions has the nature of formulas or facts, some rather simple, such as the fact that the expression $\frac{(a+b)^2}{4} - ab$ is a square, and others more delicate and complicated. The second class, which is of much greater importance, consists of propositions that became an object of investigation for later mathematicians, especially Pierre de Fermat. It is a matter of opinion whether Diophantus possessed a valid proof of these theorems or merely had an empirical intuition about them. Here is a possible reconstruction of the propositions of the second class according to Heath [6].

(a) Theorems with regard to composition of numbers as the sum of two squares. Of this kind we must mention problem II:8: *To divide a given square number into two squares,*[19] and problem II:9: *To divide a given number which is the sum of two squares into two other squares.*

[19] It is to this proposition that Fermat wrote his famous note about a marvellous proof he had no room for in the margin.

The object of decomposition of numbers is also treated in problems III:19 (*To find four rational right triangles with the same hypotenuse*) and V:9 (*To divide 1 into two parts such that, if a given number is added to either part, the result will be square.*)

(b) Theorems with regard to composition of numbers as the sum of three squares. In V:11 we are asked *To divide unity (or 1) into three parts such that, if we add the same number to each of the parts, the results are all squares*. In modern notation we will write this as:

To find x, y, and z, such that

$$x + y + z = 1$$

$$x + a = m^2$$

$$y + a = n^2$$

$$z + a = p^2,$$

where a is a given number.

If we add the last three equations we see that

$$3a + 1 = m^2 + n^2 + p^2,$$

or the number $3a + 1$ is a sum of three squares.

In the necessary condition Diophantus states that t*he given number must not be 2 or any multiple of 8 increased by 2*, in other words $3a+1$, which is a sum of three squares, must not be 7 or of the form $24k + 7$. We can observe that the factor 3 of 24 can be ignored since $24k + 7$ is of the form $3s + 1$. We are thus led to give credit to Diophantus that a number of the form $8k + 7$ cannot be a sum of three squares.

Furthermore, it must be noted, that the above necessary condition is not complete and there are numbers not of the form $8k+7$, such as numbers of the form $32k+9$ [20], which cannot be composed as a sum of three squares. A complete description of the restrictions for the given number a is given by Fermat in [6].

[20] This result is attributed to Bachet.

(c) Theorems with regard to composition of numbers as the sum of four squares. There are three problems in *Arithmetica* (IV:29, V:30 and V:14), where it is required to divide a given number into four squares. And if Diophantus states a necessary condition for dividing a number into two or three squares as in the previous case of V:11 and V:9, here he omits any restrictions for the given number. Must we then assume that Diophantus knew that any number can be divided into four squares and thus conditions are not necessary? We ought to credit him the knowledge, at least empirically, because his work, other than manuscript defects and uncertainty of translation, carries few minor imperfections and no inaccuracies. To see Diophantus' ideas about decomposition of numbers into squares in full light it took 14 centuries and the great minds of Fermat, who stated that *Every number is either a square or the sum of two, three or four squares,* and Lagrange, who proved the above statement based on Euler's work. In conclusion, we see that Diophantus implicitly states or makes assumptions of certain theorems and facts in the necessary conditions or somewhere along the proof part of his problems and does not bother to provide an evidence or justify his references, assuming the reader is familiar with them or rather more probable, assumes that his *Porisms* are at the reader's disposal and no additional explanations are necessary.

1.1.4 Other possible works

Based on a commentary on Iamblichus where the author says *thus Diophantus in the Moriastica ... for he describes as parts the progression without limit in the direction of less than a unit*, we may suppose that *Moriastica* is another work of Diophantus which consist of rules of calculating with fractions. [4, 6] With no other additional evidence or even a hint of such a work we will not make any further remarks.

In [11], Jean Christianidis suggests the possibility of another treatise by Diophantus, entitled *Teaching of the Elements of Arithmetic*. Due to the fact that mathematicians have been hindered in understanding and correctly interpreting Diophantus' method of solution because of the missing ground work for *Arithmetica*, such as *The Porisms*, and also because of the diverse nature of the problems of *Arithmetica*, we are led to believe that another collection of problems with the title *Teaching of the Elements of Arithmetic*, must have proceeded and given foundation for *Arithmetica*. Because of the lack of solid evidence on this topic we will refrain from further speculation on that possibility.

1.2 Notation and definitions in Arithmetica

1.2.1 Ancient Greek grammar highlights

In order to get a glimpse of the math symbolism used by Diophantus let us engage in a brief overview of some alphabetical and grammatical elements from [2, 10]. The Greek alphabet includes 24 symbols. The lower case Greek letters,[21] along with three Phoenician letters (digamma, koppa, and sampi), constitute the Greek number system.

1 α	10 ι	100 ρ
2 β	20 κ	200 σ
3 γ	30 λ	300 τ
4 δ	40 μ	400 υ
5 ϵ	50 ν	500 ϕ
6 ϝ	60 ξ	600 χ
7 ζ	70 o	700 ψ
8 η	80 π	800 ω
9 θ	90 ϟ	900 ϡ

Every Greek word which begins with a vowel or a diphthong[22] has a symbol over it called a *breathing mark*. This symbol resembles a comma raised above the vowel. If the diacritical mark is turned to the right the symbol is called a *rough breathing mark* and the vowel is pronounced with an initial h sound. If the mark is turned to the left, the symbol is called a *smooth breathing mark* and the vowel is pronounced normally without the h sound as in ἀριθμος.

[21] The Greek alphabet includes two symbols for sigma: σ and ς. Only σ represents the number 200. ς is called *final sigma* and is only used at the end of a word.

[22] A diphthong is a combination of two vowels which make a new sound. For example the diphthong $\upsilon\iota$ makes the sound *ui* as in queen.

Greek accent marks include the *acute accent* (ά) for emphasizing the syllable, the *grave accent* (ὰ), and the *circumflex* ($\hat{\alpha}$), which lengthens its vowel.

The Greek language is an extremely inflected language. Verbs, nouns, pronouns, adjectives, and definite articles take different forms to indicate gender (masculine, feminine, neuter), number (singular and plural), case (nominative, genitive, dative, accusative and vocative), as well as grammatical parts of the sentence (subject, verb, direct object, etc.)

Since word order in the Greek sentence is not important, the inflection of nouns indicates which word is the subject, which the direct object, etc. Personal pronouns are optional, as the verb ending absorbs the hidden pronoun.

One of the most common nouns in *Arithmetica* is, of course, *arithmos*, meaning *a number containing an undefined number of units.*[6] Since it is the main character in the treatise, we are obliged to demonstrate all cases for *arithmos*, even though the nominative and the accusative cases are most widely used in the text and the vocative case almost certainly never. *Arithmos* is a masculine noun and as most masculine nouns it belongs to the 2nd declension.[23]

Case	**Singular**	**Plural**
Nominative	ἀριθμος	ἀριθμοι
Genitive	ἀριθμου	ἀριθμων
Dative	ἀριθμῳ	ἀριθμοις
Accusative	ἀριθμον	ἀριθμους
Vocative	ἀριθμε	ἀριθμοι

We must also note that the definite article preceding the noun has different cases as well. Greek punctuation includes a floating dot "·", equivalent to our semi-colon, and ";", in place of our question mark.

[23] Greek nouns are grouped according to their endings in three declensions.

1.2.2 Notation for powers and other definitions given by Diophantus

Arithmetica starts with some specific definitions, formulas, and notations. It is interesting to note that some definitions concern unknown quantities only, while others can be used for both known and unknown quantities. For example, we use algebraic terms such as *square* and *cube* for known and unknown quantities: x squared, 3 squared, x cubed, 5 cubed. Diophantus, however, uses two different words for *square*: $\delta\acute{\upsilon}\nu\alpha\mu\iota\varsigma$, abbreviated as Δ^{Υ}, when referring to the square of the unknown, and $\tau\epsilon\tau\rho\acute{\alpha}\gamma\omega\nu$ος, with no abbreviation, when referring to the the square of a known quantity.

Cube (x^3) is defined as *kubos* and the abbreviation or symbol for it is K^{Υ}.[24]

x^4 is defined as square-square (*dynamodynamis*) with the symbol or abbreviation $\Delta^{\Upsilon}\Delta$.

x^5 is square-cube (*dynamocubus*) with ΔK^{Υ}.

x^6 is cube-cube (*cubocubus*) with $K^{\Upsilon}K$.[14]

Heath in [6] gives a complete account for the abbreviation of the unknown, *arithmos*, and the reasons behind it. Since every letter of the Greek alphabet, except the final sigma, ς, represents a number, Nesselman reasons that Diophantus uses ς for abbreviating *arithmos*. Heath, on the other hand, argues that like the rest of the notation, which in most cases consist of the first two letters of the word needed to be abbreviated or represented, Diophantus creates the new symbol from a combination of the first two letters of $'\alpha\rho\iota\theta\mu$ος, namely α and ρ. In addition, Heath reassures us that the symbol for the unknown in some manuscripts does not resemble the final sigma ς very much and therefore the connection between the symbol and the final sigma ς cannot be established.

In general, Diophantus uses full words for the known quality and abbreviation for the unknown. Thus the full word *arithmos* is used for non-technical purposes only, as in the statement of a problem; and its abbreviation ς, for technical purposes, i.e. as in the process of setting and solving an equation.

As mentioned earlier, word order in the Greek sentence is not important and therefore

[24] $K\Upsilon$ are the first two upper case letters from the Greek word $K\Upsilon$B0Σ. The same word and abbreviation is used for the cube of known and unknown quantity.

case endings absorb information such as gender, number, subject, direct object, and so on. This explains why in the manuscript even abbreviated words and symbols have the case endings as superscript, such as ς^{ov} for $\alpha\rho\iota\theta\mu o\upsilon$, the genitive case meaning *of a number*, and $\varsigma\varsigma^{oi}$ for $\alpha\rho\iota\theta\mu o\iota$, indicating nominative case, plural number. However, Tannery in his Greek edition of *Arithmetica* [14] omits case ending abbreviations of ς and doubling of ς.

Diophantus' symbol for minus, ⋔, as explained at the beginning of *Arithmetica*, resembles the Greek letter Ψ upside down. Heath in [6] speculates that even though Diophantus himself explains the symbol as an upside down Ψ, it originated from the Greek word for negation or wanting, ΛΕΙΨΙΣ. In this case, however, the first two letters could not be used except one of them superimposed as in K^{Υ} for *kubus*, because these letters already represent a number. So, if the third letter I gets inserted inside the first letter Λ, we get the desired symbol of an upside down Ψ.

Diophantus does not have a symbol for addition, instead he places the symbol $\mathring{\mathrm{M}}$ from $\mu o\nu\acute{\alpha}\varsigma$ (unit) in front of each invariable element, and for that reason he is forced to separate positive and negative terms in expressions. For example $x^3 - 2x^2 + 4x - 3$ means $K^{\Upsilon}\bar{\alpha}\varsigma\bar{\delta}$ ⋔ $\Delta^{\Upsilon}\bar{\beta}\mathring{\mathrm{M}}\bar{\gamma}$

Note that the coefficient is placed after the variable and has a bar on top to distinguish it from cardinal numbers.

Diophantus does not have a symbol for multiplication. For his problems such a symbol is unnecessary since he usually multiplies the unknown quantity and the definite number and only numerical coefficients are used. Similarly, no symbol for division is needed in the case when there is no remainder. In other cases the quotient is written as a fraction.

With regard to writing fractions with numerator 1, Diophantus follows the Greek tradition of writing only the denominator with a certain sign to distinguish the fraction from the cardinal number. In more recent manuscripts, the symbol affixed to the number resembles a double accent (δ'' means $\frac{1}{4}$). Diophantus follows this pattern in the hypothesis and analysis of his problems, but in the solutions he uses a simple accent or a symbol resembling $\wedge$, only with a shorter right side placed above the letter to the right.[6] Tannery, however, in [14] adopts the symbol $\times$ instead throughout his edition. Another common practice inherited from the Egyptians is the use of special symbols for $\dfrac{1}{2}$ and $\dfrac{2}{3}$. With regard to expressing the remaining fractions we must mention the method used by Planudes and Bachet [6], where the denominator of a fraction is written as an exponent ($\iota\delta^{\gamma}$ means $\dfrac{14}{3}$). Another commonly used way is writing the fraction as a product of numerator and reciprocal of the denominator. Thus $\bar{\gamma}\varepsilon'$ means $3\cdot\dfrac{1}{5}$. In addition, complex fractions with compound exponents are described in paragraph form as numerator divided by denominator.

For the reciprocal of the unknown $\left[\dfrac{1}{x}\right]$ Diophantus uses the name *arithmoston*, obviously derived from *arithmos* [14]. Similarly, he derives the following terms:

dynamis $[x^2]$	*dynamoston* $\left[\dfrac{1}{x^2}\right]$
cubus $[x^3]$	*cuboston* $\left[\dfrac{1}{x^3}\right]$
dynamodynamis $[x^4]$	*dynamodynamoston* $\left[\dfrac{1}{x^4}\right]$
dynamocubus $[x^5]$	*dynamocuboston* $\left[\dfrac{1}{x^5}\right]$
cubocubus $[x^6]$	*cubocuboston* $\left[\dfrac{1}{x^6}\right]$

Diophantus does not use a special symbol for equal, so the word $\acute{\iota}\sigma o\varsigma$ or the abbreviation $\acute{\iota}\sigma$, from the Greek word for equal, is used. Some manuscripts have the abbreviation "ι^{σ}". Tannery uses "$\acute{\iota}\sigma.$" in [14].

After introducing symbols and defining powers, Diophantus proceeds with examples illustrating the rules of exponents such as:[25]

$$x \cdot x = x^2$$

$$x \cdot x^2 = x^3$$

....

[25] These examples are given in modern notation from the Latin text of [14]. In the Greek text they are described in paragraph form.

$$\frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2}$$
$$\frac{1}{x} \cdot \frac{1}{x^2} = \frac{1}{x^3}$$

....

$$\frac{1}{x} \cdot x^2 = x$$
$$\frac{1}{x} \cdot x^3 = x^2$$

....

and a dozen other examples. This brings the question of why Diophantus does not simply give a formula instead of numerous examples illustrating the same principle? To this very legitimate question one might find a satisfactory explanation in the nature of *Arithmetica*, which is a collection of exercises, more like a workbook in the modern sense, and not a theoretical treatise or collection of propositions, such as *The Porisms*, or a textbook in modern terms.

The reader can clearly see that in spite of the numerous abbreviations introduced for the unknown and its powers, solving equations dressed in these symbols is still extremely difficult and inconvenient. The reason for that is that the abbreviations are derived from the words they represent and thus there is no connection between the symbols. For example Δ^{Υ} (x^2) and ς (x) have no resemblance. These and other characteristics of Diophantus' notation will be discussed in the following subsection.

1.2.3 Three stages of algebraic notation

In our efforts to present accurately the stages of algebraic notation, we will refer to Nesselman's view explained in detail in [6]. The first stage can be viewed as **Rhetorical Algebra**, or algebra in complete paragraph form with no symbols of any kind, not even symbols for numbers. Representatives of this stage are Iamblichus and all Arabian and Persian algebraists of which we have knowledge. It is also well known that when Arabian scribes translated mathematical treatises which contained symbols, they did not carry over the symbols but translated them into words. Thus the four Arabic books of *Arithmetica* were translated in Arabic in full paragraph form.

The second stage, known as **Syncopated Algebra** is a transitional stage between the ancient Rhetorical Algebra and our present symbolic algebra. If we consider Diophantus' notation as being symbols and not mere abbreviations, we may view him as one of the earliest Syncopated Algebraists. Diophantus is credited for his efforts to introduce a good number of notations for operations and quantities and as we saw in the previous subsections, this is a step toward the simplification of math problems and solutions, giving them greater clarity. As with any technological advancement that has not yet achieved the status of an *industry standard*, the symbols used are often inconsistent and not widely used. Thus, one mathematician or scribe uses one symbol for a particular operation, another uses some other symbol and there is lack of unity. Since this stage does not allow for symbol representation of all math elements, the text remains cumbersome and difficult to follow. Nevertheless, European mathematicians after Diophantus continued the tradition of syncopated algebra until the middle of the 17th century, with the exception of Vièta, who first introduced a regular system of reckoning with letters.

And finally, the present stage of **Symbolic Algebra** is a system of notations by symbols, which have no direct relationship with the things they represent. This algebra system is completely independent of words, except when the author chooses to supply them for the sake of clarity and understanding.

1.3 Manuscripts, translations, and commentaries of Arithmetica

This section is entirely based on the information provided by Heath in [6] that uses Tannery's [14] with some minor additions. According to these references we find no trace of *Arithmetica* from the day of Hypatia (415 AD) until the 8th or 9th century. The manuscripts can be divided into three classes according to their source. The First class, also known as the Non-Planudean class, includes the most ancient copies that remain faithful to the original archetype, the lost copy of the Hypatian recension. The second, or Planudean class, includes all manuscripts based on Maximus Planudes's commentary on Books I and II. Then there is a third class that consists of a few manuscripts which are a mix of the first two classes, i.e., they include Planudes's commentary in Books I and II but the remaining books follow the more genuine text from the first class.

From the **First (Non-Planudean) class** we have the following manuscripts: *Matritensis 48, Vaticanus gr. 191*, and *Vaticanus gr. 304*. The most ancient and genuine manuscript which we posses today, the *Matritensis 48* from the 13th century, is copied from a lost MS from the 8*th* or 9*th* century and is marked as type *A*. This manuscript, however, is now difficult to read because of the many corrections made to it by someone from another manuscript of the Planudean class. Fortunately, there is another copy made from the *Matritensis 48*, prior to its corrections, and thus we have the *MS. Vaticanus graecus 191*, marked as type *V*. *Vaticanus 304* is a true copy of *V*, not of *A*, and since it was very clearly written it was used to make further copies.

The manuscript *Marcianus* 308 from the 15*th* century, marked as type B_1, is a representative of the **Planudean class**, since it includes Planudes's commentary on Books I and II. This manuscript was based on an older copy from the 14*th* century, Ambrosianus Et.157 sup. (of which only 10 leaves are extant), and was seen, according to Heath [6], by Regiomontanus at Venice in 1464. Another manuscript from the 15th century, called *Guelferbytanus Gudianus I*, was used by Xylander in his Latin translation, according to Tannery. It is uncertain whether this manuscript is a copy of B_1 or its predecessor *Ambrosianus Et. 157 sup.*

Lastly, the *MS. Parisinus 2379*, marked by Tannery as type C, is the first representative of **the mixed class**. This copy, used by Bachet for his translation and publication, was written after 1545, and has the oddity of following *MS. Vaticanus gr. 200* (a MS of the Planudean class) in the first two books, obviously for the purpose of including Planudes' commentary, while the rest of the text is copied from a manuscript of the First class, namely *MS. Vatican 304*.

In addition to the above manuscripts, there are a dozen other more recent manuscripts which call for little notice. The latest found manuscript is the *Cracow MS*, rediscovered in the library of the University of Cracow, which obviously agrees with the three major class representatives *Matritensis 48* (A), *Marcianus 308* (B), and *Parisinus 2379* (C), but it has some passages not found in the last two, so it probably was not copied from either of them.

Of the early writers and commentators of *Arithmetic*a we must mention Hypatia, daughter of Theon of Alexandria and the first known woman mathematician, Georgius Pachymeres (1240-1310), who wrote a paraphrase in Greek of at least a portion of Diophantus, and Maximum Planudes (1260-1310), who added a systematic commentary of the first two books as mentioned above.

In addition, Diophantus' work was an object of study by a number of Arabian scholars. *Abū'l Wafā al-Būzjānī* (940 – 998) and *Qustā ibn Lūqā* (912) wrote commentaries on a portion of *Arithmetica*. Others used Diophantus' methods in their own Algebra treatises.

It is interesting to note that in Europe, Diophantus' work continues the *hide and seek* pattern for quite awhile. After Regiomontanus (15th C.) calls attention to this very ancient extant work of the Greeks and remarks that it is not yet translated into Latin, there is no mentioning of it by Luca Paciolo (end of 15th C.), Cardano or Tartaglia (end of 16th C.). Rafael Bombelli, with the help of Antonio Maria Pazzi, were the first to attempt the gigantic task of translating the first five books, but, overwhelmed and distracted by other tasks, they never published. Bombelli, however, included numerous problems and solutions from Books I-V in his Algebra, mixing them up with his own problems. We are assured by Bachet that Bombelli's translation, even though never published, is far superior to the one of his successor Xylander.

Wilhelm Holzmann, known by the name Xylander, was the first to translate and publish *Arithmetica*. A man of many interests and noble character, who taught himself from arithmetical books, he progressed to a level of modifying and improving what he found in these books. But when he first became acquainted with the *Arithmetica*, Xylander realized his inferiority and declared his own ignorance in the light of Diophantus' book.

Xylander's translation of *Arithmetica*, based on *Guelferbytanus Gudianus I* manuscript, has endured much fair and unfair criticism. This can best be summarized with Nesselman's words quoted in [6]: *Xylander's work remains, in spite of various defects which are unavoidable in a first edition of so difficult an author, especially when based on only one MS. and that full of errors, a highly meritorious achievement, and does not deserve the severe strictures which it has sometimes had passed upon it. It is true that Xylander has in many places not understood his author, and has misrepresented him in others; his translation is often rough and un-Latin, this being due to a too conscientious adherence to the actual wording of the original; but the result was none the less brilliant on that account. The mathematical public was put in possession of Diophantus' work, and the appearance of the translation had an immediate and enormous influence on the development and shaping of Algebra.*

The next major step toward reviving the work of Diophantus for a European audience was the Latin translation of Bachet in 1621, based on the *MS. Parisinus 2379*, a manuscript from the mixed class. Even though Bachet's translation appears to be significantly superior to Xylander's (it also includes the Greek text with Bachet's additions in brackets), it closely follows Xylander's version correcting the inaccuracies when necessary, and we must mention the ungrateful attitude of Bachet toward the pioneer work of his predecessor demonstrated by his own words *that it is almost fairer to attribute the translation to me than to Xylander.* Fermat used that same edition of Bachet when he wrote his famous note on the margin (see section 1.5). His son, Clement Samuel, published in 1670 a second edition of Bachet's work with the addition of Fermat's notes. The reprint, however, contains numerous mistakes in the Greek text and Bachet's additions in brackets were not included.[1, 6]

In addition to Latin, Diophantus' work is translated into modern languages as well. Among these we must mention *Arithmetica's* reproduction (not a literal translation) of Simon Steven (1585) of Books I-IV in French, based on Xylander's version, with the addition of Books V and VI by Albert Girard (1625); the translation of the fragment *On Polygonal Numbers* by Poselger (1810) in German; *Arithmetica's* translation and commentary by Otto Schulz (1822), based on the second Bachet's edition; and the German translation of G. Wertheim (1890). In 1893, Paul Tannery published a Latin translation of *Arithmetica* and *On Polygonal Numbers* with the Greek text side by side, and in 1895 he added a second volume with Planudes's commentary, other ancient commentaries and 38 arithmetical epigrams in Greek with notes. Tannery's work is with no doubt the most extensive and complete commentary on Diophantus ever published, but written in Latin it is not easily accessible to most modern readers. That is why we are very fortunate to have Heath's work [6] in English, published in 1910, which includes Diophantus' work in modern notation, commentary summarizing Tannery's and other previous writers' findings, along with some significant new research and additions (notes, theorems, and problems by Fermat and Euler).

The French translation of Diophantus' work by Paul Ver Eecke, which appeared in 1926, is a literal word for word translation based on Tannery's Greek edition. Paul Ver Eecke gives us the text without any modifications of symbols or abbreviations. In our opinion the lack of modern notation and questionable symbolism[26] represents the ancient text most accurately, and for this reason we have chosen Eecke's work to translate into English 42 problems of first degree (see Chapter 2).

1.4 The four Arabic Books and Arithmetica

The question of the missing books of *Arithmetica* has puzzled mathematicians for quite awhile. Out of the 13 books of *Arithmetica* only six survive to our time in Greek. According to Heath [6], the missing seven books were lost at an early time, perhaps shortly after they were written. In [14], Paul Tannery suggests that Hypatia's commentary does not extend beyond the six known books, most likely because they were already lost. There are

[26] Manuscripts are very inconsistent and disagreeable on the nature and origin of the symbols used. For example, Tannery uses ς for *arithmos* in the technical sense, but due to scribes' negligence, or other reasons, that symbol is not always present in the manuscripts.

several Arabic translations of *Arithmetica* but they again contain only the material from the existing six books.

However, in 1968 a remarkable discovery of an Arabic manuscript labeled *Books IV-VII of Atithmetica by Diophantus* and translated by *Qustā ibn Lūqā*, found in the library Astan-i Quds (The Holy Shrine Library) in Meshed, Iran, stirred up a lot of questions and dilemmas. [12] Are these manuscripts a translation of the lost books? What is their place with regard to the Greek books already known as Books I-VI?

To answer these and other similar questions we must carefully examine the recent manuscript, its origin, and the nature of the problems presented there.

The newly found Arabic books were made around or after the 9th century by *Qustā ibn Lūqā*, a Christian of Greek origin, who in addition to being a mathematician, was interested in medicine, philosophy, and astrology. His translation is an excellent one with only minor mistakes indicating that the work was done quickly and from a manuscript well preserved and easy to read.[12]

Without a doubt the four books were once part of *Arithmetica*. Each Arabic book starts and finishes with the statement that the author is Diophantus. Thus the first Arabic book, Book IV begins with *Fourth Book of the treatise of Diophantus the Alexandrian on squares and cubes* and ends with *End of the fourth Book of the treatise of Diophantus on squares and cubes, and it contains forty-four problems.* In addition, a problem from the Greek books (Greek IV:1) appears to be an interpolation of problems stated in the Arabic books (Arabic V:7, 8)[12]. It is also apparent that the general structure of the Arabic problems quite resembles that of the Greek problems. For illustration let us examine a sample problem.

Arabic IV:1[27] *We wish to find two cubic numbers the sum of which is a square number.*

We put x *as the side of the smaller cube, so that its cube is* x^3*, and we put as the side of the greater cube an arbitrary number of* x*'s, say* $2x$*; then the greater cube is* $8x^3$*. Their sum is* $9x^3$*, which must be equal to a square. We make the side of that square any number*

[27] This problem is reproduced from [12] where Sesiano uses modern notation for the powers of the unknown. The Arabic translation is in complete paragraph form.

of x's we please, say $6x$*, so the square is* $36x^2$*. Therefore,* $9x^3$ *is equal to* $36x^2$*. Then, since the side (of the equation) containing the* x^2*'s is lesser in degree that the other, we divide the whole by* x^2*;* $9x^3$ *divided by* x^2 *gives* $9x$*, that is 9 roots of* x^2*,*[28] *and the result from the division of the* $36x^2$ *by* x^2 *is a number, namely* 36*. Thus* $9x$*, that is (nine) roots, equals* 36*; hence* x *is equal to* 4*. Since we assumed the side of the smaller cube to be* x*, the side is* 4*, and the smaller cube is* 64*; and since we assumed the side of the greater cube to be* $2x$*, the side is* 8*, and the greater cube is* 512*. The sum of the two cubes is* 576*, which is a square with* 24 *as its side. Therefore, we have found two cubic numbers the sum of which is a square, the lesser being* 64 *and the larger,* 512*. This is what we intended to find.*

It is obvious for the reader that here, as in the Greek books, Diophantus starts with a general problem, names the unknowns, assumes certain values for the sake of convenience, sets an equation, solves it, and at the end returns to the conditions to verify his solutions. The text is overwhelming with unnecessary explanations and repetitions, which is probably a result of translation to Arabic as well as additions by scribes. The problems following Arabic IV:1 are very similar in nature, they ask to find *two cubic numbers whose difference is a square, to find two square numbers, whose difference is a cubic number* and so on, just in the spirit of Diophantus.

With regard to their place within the Greek books most commentators agree the Arabic books must be placed between the third and fourth of the Greek books.[12] The first Arabic book carries the name *Book of the Squares and Cubes* with the very first problem treating the sum of two cubes as being equal to a square (See the above problem Arabic IV:1). It makes sense that the newly found books should precede the Greek Book IV, where we find problems with cubes for the very first time.

In conclusion, the Greek text appears to be more genuine than the Arabic text and, with the exception of some interpolated problems most likely included later by an unknown commentator or copyist, shows no evidence of commentary or reworking of problems within

[28] This seems to be an Arabic addition.

the text. Tannery suggests in [14] that perhaps the Greek text we have today was once Hypatia's commentary which was later removed by Planudes's suggestion. Since the Arabic text resembles a mixture of original problems and a commentary, one is led to wonder if that could be a translation of Hypatia's lost commentary.

1.5 Arithmetica and Fermat's Last Theorem

Diophantus' work led to one of the greatest mathematical challenges of all times: Fermat's Last Theorem. The 17th century jurist and amateur mathematician Pierre de Fermat was intrigued by the ancient wisdom of the Greeks, and especially the works of Archimedes, Eudoxus, and Diophantus. Fermat was trying to generalize the problems in *Arithmetica*, and in the process wrote in the margin of problem II:8 around the year 1637 a tantalizing statement which sent mathematicians around the world for the next 350 years in frantic search for a proof.

The problem which received Fermat's special attention is quite innocuous: *To divide a given square number into two squares.* And here is Diophantus' solution as stated by Heath in [6]:

Given square number 16.

x^2 *one of the required squares. Therefore,* $16 - x^2$ *must be equal to a square.*

Take a square of the form $(mx - 4)^2$, m *being any integer and* 4 *the number which is the square root of* 16, *e.g. take* $(2x-4)^2$, *and equate it to* $16-x^2$. *Therefore* $4x^2-16x+16 = 16 - x^2$, *or* $5x^2 = 16x$, *and* $x = \frac{16}{5}$.

The required squares are therefore $x = \frac{256}{25}$, $x = \frac{144}{25}$.

And so in the margin of the above problem Fermat wrote: *On the other hand, it is impossible to separate a cube into two cubes, or biquadrate into two biquadrates, or generally any power except a square into two powers with the same exponent. I have discovered a truly marvellous proof of this, which, however, the margin is not large enough to contain.* [1]

The Latin copy of *Arithmetica* with Fermat's statement was published by his son Clement Samuel in 1670 and the race for finding the *truly marvellous proof* began. Since all Fermat's proofs were either proven or disproved by the early 1800's, except this one, mathematicians began to wonder whether Fermat really possessed such a marvellous proof as he had promised.[1]

In modern notation *Fermat's Last Theorem (FLT)* is stated as: $x^n + y^n = z^n$ *has no whole number solution for* $n > 2$.

Despite the fact that *FLT* has no application in science, engineering, nor even number theory, it attracted the attention of mathematicians around the world. Fermat proved his theorem for $n = 4$ by a method of *infinite descent*; the Swiss mathematician Euler established the theorem for $n = 3$ and $n = 4$, independently of Fermat; Dirichlet and Legendre proved separately the case $n = 5$; Gabriel Lamé and Henri Lebesgue proved the case $n = 7$; and even though computers can verify the theorem for very large numbers, a traditional mathematical proof was needed to show that the statement is false for all $n > 2$.[1] As decades and centuries started to roll over, *FLT* seem to have become forgotten and forsaken. But not by Andrew Wiles. In 1993, after 7 long years working in his attic in complete secrecy, the Princeton professor was ready to present to the audience of a conference on Iwasawa Theory in Cambridge University and to the whole world the key to unlock the old mystery. Wiles's 200 page presentation on *Modular Forms, Elliptic Curves, and Galois Representations*, followed by his final remark, "And this proves Fermat's Last Theorem", threw the audience and the rest of the mathematical world into an uproar. But alas, this was a short victory, for within weeks a hole in Wiles's proof was discovered and as hard as he tried to patch it, the hole would simply not go away. The next 15 months was a time of anger, frustration and humiliation for Dr. Wiles. He frantically tried to check every line and every stroke of his proof and just when he was ready to give up he made a remarkable discovery that changed the course of his proof. The new proof, now bearing the names of Andrew Wiles and his friend Richard Taylor, who came from England to help him correct the proof, had no flaws and finally the three and half century old theorem was put to rest. [1]

And so we see that the spirit of Diophantus has haunted the twisted paths of mathematics into our very day.

Chapter 2

FORTY TWO PROBLEMS OF FIRST DEGREE FROM ARITHMETICA

The first book of *Arithmetica*, which includes mostly equations of first degree and systems of equations of 1st degree, has long been overshadowed by the fame of Diophantine equations of higher degree, about one of which the Byzantine mathematician Maximum Planudes has written: *Thy soul, Diophantus, be with Satan because of the difficulties of your other theorems and this one in particular.* The problems involving first degree equations (42 genuine and 9 interpolated) are not intimidating at all; in fact they appear to be relatively simple and yet elegant and delightful even for a not very mathematically sophisticated mind.

2.1 First degree equations with one unknown

Book I:7. To subtract two given numbers from the same number so as to make the remainders have to one another a given ratio. Let it be required to subtract 100 and 20 from the same number so as to make the greater remainder be three times the lesser.

Let the required number be 1 arithmos. If we subtract 100, the remainder is 1 arithmos minus 100 units; and if we subtract 20, the remainder is 1 arithmos minus 20 units. Now, it is necessary that the greater remainder be three times the lesser; so three times the lesser is equal to the greater. Now, three times the lesser gives 3 arithmoi[1] minus 300 units, which will equal to 1 arithmos minus 20 units. Add the negative terms on both sides and we obtain 3 arithmoi equal to 1 arithmos plus 280 units. Subtract like from like; it remains that 2 arithmoi are equal to 280 units; and the arithmos becomes 140 units.

Returning to the hypothesis, it was posed that the required number be 1 arithmos; so it will be 140 units. If we subtract 100, 40 units are left; at the same time if we subtract 20, 120 units are left; thus it is established that the greater remainder is three times the lesser.

[1] Arithmoi is the plural for arithmos

Book I:8. To two given numbers add the same number so as to make the resulting numbers have to one another a given ratio.

It is necessary, however, that the given ratio must be less that the ratio which the greater of the given numbers has to the lesser.

Let it be required to add the same number to 100 and to 20, so as to make the greater number be three times the lesser.

Let the number added to each number be 1 arithmos. If we add it to 100, we will get 1 arithmos plus 100 units, at the same time if we add it to 20, we will get 1 arithmos plus 20 units. It is also necessary that the greater number be three times the lesser; so three times the lesser will be equal to the greater. Now, three times the lesser becomes 3 arithmoi plus 60 units. Equate them to 1 arithmos plus 100 units. Subtract like from like, and the two remaining arithmoi are equal to 40 units, from which the arithmos becomes 20 units.

Returning to the conditions, it was proposed that the number to add to each number be 1 arithmos; so it will be 20 units. If it be added to 100, it becomes 120 units; at the same time if be is added to 20, it becomes 40 units; thus it is established that the greater number is three times the lesser.

Book I:9. From two given numbers subtract the same number so as to make the remainders have to one another a given ratio.

It is necessary, however, that the given ratio be greater than the ratio which the greater of the given numbers has to the lesser.

Let it be required to subtract the same number from 20 and from 100 and to make the greater remainder be six times the lesser.

Let the number to be subtracted from each number be 1 arithmos. If it be subtracted from 100, it remains 100 units minus 1 arithmos; and if it be subtracted from 20, it remains 20 units minus 1 arithmos. Now, it is necessary that the greater remainder be six times the lesser, so six times the lesser will be equal to the greater. Now, six times the lesser gives 120 units minus 6 arithmoi which will be equal to 100 units minus 1 arithmos. Add the negative terms on both sides and subtract like from like; the 5 remaining arithmoi are equal to 20 units, and the arithmos becomes 4 units.

Returning to the conditions, it was posed that the number to subtract from each given number be 1 arithmos, so it will be 4 units. If it be subtracted from 100, it remains 96 units, and if it be subtracted from 20, it remains 16 units; thus it is established that the greater remainder is six times the lesser.

Book I:10. Given two numbers, add to the lesser and subtract from the greater the same number so as to make the resulting numbers be in a given ratio.

Let it be required to add to 20 and to subtract from 100 a number that is the same in both cases so as to make the greater number be four times the lesser.

Let the number to add to and to subtract from each number be 1 arithmos. From then on, if we add it to 20 we get 1 arithmos plus 20 units, and if we subtract it from 100, we get 100 units minus 1 arithmos. Now, it is necessary that the greater number be four times the lesser, so four times the lesser is equal to the greater. Now, four times the lesser becomes 400 units minus 4 arithmoi, which will be equal to 1 arithmos plus 20 units.

Add the negative terms on both sides and subtract like from like; the 5 remaining arithmoi are equal to 380 units, and the arithmos becomes 76 units.

Returning to the conditions, it was posed that the number to add and to subtract from each number be 1 arithmos; so it will be 76 units. If we add 76 units to 20, we obtain 96 units, and if we subtract it from 100, we get 24 units; and thus it is established that the greater number is four times the lesser.

Book I:11. Given two numbers, add the first to and subtract the second from the same number so as to make the resulting numbers have to one another a given ratio.

Let it be required to add 20 to a number and to subtract 100 from the same number, so as to make the greater number be three times the lesser.

Let the required number be 1 arithmos. If we add it to 20 units, it becomes 1 arithmos plus 20 units, and if we subtract 100 units from it, it remains 1 arithmos minus 100 units. Now, it is necessary that the greater number be three times the lesser; so the lesser number, taken three times, is equal to the greater. But the lesser number taken three times becomes 3 arithmoi minus 300 units; so 3 arithmoi minus 300 units are equal to 1 arithmos plus 20 units. Add the negative terms on both sides and subtract like from like; it follows that 320 units are equal to 2 arithmoi, and the arithmos becomes 160 units.

Returning to the hypothesis, the greater number will be 180 units, the lesser 60 units, and thus it is established that the greater number is three times the lesser.

Book I:39. Given two numbers, to find a third number such that if we take the sum of any two of these three numbers and we multiply it by the remaining number, we obtain three numbers which have equal differences.[2]

Let 3 and 5 be the two given numbers; it is necessary then to find a third number such that if we take the sum of any two of these numbers and multiply it by the remaining number, we obtain three numbers which have equal differences.

Let the required number be 1 arithmos. From then on, if we add it to 5 units, we obtain 1 arithmos plus 5 units, which multiplied by the remaining number, that is by 3 units, gives us 3 arithmoi plus 15 units. Again, if 1 arithmos is added to 3 units, we obtain 1 arithmos plus 3 units, which multiplied by 5 units gives 5 arithmoi plus 15 units. Finally, if 5 units are added to 3 units and the obtained 8 units are multiplied by 1 arithmos, we obtain 8 arithmoi.

From then on, it is clear that 3 arithmoi plus 15 units can never be the greatest number, because 5 arithmoi plus 15 units is greater than that. Accordingly, 3 arithmoi plus

[2] in arithmetic progression

15 units must be either the middle number or the least number, while 5 arithmoi plus 15 units must be either the greatest number or the middle number; and 8 arithmoi could be the greatest number, the middle number or the least number, given that the value of the arithmos is unknown.

So, to begin with, let 5 arithmoi plus 15 units be the greatest number; 3 arithmoi plus 15 units be the least number; and 8 arithmoi be, obviously, the middle number.

On the other hand, if these three numbers have equal differences, the sum of the greatest and the least is twice the middle number. And now, the greatest number added to the least, is 8 arithmoi plus 30 units, which we equate to 16 arithmoi, and the arithmos becomes $\frac{15}{4}$ and this will be the required number which satisfies the conditions.

Now, let 5 arithmos plus 15 units be the greatest number; 3 arithmos plus 15 units be the middle number, and 8 arithmoi the least number. If these three numbers have equal differences, the middle number exceeds the least by the same amount the greatest exceeds the middle. So, the greatest number exceeds the middle by 2 arithmoi, and the middle exceeds the least by 15 units minus 5 arithmoi, then 15 units minus 5 arithmoi are equal to 2 arithmoi and the arithmos becomes $\frac{15}{7}$. This will be the required number which solves the problem.

But now suppose that 8 arithmoi is the greatest number; 5 arithmoi plus 15 units the middle number, and 3 arithmoi plus 15 units the least number.

From then on, since the greatest number added to the least is again twice the middle number, we have 11 arithmoi plus 15 units is twice the middle number. Now, the middle number is 5 arithmoi plus 15 units; so 10 arithmoi plus 30 units are equal to 11 arithmoi plus 15 units; so the required number is 15 units and this number solves the problem.

2.2 Determinate systems of equations of first degree

Book I:1. To divide a given number into two numbers whose difference is given.

Let the given number be 100 and the difference 40 units; find the numbers.

Let the lesser number be 1 arithmos, then the greater number will be 1 arithmos plus 40 units. Accordingly, the sum of the two numbers becomes 2 arithmoi plus 40 units. Now, the 100 given units is that sum, so 100 units are equal to 2 arithmoi plus 40 units. Subtract like from like, that is, 40 units from 100, and similarly, 40 units from 2 arithmoi plus 40 units. The two remaining arithmoi are worth 60 units, and each arithmos becomes 30 units.

Returning to the conditions, the lesser number will be 30 units while the greater number will be 70 units and the proof is obvious.

Book I:2. To divide a given number into two numbers whose ratio is given.

Let it be required to divide 60 into two numbers which are in ratio three to one.

Let the lesser number be 1 arithmos; then the greater number will be 3 arithmos, and thus the greater number is three times the lesser number. Also, it is necessary that the sum of the two numbers be 60 units. But the sum of the numbers is 4 arithmoi, so 4 arithmoi are equal to 60 units, and then the arithmos is 15 units. Accordingly, the lesser number will be 15 units and the greater number 45 units.

Book I:3. To divide a given number into two numbers one of which is in a given ratio with the other plus a given difference.

Let it be required to divide 80 into two numbers such that the greater number be three times the lesser number plus 4 units.

Let the lesser number be 1 arithmos. Then the greater number will be 3 arithmoi plus 4 units, and thus the greater number will be three times the lesser plus 4 units. We also want that the sum of the two numbers be equal to 80 units. Now, the sum of these two numbers is 4 arithmoi plus 4 units, so 4 arithmoi plus 4 units are equal to 80 units. Subtract like from like. The remaining 76 units are equal to 4 arithmoi, and the arithmos becomes 19 units.

Returning to the conditions, the lesser number will be 19 units and the greater number 61 units.

Book I:4. To find two numbers in a given ratio such that their difference is also given.

Let it be required that the greater number be five times the lesser and the difference of the numbers be 20 units.

Let the small number be 1 arithmos, then the big will be 5 arithmoi. Finally, we want 5 arithmoi to exceed 1 arithmos by 20 units. Now, their difference is 4 arithmoi which will become equal to 20 units. From then on, the small number will be 5 units, and the big will be 25 units; it is established that the big number is 5 times the small, and their difference is 20 units.

Book I:5. To divide a given number into two numbers such that given fractions (not the same) of each number when added together produce a given number.

The latter given number, however, must be such that it lies between the two numbers obtained when the given fractions (respectively) are taken of the first given number.

From then on, let it be required to divide 100 into two numbers, such that one third of the first number and one fifth of the second number added together produce 30 units.

Let one fifth of the second number be 1 arithmos, and so this second number will be 5 arithmoi. Then one third of the first number will be 30 units minus 1 arithmos, and so the first number will be 90 units minus 3 arithmoi. Finally, we want the two numbers added together to produce 100 units. Now, the two numbers added together produce 2 arithmoi plus 90 units, which becomes equal to 100 units. Subtract like from like, and so the remaining 10 units are equal to 2 arithmoi [so the arithmos will be 5 units.]1[3]

Returning to the conditions, we supposed that one fifth of the second number be 1 arithmos, that is 5 units; so the second number will be 25 units. Now, one third of the first number is 30 units minus 1 arithmos, which is 25 units, so the first number will be 75 units.

[3]This is a gap filled in by Bachet [16].

Thus it is established that one third of the first number plus one fifth of the second number is 30 units, and that the sum of the numbers produces the given number.

Book I:6. To divide a given number into two numbers such that a given fraction of the first number exceeds a given fraction of the second number by a given number.

It is necessary, that the later given number be smaller than the number obtained when the greater given fraction is taken of the first given number.

Let it be required to divide 100 into two numbers such that the difference of one fourth of the first number and one sixth of the second number be 20 units.

Let one sixth of the second number be 1 arithmos, then this number will be 6 arithmoi. From then on, one fourth of the first number will be 1 arithmos plus 20 units; so the second number will be 4 arithmoi plus 80 units. Moreover, we want the numbers added together to produce 100 units. Now, these two numbers added together form 10 arithmoi plus 80 units, which equal to 100 units.

Subtract like from like: it remains that 10 arithmoi are equal to 20 units, and the arithmos becomes 2 units.

Returning to our conditions, it was proposed that one sixth of the second number be 1 arithmos, which is 2 units, then the second number will be 12 units. On the other hand, since one fourth of the first number is 1 arithmos plus 20 units, which is equal to 22 units, then the first number will be 88 units. From then on, it is established that one fourth of the first number exceeds one sixth of the second number by 20 units and that the sum of the required numbers is the given number.

Book I:12. To divide a given number twice into two numbers in such a way that the first number of the first pair may have to the first number of the second pair a given ratio and also the second of the second pair to the second of the first pair another given ratio.

Let it be required to divide 100 twice into two numbers in such a way that the greater number of the first pair be twice the lesser number of the second pair, and the greater number of the second pair be three times the lesser number of the first pair.

Let the lesser number of the second pair be 1 arithmos. From then on, the greater number of the first pair will be 2 arithmoi. Accordingly, the lesser number of the first pair will be 100 units minus 2 arithmoi. And since the greater number of the second pair is three times the previous numner, it wil be 300 units minus 6 arithmoi. Also, it is necessary that the sum of the numbers from the second pair be 100 units. But this sum is 300 units minus 5 arithmoi. Equate that to 100 units and the arithmos becomes 40 units.

Returning to the conditions, the greater number of the first pair, being 2 arithmoi, will become 80 units; the lesser number from the same pair, being 100 units minus 2 arithmoi, will become 20 units; the greater number of the second pair, being 300 units minus 6 arithmoi, will become 60 units, and the lesser number of the second pair, being 1 arithmos, will become 40 units and the proof is obvious.

Book I:13. To divide a given number three times into two numbers in such a way that one of the first pair has to one of the second pair a given ratio, the second of the second pair to one of the third pair another given ratio, and the second of the third pair to the second of the first pair another given ratio.

Let it be required to divide 100 three times into two numbers in such a way that the greater number of the first pair be three times the lesser number of the second pair; the greater number of the second pair be twice the lesser number of the third pair, and the greater number of the third pair be four times the lesser number of the first pair.

Let the lesser number of the third pair be 1 arithmos. From then on, the greater number of the second pair will be 2 arithmoi and since the number to be divided is 100 units it follows that the lesser number of the second pair will be 100 units minus 2 arithmoi. Since the greater number of the first pair is three times the previous number, it will be 300 units minus 6 arithmoi. Accordingly, the lesser number of the first pair will be 6 arithmoi minus

200 units. Since the greater number of the third pair is 4 times the previous number, it will be 24 arithmoi minus 800 units. Also, it is necessary that the sum of the numbers in the third pair be 100 units. But this sum is 25 arithmoi minus 800 units. Equate that to 100 units and the arithmos becomes 36 units.

Returning to the conditions, the lesser number of the third pair will be 36 units, while the greater number will be 64 units; the lesser number of the first pair will be 16 units, while the greater number will be 84 units; and the lesser number of the second pair will be 28 units, while the greater number will be 72 units. It is clear that these numbers solve the problem.

Book I:15. To find two numbers such that each, after receiving from the other a given number, is in a given ratio with the remaining number.

Let it be required that the first number, after receiving 30 units from the second number, becomes twice as big as the second number, and the second number, after receiving 50 units from the first, becomes three times the first number.

Let the second number be 1 arithmos plus 30 units, which that number will give. From then on, the first number will be 2 arithmoi minus 30 units, and if it receives 30 units from the second number it becomes twice as big as the second number after giving its 30 units. Also, it is necessary that the second number after receiving 50 units from the first becomes three times the first after giving its 50 units. But if the first number gives 50 units, the remainder will be 2 arithmos minus 80 units, while the second number after receiving 50 units becomes 1 arithmos plus 80 units. Finally, it is necessary that 1 arithmos plus 80 units be 3 times 2 arithmoi minus 80 units; so the least of these numbers, taken 3 times will be equal to the greater number and the arithmos becomes 64 units.

Accordingly, the first number will be 98 units, the second 94 units and these numbers solve the problem.

Book I:16. To find three numbers which added two by two produce numbers that are given.

It is necessary, however, that half of the sum of the given numbers be greater than each of these numbers.

Let it be required that the first number added to the second make 20 units; the second added to the third make 30 units, and the third added to the first make 40 units.

Let the sum of the three numbers be 1 arithmos. From then on, since the first number plus the second number is 20 units, if we subtract 20 units from 1 arithmos we will have the third number as 1 arithmos minus 20 units. For the same reason, the first number will be 1 arithmos minus 30 units and the second number will be 1 arithmos minus 40 units. It is necessary, also, that the sum of the three numbers become equal to 1 arithmos. But the sum of the three numbers is 3 arithmoi minus 90 units. Equate that to 1 arithmos and the arithmos becomes 45 units.

Returning to the hypothesis, the first number will be 15 units, the second will be 5 units, the third will be 25 units and the proof is clear.

Book I:17. To find four numbers which added three by three produce numbers that are given.

It is necessary, however, that one third of the sum of the four numbers be greater than each of the numbers.

Let it be required that the sum of the first three numbers be 20 units; the sum of the second, third and fourth numbers be 22 units; the sum of the third, fourth and first be 24 units; and the sum of the fourth, first and second be 27 units.

Let the sum of the four numbers be 1 arithmos. From then on, if we subtract the first three numbers, that is to subtract 20 units from 1 arithmos, it remains as the fourth number 1 arithmos minus 20 units. For the same reason, the first number will be 1 arithmos minus 22 units; the second will be 1 arithmos minus 24 units; and the third 1 arithmos minus 27 units. At last, it is necessary that the four numbers added together be equal to 1 arithmos. But the four numbers added together make 4 arithmoi minus 93 units, which we equate to 1 arithmos and the arithmos becomes 31 units.

Returning to the hypothesis, the first number will be 9 units, the second number will be 7 units, the third 4 units, the fourth 11 units and these numbers solve the problem.

Book I:18. To find three numbers such that the sum of any two exceeds the third by a given number.

Let it be required that the sum of first two numbers exceeds the third by 20 units; the sum of second and third exceeds the first by 30 units, and the sum of third and first exceeds the second by 40 units.

Let the sum of the three numbers be 2 arithmoi. From then on, since the sum of first and second number exceeds the third by 20 units, if we add the third number to both sides we have that the sum of the three numbers is twice the third number plus the excess of 20 units. Accordingly, if we subtract 20 units from the sum of the three numbers, that is from 2 arithmoi, we will have twice the third number is equal to 2 arithmoi minus 20 units, and thus the third number will be 1 arithmos minus 10 units. For the same reason, the first number will be 1 arithmos minus 15 units and the second number 1 arithmos minus 20 units. It remains to equate the sum of the three numbers to 2 arithmoi. But the three numbers added together are 3 arithmoi minus 45 units which we equate to 2 arithmoi and the arithmos becomes 45 units.

Returning to the conditions, the first number will be 30 units, the second number will be 25 units, the third number 35 units and these numbers satisfy the proposition.

Book I:19. To find four numbers such that the sum of any three exceeds the fourth by a given number.

It is necessary that half the sum of the four given differences be greater than any one of them.

Let it be required that the sum of first three numbers exceeds the fourth by 20 units; the sum of second, third and fourth number exceeds the first by 30 units, the sum of third, fourth and first number exceeds the second by 40 units and finally, the sum of fourth, first and second number exceeds the third by 50 units.

Let the sum of all four numbers be 2 arithmoi. From then on, since the sum of the first three numbers exceeds the fourth by 20 units, the sum of all four numbers exceeds twice the fourth number in the same way that the sum of the first three numbers exceeds the fourth, and since the sum of the four numbers is 2 arithmoi, it follows that 2 arithmoi exceeds twice the fourth number by 20 units. Accordingly, twice the fourth number will be 2 arithmoi minus 20 units and the fourth number will be 1 arithmos minus 10 units. For the same reason, the first number will be 1 arithmos minus 15 units, the second number will be 1 arithmos minus 20 units, and the third number will be 1 arithmos minus 25 units. It is necessary, however, that the sum of all four numbers be equal to 2 arithmoi. But the sum of the four numbers is 4 arithmoi minus 70 units. Equate that to 2 arithmoi and the arithmos becomes 35 units.

Returning to the hypothesis, the first number will be 20 units, the second will be 15 units, the third 10 units, the fourth 25 units and these numbers solve the problem.

Book I:20. To divide a given number into three numbers is such a way that that the sum of each extreme and the mean has to the other extreme a given ratio.

Let it be required to divide 100 into three numbers in such a way that the sum of the first two numbers be three times the third and the sum of the second and third number be four times the first.

Let the third number be 1 arithmos. From then on, since the sum of the first two numbers is three times the third, let us set the sum of these two numbers equal to 3 arithmoi.

Accordingly, the sum of the three numbers is 4 arithmoi, which we equate to 100 units and the arithmos becomes 25 units.

Returning to the conditions, the third number being 1 arithmos, will be 25 units and the sum of the first and second number being 3 arithmoi, will be 75 units. On the other hand, since the sum of second and third number is four times the first, let the first number be 1 arithmos.[4] From then on, the sum of the second and third numbers will be 4 arithmoi and the sum of the three numbers will be 5 arithmoi. But, this sum is also 100 units and the arithmos becomes 20 units. Accordingly, the first number will be 20 units, the sum of second and third number will be 80 units; and since the third number is 25 units, the second number remains to be 55 units and these numbers satisfy the conditions.

Book I:21. To find three numbers in such a way that the greatest exceeds the middle number by a given fraction of the least, the middle exceeds the least by a given fraction of the greatest, and the least exceeds a given number by a given fraction of the middle.

It is necessary that the middle number exceeds the least by such a fraction of the greatest that if its denominator be multiplied by the excess of the middle number over the least, the coefficient of the arithmos in the product is greater than the coefficient of the arithmos in the expression for the middle number resulting from the assumption made.

Suppose that the greatest number exceeds the middle number by one third of the least; the middle number exceeds the least by one third of the greatest; and the least exceeds 10 units by one third of the middle.

Let the least number, which exceeds the middle by one third, be 1 arithmos plus 10 units. From then on, the middle number will be 3 arithmoi, and thus the least number is formed by one third of the middle plus 10 units. And also the middle number will be 3 arithmoi. From then on, since we want the least number to exceed one third of the middle by 10 units, it will be 1 arithmos plus 10 units. It is necessary that the middle number exceeds the

[4] This is a second or new arithmos. Diophantus does not distinguish between different unknowns which he introduces in the solution of the same problem. He uses the same word "arithmos" for different unknowns.

least by one third of the first number.[5] But, the middle number exceeds the greatest number by 2 arithmoi minus 10 units, which is then one third of the greatest number. Accordingly, the greatest number will be 6 arithmoi minus 30 units. Also, it is necessary, that the greatest number exceeds the middle by one third of the least. Now, the greatest number exceeds the middle by 3 arithmoi minus 30 units, which is then one third of the least number, so the least number will be 9 arithmoi minus 90 units. But, we have found that this number is also 1 arithmos plus 10 units, so the arithmos becomes $12\frac{1}{2}$ units. From then on, the third number[6] will be $22\frac{1}{2}$ units, the middle number will be $37\frac{1}{2}$ units, the greatest number will be 45 units and these problems satisfy the conditions.

2.3 Determinate systems of equations reducible to first degree

Book I:26. Given two numbers to find a number which, when multiplied by the given numbers respectively, make one product a square and the other the side of that square.

Let the two given numbers be 200 and 5 and the required number be 1 arithmos. From then on, if the required number is multiplied by 200 units, it gives 200 arithmoi, and if it is multiplied by 5 units, it gives 5 arithmos. Now, it is necessary that one of the numbers be a square and the other a side of that square; so if we square 5 arithmoi we obtain 25 arithmoi squared equal to 200 arithmoi. Divide both sides by the arithmos and it follows that 25 arithmoi are equal to 200 units and the arithmos becomes 8 units, which satisfies the conditions.

Book I:29. To find two numbers such that their sum and the difference of their squares are given numbers.

Let it be required that the sum of the numbers is 20 units and the difference of their squares is 80 units.

[5] the greatest number
[6] the least number

Let the difference of the numbers be 2 arithmoi. The greater number will be 1 arithmos plus 10 units, and the lesser number 10 units minus 1 arithmos, which establishes again that the sum of the numbers is 20 units and their difference is 2 arithmoi.

It is necessary also that the difference of the squares of the numbers be 80 units. But the difference of their squares is 40 arithmoi, which we equate to 80 units. From then on, it follows again that the greater number is 12 units, the lesser number is 8 units and these numbers satisfy the problem.

Book I:31. To find two numbers in a given ratio and such that the sum of their squares has to their sum a given ratio.

Let it be required that the greater number be three times the lesser and the sum of the squares of these numbers be five times their sum.

Let the lesser number be 1 arithmos, then the greater will be 3 arithmoi. Also, it is necessary, that the sum of the squares of the numbers [be five times these numbers taken together. But the sum of the squares of the numbers][7] is 10 arithmoi squared, while the sum of the numbers is 4 arithmoi, so that 10 arithmoi squared are five times 4 arithmoi. Accordingly, 20 arithmoi are equal to 10 arithmoi squared and the arithmos becomes 2 units.

So, the lesser number will be 2 units, the greater will be 6 units and these numbers satisfy the conditions.

Book I:32. To find two numbers in a given ratio and such that the sum of their squares has to their difference a given ratio.

Let it be required that the greater number be three times the lesser, and the sum of their squares be ten times their difference.

Let the lesser number be 1 arithmos. From then on, the greater number will be 3 arithmoi. Also, we want that the sum of the squares of the numbers to be ten times their

[7]This text is added by Bachet [16].

difference. But the sum of the squares of the numbers is 10 arithmoi squared, while the difference of the numbers is 2 arithmoi; so 10 arithmoi squared is 10 times 2 arithmoi. Divide both sides by the arithmos; it follows that 10 arithmoi are equal to 20 units and the arithmos becomes 2 units. From then on, the lesser number will be again 2 units, the greater number 6 units and these numbers satisfy the conditions.

Book I:33. To find two numbers in a given ratio and such that the difference of their squares has to their sum a given ratio.

Let it be required that the greater number be three times the lesser and the difference of the squares of the numbers be six times the sum of these numbers.

Let the lesser number be 1 arithmos. From then on, the greater will be 3 arithmoi. It remains to have that the difference of the squares of the numbers is 6 times the sum of the numbers. But the difference of the squares of the numbers is 8 arithmoi squared, while the sum of the numbers is 4 arithmoi, so 8 arithmoi squared are six times 4 arithmoi; so 24 arithmoi are equal to 8 arithmoi squared and the arithmos becomes 3 units. [From then on, the lesser number will be 3 units, the greater number will be 9 units][8] and these numbers solve the problem.

Book I:34. To find two numbers in a given ratio and such that the difference of their squares has to their difference a given ratio.

Let it be required that the greater number be three times the lesser and the difference of squares of the numbers be twelve times the difference of these numbers.

Let the lesser number be again 1 arithmos. From then on, the greater number will be 3 arithmoi. Also, it is necessary that the difference of squares of the numbers be twelve times the difference of these numbers. But the difference of squares of the numbers is 8 arithmoi squared, which is then twelve times 2 arithmoi. Accordingly, 24 arithmoi are equal

[8] This text is inserted by Bachet [16].

to 8 arithmoi squared, the arithmos becomes again 3 units and the proof is obvious.

Corollaries. Similarly, by the same method, can be found:

Two numbers in a given ratio and such that their product is to their sum in a given ratio.

On the other hand, two numbers in a given ratio and such that their product is to their difference in a given ratio.

Book I:35. To find two numbers in a given ratio and such that the square of the lesser number has to the greater number a given ratio.

Let it be required that the greater number be three times the lesser, and the square of the lesser be six times the greater number.

Let the lesser number again be 1 arithmos. From then on, the greater number will be 3 arithmoi. It is necessary also, that the square of the lesser number be six times the greater number.But the square of the lesser number is 1 arithmos squared; so 1 arithmos squared is six times 3 arithmoi, so that 18 arithmoi are equal to 1 arithmos squared and the arithmos becomes 18 units. From then on, the lesser number will be 18 units, the greater number will be 54 units, and these numbers solve the problem.

Book I:36. To find two numbers in a given ratio and such that the square of the lesser has to the lesser itself a given ratio.

Let it be required that the greater number be three times the lesser and the square of the lesser be six times the lesser itself.

The greater number will be as before 3 arithmoi and the lesser 1 arithmos, which establishes that the greater number is three times the lesser. Also, it is necessary that the square of the lesser number be six times the lesser itself. Accordingly, 1 arithmos squared is six times 1 arithmos, so that 6 arithmoi are equal to 1 arithmos squared and the arithmos becomes 6 units. So, the lesser number will be 6 units, the greater number will be 18 units and these numbers solve the problem.

Book I:37. To find two numbers in a given ratio and such that the square of the lesser has to the sum of both numbers a given ratio.

Let it be required that the greater number be three times the lesser and the square of the lesser be twice the sum of the two numbers.

The greater number will be again as before 3 arithmoi and the lesser number 1 arithmos. It remains also to have that the square of the lesser number be twice the sum of the two numbers. But the square of the lesser number is 1 arithmos squared, while the sum of the two numbers is 4 arithmoi, so 1 arithmos squared is twice 4 arithmoi, so that 8 arithmoi are equal to 1 arithmos squared and the arithmos becomes 8 units. From then on, the lesser number will be 8 units, the greater will be 24 units and these numbers solve the problem.

Book I:38. To find two numbers in a given ratio and such that the square of the lesser number has to the difference of the numbers a given ratio.

Let it be required that the greater number be three times the lesser and the square of the lesser be six times the difference of the numbers.

The greater number will be again as before 3 arithmoi while the lesser number will be 1 arithmos. It remains also to have that the square of the lesser number be six times the difference of the numbers. Accordingly, 1 arithmos squared is six times 2 arithmoi, so that 12 arithmoi are equal to 1 arithmos squared and the arithmos becomes 12 units. From then on, the lesser number will be 12 units, the greater number will be 36 units and these numbers satisfy the conditions.

Corollaries. Similarly can be found by the same method:

Two numbers in a given ratio and such that the square of the greater has to the lesser a given ratio.

And again, two numbers in a given ratio and such that the square of the greater has to the greater itself a given ratio.

Similarly, two numbers in a given ratio and such that the square of the greater has to the sum of the two numbers a given ratio.

And finally, two numbers in a given ratio and such that the square of the greater has to the difference of the two numbers a given ratio.

Book IV:36. To find three numbers such that the product of any two has to the sum of those two a given ratio.

Let it be required that the product of the first two numbers be three times their sum; the product of second and third numbers be four times their sum, and the product of first and third numbers be five times their sum.

Let the second number be 1 arithmos, then with accordance with the Lemma,[9] the first number will be 3 arithmoi over 1 arithmos minus 3 units, and similarly, the third number will be 4 arithmoi over 1 arithmos minus 4 units.

It is necessary also that the product of first and third numbers be five times their sum. But the product of first and third numbers is 12 arithmoi squared over 1 arithmos squared plus 12 units minus 7 arithmoi. On the other hand, the sum of first and third numbers is 7 arithmoi squared minus 24 arithmoi over 1 arithmos squared plus 12 units minus 7 arithmoi, which is obtained by the following way. When it is necessary to add fractions, such as 3 arithmoi over 1 arithmos minus 3 units, and 4 arithmoi over 1 arithmos minus 4 units, we conversely multiply the arithmos of the fractions by the denominator, in this case 3 arithmoi by the denominator of the other fraction, that is by 1 arithmos minus 4 units, and on the other hand, 4 arithmoi by the denominator of the other fraction, that is by 1 arithmos minus

[9] Lemma to IV:36

3 units. Thus, we obtain as a sum 7 arithmoi squared minus 24 arithmoi over the product of the denominators, that is 1 arithmos squared plus 12 units minus 7 arithmoi. Now, we have as product of first and third numbers 12 arithmoi squared over 1 arithmos squared plus 12 units minus 7 arithmoi, so 12 arithmoi squared [over 1 arithmos squared plus 12 units][10] minus 7 arithmoi are five times the sum of the numbers. So, let's take 5 times this sum and we obtain 35 arithmoi squared minus 120 arithmoi over 1 arithmos squared plus 12 units minus 7 arithmoi. Multiply both sides by the common denominator of the expressions, that is by 1 arithmos squared plus 12 units minus 7 arithmoi, and 12 arithmoi squared becomes equal to 35 arithmoi squared minus 120 arithmoi, hence, the arithmos becomes $\frac{120}{23}$.

Returning to the conditions, we had as first number 3 arithmoi over 1 arithmos minus 3 units, as second number 1 arithmos, and as third number 4 arithmoi over 1 arithmos minus 4 units. Now, we found that the arithmos is $\frac{120}{23}$, and if we substitute it in 3 arithmoi, they become 360 units, then substitute in the denominator, 120 units substituted in 1 arithmos minus 3 units gives 51 units. Finally, the first number will be $\frac{360}{51}$ and the second number will be $\frac{120}{23}$, because it does not have arithmos in the denominator. As for the third number, $\frac{120}{23}$ likewise substituted in 4 arithmoi gives 480, and likewise in the denominator, 120 substituted in 1 arithmos minus 4 units gives 28 units, so that finally, the third number will be $\frac{480}{28}$ units, and the proof is obvious.

2.4 Systems of equations apparently indeterminate but really reduced, by arbitrary assumptions, to determinate equations of first degree

Book I:14. To find two numbers such that their product has to their sum a given ratio.

It is necessary, however, that the assumed value of one of the two numbers be greater than the number representing the given ratio.

Let it be required that the product of the numbers be three times their sum.

Let one of the numbers be 1 arithmos, and the other, greater than 3 units, which

[10] added by Bachet [16]

follows from the proceeding conditions, be 12 arithmoi. From then on, the product of the numbers is 12 arithmoi and their sum is 1 arithmos plus 12 units. Finally, it is necessary that 12 arithmoi be three times 1 arithmos plus 12 units, so the lesser number taken three times is equal to the greater number and the arithmos becomes 4 units.

One of the numbers will be 4 units, the other 12 units, and they solve the problem.

Book I:22. To find three numbers such that if each give to the next following a given fraction of itself, in order, the results after each had given and taken may be equal.

Let it be required that the first number gives a third of itself to second; the second number gives a fourth of itself to third; and the third gives a fifth of itself to first number, and after the mutual transfer we obtain equal numbers.

Since the first number gives a third of itself, let this be some number of arithmoi divisible by 3, let's say 3 arithmoi; and since the second number gives one fourth of itself, let it be some number of units divisible by 4, let's say 4 units. From then on, the second number after giving and receiving becomes 1 arithmos plus 3 units.

It is necessary that the first number after giving and receiving becomes as well 1 arithmos plus 3 units. But this first number after giving a third of itself, that is 1 arithmos, and receiving 3 units minus 1 arithmos, becomes 1 arithmos plus 3 units, so 3 units minus 1 arithmos are one fifth of the third number and this third number is then 15 units minus 5 arithmoi.

Finally, it is necessary that the third number after giving a fifth of itself and receiving a fourth of the second number, that is 1 unit, becomes also 1 arithmos plus 3 units. Now, this third number giving a fifth of itself, that is 3 units minus 1 arithmos, remains 12 units minus 4 arithmoi, and receiving a fourth of the second number, that is 1 unit, becomes 13 units minus 4 arithmoi, which we equate to 1 arithmos plus 3 units, and the arithmos becomes 2 units.

Returning to the hypothesis, the first number will be 6 units, the second number 4 units, the third number 5 units and the proposed things are demonstrated.

Book I:23. To find four numbers such that if each give to the next following a given fraction of itself, the results may all be equal.

Let it be required that the first number gives a third of itself to second; the second gives a fourth of itself to third; the third gives a fifth of itself to fourth; the fourth gives a sixth of itself to first, and after the mutual transfer we obtain equal numbers.

Since the first number gives a third of itself, let this be some number of arithmoi divisible by 3, say 3 arithmos; and since the second number gives a fourth of itself, let it be some number of units divisible by 4, say 4 units. From then on, the second number giving a fourth of itself, that is 1 unit, and receiving one third from the first, that is 1 arithmos, becomes 1 arithmos plus 3 units.

It is necessary that the first number after giving a third of itself, that is 1 arithmos, and receiving a sixth of the fourth, becomes also 1 arithmos minus 3 units. But the first number after giving 1 arithmos, remains 2 arithmoi, so it is necessary that after receiving a sixth of the fourth number, it becomes 1 arithmos plus 3 units. Accordingly, 3 units minus 1 arithmos are the one sixth part of the fourth number. And the fourth number will be 18 units minus 6 arithmoi.

It remains to obtain that the fourth number after giving a sixth of itself and receiving a fifth of the third number, becomes also 1 arithmos plus 3 units. But after giving one sixth of itself, that is 3 units minus 1 arithmos, it remains 15 units minus 5 arithmoi, so it is necessary that after receiving a fifth of the third number, it becomes 1 arithmos plus 3 units. Now, if this remainder receives 6 arithmos minus 12 units, it becomes 1 arithmos plus 3 units; so that 6 arithmoi minus 12 units are the one fifth part of the third number, and this third number will be 30 arithmoi minus 60 units.

Finally, it is necessary that the third number after giving a fifth of itself and receiving one fourth of the second number, becomes also 1 arithmos plus 3 units. But the third number after giving a fifth of itself, that is 6 arithmoi minus 12 units, remains 24 arithmoi minus 48 units, and after receiving a fourth from the second number, becomes 24 arithmoi minus 47 units, which we equate to 1 arithmos plus 3 units and the arithmos becomes $\frac{50}{23}$.

Returning to the hypothesis, the first number will be $\frac{150}{23}$; the second number will be $\frac{92}{23}$; the third number will be $\frac{120}{23}$; and the fourth number will be $\frac{114}{23}$. Eliminate the fractions and the first number becomes 150, the second 92, the third 120, and the fourth 114. These numbers satisfy the conditions.

Book I:24. To find three numbers such that if each receives a given fraction of the sum of the other two, the results are all equal.

Let it be required that the first number receives a third of the sum of the two remaining numbers, the second receives a fourth of the sum of the two remaining numbers, the third receives a fifth of the sum of the two remaining numbers and the resulting numbers are equal.

Let the first number be 1 arithmos, and since the sum of the remaining two numbers give him one third of itself, this sum must be, for convenience, a number of units divisible by 3, such as 3 units. From then on, the sum of the three numbers will be 1 arithmos plus 3 units and it is established that the first number after receiving a third of the sum of the remaining two numbers, is 1 arithmos plus 1 unit.

It is necessary also, that the second number after receiving a quarter of the sum of the [remaining][11] two numbers becomes one arithmos plus 1 unit. Take all this four times. [12] Accordingly, four times the second number plus the other two is three times the second number plus the other three numbers, so three times the second number plus the three numbers becomes 4 arithmoi plus 4 units. From then on, if we subtract from both sides the sum of the three numbers, the 3 arithmoi plus 1 unit that remain are three times the second number, and the second number will be 1 arithmos plus $\frac{1}{3}$ of a unit.

Also, it is necessary that the third number after receiving a fifth of the sum of the

[11] Tannery fills in this word.[6]

[12] Multiply both sides by 4.

remaining two numbers, becomes 1 arithmos plus 1 unit. Similarly, take both sides five times,[13] and we conclude, by the same reasoning, that the third number is 1 arithmos plus $\frac{1}{2}$ unit.

Finally, it is necessary that the sum of the three numbers is 1 arithmos plus 3 units, and the arithmos becomes $\frac{13}{12}$. From then on, after eliminating the fractions, the first number will be 13 units, the second number 17 units, the third number 19 units, and these numbers satisfy the conditions.

Book I:25. To find four numbers such that, if each receives a given fraction of the sum of the remaining three, the four resulting numbers are equal.

Let it be required that the first number receives a third of the sum of the remaining three numbers; the second number receives a quarter of the sum of the remaining three numbers, in the same way the third also receives a fifth and finally the fourth number receives a sixth, and the resulting numbers are equal.

Let the first number be 1 arithmos, and since the sum of the remaining three numbers give a third of itself, this sum must be a number of units divisible by 3, such as 3 units. From then on, the first number after receiving a third of the sum of the remaining three numbers, becomes 1 arithmos plus 1 unit.

In the same way, we do this again four times and conclude that the second number is 1 arithmos plus $\frac{1}{3}$ of a unit, the third number is 1 arithmos plus $\frac{1}{2}$ unit, and the fourth number is 1 arithmos plus $\frac{3}{5}$ of a unit.

It is necessary that the sum of the four numbers is equal to 1 arithmos plus 3 units, and we conclude that the arithmos is 47 units over 90 units. From then on, the first number will be 47 units, the second 77 units, the third 92 units, the fourth 101 units, and these numbers satisfy the conditions.

[13] Multiply both sides by 5.

Book IV:33. To find two numbers such that each after receiving from the other the same part or parts, has to the remainder of the giving number a given ratio.

Let it be required that the first number, having received a part or parts of the second, be three times the remainder [of the second number][14], and the second number having received the same part or parts of the first number, be five times the remainder[15].

Let the second number be 1 arithmos plus 1 unit and its part or parts be 1 unit. Accordingly, the first number will be 3 atithmoi minus 1 unit, and if the first number receives part or parts, that is 1 unit, from the second number, it becomes thus three times the remainder.

We also want that the second number after receiving the same part or parts [from the first number][16], be five times the remainder. But since the sum of both numbers is 4 arithmoi, the second number received what was given from the first, and thus the obtained number became five times the remainder, then it follows that the sum of the obtained numbers and the remainder must be 4 arithmoi, so that we will have this remainder if we take one sixth of 4 arithmoi, that is $\frac{2}{3}$ of an arithmos. Accordingly, if we subtract $\frac{2}{3}$ of an arithmos from 3 arithmoi minus 1 unit, we will have the part or parts of the first number. From then on, if we subtract, the remainder becomes $\frac{7}{3}$ of an arithmos minus 1 unit, and the second number, that is 1 arithmos plus 1 unit, after receiving $\frac{7}{3}$ of an arithmos minus 1 unit of the first number, becomes then five times the remainder of the first number.

It remains now to find if $\frac{7}{3}$ of an arithmos minus 1 unit is part or parts of 3 arithmoi minus 1 unit, as 1 unit is part or parts of 3 arithmoi minus 1 unit. Now, as we search if it is, the product of $\frac{7}{3}$ of an arithmos minus 1 unit and 1 arithmos plus 1 unit is equal to the product of 3 arithmoi minus 1 unit over 1 unit, that is the fractions are multiplied conversely, from which $\frac{7}{3}$ of an arithmos squared plus $\frac{4}{3}$ of an arithmos minus 1 unit are equal to 3 arithmoi minus 1 unit and the arithmos becomes $\frac{5}{7}$.

[14] of the second number
[15] of the first number
[16] A phrase filled in by Bachet.[16]

Returning to the hypothesis, the first number will be $\frac{8}{7}$ and the second $\frac{12}{7}$. Now, the part of the second number was 1 unit. Consider 1 unit compared to the second number and we have $\frac{7}{12}$. Take these two numbers 7 times and the first will become 8 units, the second 12 units and the parts will be $\frac{7}{12}$. But since the first number is not a multiple of 12, take the numbers three times, in order to avoid the parts, and so the first number will be 24, the second will be 36, their parts will be $\frac{7}{12}$ and the proof is obvious.

2.5 Indeterminate equations of first degree

Lemma to IV:34. To find two numbers indeterminately such that their product together with their sum is a given number.

Let the given number be 8 units. Let the first number be 1 arithmos and the second number be 3 units. From then on, the product of these numbers added to their sum is 4 arithmoi plus 3 units, which we equate to 8 units and the arithmos becomes $\frac{5}{4}$. Returning to the hypothesis, the first number will be $\frac{5}{4}$ and the second number 3 units.

Consider now how does the arithmos become $\frac{5}{4}$. It arises from 5 divided by 4. But 5 comes from 8 minus 3, while 4 exceeds the second number by 1 unit. We may accordingly put for the second number any number; and if we subtract it from 8 units and divide the remainder by the second number increased by 1 unit, we will have the 1st number.

Let the second number be, for example, 1 arithmos minus 1 unit. Subtract this expression from 8 units, it remains 9 units minus 1 arithmos, which we divide by the number that exceeds the second number by 1 unit, that is 1 arithmos, and we obtain 9 arithmoston[17] minus 1, which will be the first number.

Thus we have determined that the product of the numbers plus their sum is 8 units. Moreover, we can get one solution of the indeterminate problem if we make the arithmos to have as many units as we wish according to the conditions, and the problem is solved.

[17] Arithmoston is the reciprocal of the arithmos, or $\frac{1}{x}$

Lemma to IV:35. To find two numbers indeterminately such that their product minus their sum is a given number.

Let the given number be 8. Let it be supposed that the first number is 1 arithmos and the second is 3 units. From then on, the product of the numbers minus their sum, is 2 arithmoi minus 3 units, which we equate to 8 units, and the arithmos becomes $5\frac{1}{2}$ units. Returning to the conditions, the first number will be $5\frac{1}{2}$ units and the second number will be 3 units. Let's consider again how does the arithmos become $5\frac{1}{2}$ units. It comes from 11 divided by 2. But 11 is the sum of the given number and the second number, while 2 is the second number minus 1. If we suppose, accordingly, that the second number is any number, and if we add it to the given number and then divide the obtained expression by the second number minus 1 unit, we will find the first number.

Let the second number be 1 arithmos plus 1 unit; this expression added to 8 units, is 1 arithmos plus 9 units, which we divide by the second number minus 1 unit, that is by 1 arithmos, and the first number becomes 1 unit plus 9 arithmoston.[18] Thus, we have determined in the indeterminate problem, that the product of the numbers minus their sum, is 8 units.

Lemma to IV:36. To find two numbers indeterminately such that their product has to their sum a given ratio.

Let it be required that the product of the numbers be three times their sum.

Let the first number be 1 arithmos and the second number be 5 units. From then on, the product of the numbers is 5 arithmoi. Now, we want this to be three times 1 arithmos plus 5 units, so that 3 arithmoi plus 15 units are equal to 5 arithmoi; the arithmos becomes $7\frac{1}{2}$ units and the second number 5 units.

Let us examine how the arithmos become $7\frac{1}{2}$ units. It comes from 15 divided by 2. But 15 is the second number multiplied by the ratio, while 2 comes from the difference

[18] Arithmoston is the reciprocal of the arithmos, or $\frac{1}{x}$

between the second number and the ratio. If we express accordingly the second number arbitrarily by an arithmos, and if we multiply it by the ratio, which is 3 arithmos, and if we divide by the difference of the second number and the ratio, that is by 1 arithmos minus 3 units, the first number becomes 3 arithmoi over the expression 1 arithmos minus 3 units.

Chapter 3

MATHEMATICAL COMMENTARY

3.1 First degree equations with one unknown

Book I:7. *To subtract two given numbers from the same number so as to make the remainders have to one another a given ratio.*

Let it be required to subtract 100 and 20 from the same number so as to make the greater remainder be three times the lesser.

Solution. Let the required number be x. Since $x - 20 > x - 100$ and their ratio is 3:1 in order to equate them we must multiply the smaller number $(x - 100)$ by three. Thus we have:

$$3(x - 100) = x - 20$$

and $x = 140$.

Diophantus is dealing with positive rational numbers only, so the given numbers are carefully selected in order to give positive solutions.

Note also that Diophantus always checks his solution at the end: "Returning to the hypothesis, it was posed that the required number is 1 arithmos; so it will be 140 units. If we subtract 100, 40 units are left; at the same time if we subtract 20, 120 units are left; thus it is established that the greater remainder is three times the lesser."

General Problem and Solution.

Given numbers: a, b $(a > b)$

Given ratio: $p : q$ $(p > q)$

Required number: x

Since $a > b$ then $x - b > x - a$ and $q(x - b) = p(x - a)$. So,

$$x = \frac{ap - bq}{p - q}.$$

Book I:8. *To two given numbers to add the same number, so as to make the resulting numbers have to one another a given ratio.*

It is necessary, however, that the given ratio must be less that the ratio which the greater of the given numbers has to the lesser.

Let it be required to add a same number to 100 and to 20, so as to make the greater number be three times the lesser.

Solution. Let the required number be x. Then

$$x + 100 = 3(x + 20)$$

and $x = 20$.

General Problem and Solution.

Given numbers: a, b $(a > b)$

Given ratio: $p : q$ $(p > q)$

Required number: x

Since $a > b$ then $a + x > b + x$ and $q(a + x) = p(b + x)$. So, $x = \dfrac{aq - bp}{p - q}$.

The necessary condition is *that the given ratio must be less that the ratio which the greater of the given numbers has to the lesser*, or in our notation $p : q < a : b$ guarantees a positive solution for x.

Book I:9. *From two given numbers, subtract the same number so as to make the remainders have to one another a given ratio.*

It is necessary, however, that the given ratio be greater than the ratio which the greater of the given numbers has to the lesser.

Let it be required to subtract the same number from 20 and from 100 and to make the greater remainder be six times the lesser.

Solution. Let the required number be x. Then

$$6(20 - x) = 100 - x$$

and $x = 4$.

General Problem and Solution.

Given numbers: a, b $(a < b)$

Given ratio: $p : q$ $(p > q)$

Required number: x

Since

$$p(a - x) = q(b - x),$$

we have

$$x = \frac{ap - qb}{p - q}.$$

The necessary condition is *that the given ratio be greater than the ratio which the greater of the given numbers has to the lesser*, or $p : q > b : a$ guarantees a positive solution for x.

Book I:10. *Given two numbers, to add to the lesser and to subtract from the greater the same number so as to make the resulting numbers be in a given ratio.*

Let it be required to add to 20 and to subtract from 100 a number that is the same in both cases so as to make the greater number be four times the lesser number.

Solution. Let the required number be x. Then

$$20 + x = 4(100 - x)$$

and $x = 76$.

General Problem and Solution.

Given numbers: a, b $(a < b)$

Given ratio: $p : q$ $(p > q)$

Required number: x

Since sum: difference $= p : q$ then

$$q(a + x) = p(b - x)$$

and

$$x = \frac{bp - aq}{p + q}.$$

Remark. Diophantus does not give a necessary condition here, however, if he had given one it would be that the given ratio is greater than the ratio which the lesser of the given numbers has to the greater.

Book I:11. *Given two numbers, add the first to and subtract the second from the same number so as to make the resulting numbers have to one another a given ratio.*

Let it be required to add 20 to a number, to subtract 100 from the same number, so as to make the greater number be three times the lesser.

Solution. Let the required number be x. Then

$$3(x - 100) = x + 20$$

and $x = 160$.

General Problem and Solution.

Given numbers: a, b $(a < b)$

Given ratio: $p : q$ $(p > q)$

Required number: x

$$p(x - b) = q(x + a)$$

and

$$x = \frac{aq + bp}{p - q}.$$

Book I:39. *Given two numbers, to find a third number such that if we take the sum of any two of these three numbers and we multiply it by the remaining number, we obtain three numbers which have equal differences.*[1]

[1] in arithmetic progression

Let 3 and 5 be the two given numbers.

Solution. Let the required number be x. Consider the three products: $(5+3)x$, $(5+x)3$ and $(3+x)5$, or as in the text, $8x$, $3x+15$, $5x+15$. These three expressions are in one of the following orders: $5x+15 > 8x > 3x+15$, $5x+15 > 3x+15 > 8x$ or $8x > 5x+15 > 3x+15$. Since these expressions are in arithmetic progression we obtain:

In the first case: $(5x+15)+(3x+15) = 16x$ and $x = \frac{15}{4}$.

In the second case: $(3x+15) - 8x = (5x+15) - (3x+15)$ and $x = \frac{15}{7}$.

In the third case: $8x + (3x+15) = 2(5x+15)$ and $x = 15$.

3.2 Determinate systems of equations of first degree

Book I:1. *To divide a given number into two numbers whose difference is given.*

Let the given number be 100 and the difference 40 units; find the numbers.

Solution. Let the lesser number be x, then the greater number will be $(x+40)$. So then

$$x + (x+40) = 100$$

and $x = 30$. The numbers are 30 and 70.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} x + y &= 100 \\ y - x &= 40 \end{aligned}$$

When we solve the above system by elimination or substitution, we get $x = 30$ and $y = 70$.

General Problem and Solution.

Given numbers: a, b $(a > b)$

Required number: x, y $(y > x)$

$$\begin{aligned} x + y &= a \\ y - x &= b \end{aligned}$$

By solving the above system we get $x = \dfrac{a-b}{2}$ and $y = \dfrac{a+b}{2}$.

Book I:2. *To divide a given number into two numbers whose ratio is given.*

Let it be required to divide 60 into two numbers which are in ratio three to one.

Solution. Let the lesser number be x, then the greater number will be $3x$. So then $x+3x = 60$ and $x = 15$. The required numbers are 15 and 45.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} x + y &= 60 \\ \frac{x}{y} &= \frac{1}{3} \end{aligned}$$

When we solve the above system, we get $x = 15$ and $y = 45$.

General Problem and Solution.

Given number: a

Given ratio: $p : q$ $(p > q)$

Required numbers: x, y

$$\begin{aligned} x + y &= a \\ \frac{x}{y} &= \frac{q}{p} \end{aligned}$$

By solving the above system we get $x = \dfrac{aq}{p+q}$ and $y = \dfrac{ap}{p+q}$.

Book I:3. *To divide a given number into two numbers one of which is in a given ratio with the other plus a given difference.*

Let it be required to divide 80 into two numbers such that the greater number be three times the lesser number plus 4 units.

Solution. Let the lesser number be x, then the greater number will be $3x + 4$. So then $x + (3x + 4) = 80$, $4x = 76$ and $x = 19$. The required numbers are 19 and 61.

Modern Solution. Let the required numbers be x and y ($x < y$). Then

$$\begin{aligned} x + y &= 80 \\ y &= 3x + 4 \end{aligned}$$

When we solve the above system by substitution, we get $x = 19$ and $y = 61$.

General Problem and Solution.

Given number: a

Given ratio: $p : q$ ($p > q$)

Given difference: b ($b < a$)

Required numbers: x, y

$$\begin{aligned} x + y &= a \\ y &= \frac{p}{q}x + b \end{aligned}$$

By solving the above system we get $x = \dfrac{(a-b)q}{p+q}$ and $y = \dfrac{ap+bq}{p+q}$.

Book I:4. *To find two numbers in a given ratio such that their difference is also given.*

Let it be required that the greater number be five times the lesser and the difference of the numbers be 20 units.

Solution. Let the lesser number be x, then the greater number will be $5x$. So then $5x - x = 20$, $4x = 20$ and $x = 5$. The required numbers are 5 and 25.

Modern Solution. Let the required numbers be x and y ($x < y$). Then

$$\begin{aligned} \frac{x}{y} &= \frac{1}{5} \\ y - x &= 20 \end{aligned}$$

When we solve the above system by substitution, we get $x = 5$ and $y = 25$.

General Problem and Solution.

Given ratio: $p : q$ ($p > q$)

Given difference: a

Required numbers: x, y

$$\begin{aligned} \frac{x}{y} &= \frac{q}{p} \\ y - x &= a \end{aligned}$$

By solving the above system we get $x = \dfrac{aq}{p-q}$ and $y = \dfrac{ap}{p-q}$.

Book I:5. *To divide a given number into two numbers such that given fractions (not the same) of each number when added together produce a given number.*

The latter given number, however, must be such that it lies between the two numbers obtained when the given fractions (respectively) are taken of the first given number.

From then on, let it be required to divide 100 into two numbers, such that one third of the first number and one fifth of the second number added together produce 30 units.

Solution. Let the second required number be $5x$. Then $\frac{1}{3}$ of the first number will be $30 - x$, so the first required number will be $90 - 3x$. So then $5x + (90 - 3x) = 100$, $2x + 90 = 100$ and $x = 5$. The required numbers are 75 and 25.

Modern Solution. Let the required numbers be x and y. Then

$$\begin{aligned} x + y &= 100 \\ \frac{1}{3}y + \frac{1}{5}x &= 30 \end{aligned}$$

When we solve the above system, we get $x = 25$ and $y = 75$.

General Problem and Solution.

Given number: a

Given fractions: $\dfrac{1}{p}, \dfrac{1}{q}$ $(p < q)$

Given sum: b

Required numbers: x, y

$$\begin{aligned} x + y &= a \\ \frac{1}{p}y + \frac{1}{q}x &= b \end{aligned}$$

By solving the above system we get $x = \dfrac{q(a-bp)}{q-p}$ and $y = \dfrac{p(bq-a)}{q-p}$.

The necessary condition is *The latter given number, however, must be such that it lies between the two numbers obtained when the given fractions (respectively) are taken of the first given number*, or in our notation $\frac{1}{q}a < b < \frac{1}{p}a$, which simplifies to $a - bp > 0$ and $bq - a > 0$, guarantees positive solutions only.

Book I:6. *To divide a given number into two numbers such that a given fraction of the first number exceeds a given fraction of the second number by a given number.*

It is necessary, that the later given number be smaller than the number obtained when the greater given fraction is taken of the first given number.

Let it be required to divide 100 into two numbers such that the difference of one fourth of the first number and one sixth of the second number is 20 units.

Solution. Let the second required number be $6x$. Then $\frac{1}{4}$ of the first number will be $x + 20$, so the first required number will be $4x + 80$. So then $6x + (4x + 80) = 100$, $10x = 20$ and $x = 2$. The required numbers are 88 and 12.

Modern Solution. Let the required numbers be x and y. Then

$$\begin{aligned} x + y &= 100 \\ \frac{1}{4}x - \frac{1}{6}y &= 20 \end{aligned}$$

When we solve the above system, we get $x = 88$ and $y = 12$.

General Problem and Solution.

Given number: a

Given fractions: $\frac{1}{p}$, $\frac{1}{q}$ $(p < q)$

Given difference: b

Required numbers: x, y

$$\begin{aligned} x + y &= a \\ \frac{1}{p}x - \frac{1}{q}y &= b \end{aligned}$$

By solving the above system we get $x = \dfrac{p(a+bq)}{p+q}$ and $y = \dfrac{q(a-bp)}{p+q}$.

The necessary condition is *that the later given number be smaller than the number obtained when the greater given fraction is taken of the first given number*, or in our notation $b < \dfrac{1}{p}a$, which simplifies to $a - bp > 0$, guarantees positive solutions only.

Book I:12. *To divide a given number twice into two numbers in such a way that the first number of the first pair may have to the first number of the second pair a given ratio and also the second of the second pair to the second of the first pair another given ratio.*

Let it be required to divide 100 twice into two numbers in such a way that the greater number of the first pair be twice the lesser number of the second pair, and the greater number of the second pair be three times the lesser number of the first pair.

Solution. Let x be the lesser number of the second pair. Then the greater number of the first pair is $2x$ and the lesser of the same pair is $100 - 2x$. Also, from the second given ratio we have the greater number of the second pair is $300 - 6x$.

1st pair: $(2x,\, 100 - 2x)$

2nd pair: $(300 - 6x,\, x)$

Since the sum of the numbers in the second pair is 100 we have: $300 - 6x + x = 100$ and $x = 40$. The required numbers are $(80, 20)$ and $(60, 40)$.

Modern Solution. Let the required pairs be (s, t) and (u, w). Then

$$\begin{aligned} s+t &= 100 \\ u+w &= 100 \\ s &= 2w \\ u &= 3t \end{aligned}$$

When we solve the above system, we get $s = 80$, $t = 20$, $u = 60$ and $w = 40$.

General Problem and Solution.

Given number: a

Given ratios: p, q

Required numbers: s, t, u, w. Then

$$\begin{aligned} s+t &= a \\ u+ww &= a \\ s &= pw \\ u &= qt \end{aligned}$$

By solving the above system we get:

$$\begin{aligned} s &= \frac{ap(q-1)}{pq-1} \\ t &= \frac{a(p-1)}{pq-1} \\ u &= \frac{aq(p-1)}{pq-1} \\ w &= \frac{a(q-1)}{pq-1} \end{aligned}$$

Book I:13. *To divide a given number three times into two numbers in such a way that one of the first pair has to one of the second pair a given ratio, the second of the second pair to one of the third pair another given ratio, and the second of the third pair to the second of the first pair another given ratio.*

Let it be required to divide 100 three times into two numbers in such a way that the greater number of the first pair be three times the lesser number of the second pair; the greater number of the second pair be twice the lesser number of the third pair, and the greater number of the third pair be four times the lesser number of the first pair.

Solution. Let x be the lesser number of the third pair. Then the greater number of the second pair is $2x$ and the lesser of the same pair is $(100 - 2x)$. The greater number of the first pair is $(300 - 6x)$ and the lesser of the same pair is $(6x - 200)$. Also, the greater number of the third pair is $(24x - 800)$.

1st pair: $(300 - 6x,\ 6x - 200)$

2nd pair: $(2x,\ 100 - 2x)$

3rd pair: $(24x - 800,\ x)$

Since the sum of the numbers in the third pair is 100 we have: $24x - 800 + x = 100$, $x = 36$ and the respective pairs are $(84, 16)$, $(72, 28)$ and $(64, 36)$.

Modern Solution. Let the required pairs be (s, t), (z, w) and (x, y). Then

$$\begin{aligned} x + y &= 100 \\ z + w &= 100 \\ s + t &= 100 \\ w &= 2x \\ t &= 3z \\ y &= 4s \end{aligned}$$

Substitute w, t, and y and the new system is:

$$\begin{aligned} x + 4s &= 100 \\ z + 2x &= 100 \\ s + 3z &= 100 \end{aligned}$$

And the solutions are $s = 16$, $t = 84$, $z = 28$, $w = 72$, $x = 36$ and $y = 64$.

Book I:15. *To find three numbers such that each after receiving from the other a given number is in a given ratio with the remaining number.*

Let it be required that the first number after receiving 30 units from the second number, becomes twice as big as the second number, and the second number after receiving 50 units from the first becomes three times the first number.

Solution. Let the second number be $(x + 30)$. Then the first number must be $(2x - 30)$ in order to be twice as big as the second after receiving the 30 units.

After the first number gives 50 units to the second it becomes $(2x - 80)$ and the second number becomes $(x + 80)$. Then $x + 80 = 3(2x - 80)$ and $x = 64$.

The required numbers are 98 and 94.

Modern Solution. Let the required numbers be y and z. Then

$$\begin{aligned} y + 30 &= 2(z - 30) \\ z + 50 &= 3(y - 50) \end{aligned}$$

Simplify the above system to:

$$\begin{aligned} 2z - y &= 90 \\ z - 3y &= -200 \end{aligned}$$

And the solutions are $z = 94$ and $y = 98$.

Book I:16. *To find three numbers which taken two by two produce numbers that are given.*

It is necessary, however, that half of the sum of the given numbers be greater than each of these numbers.

Let it be required that the first number added to the second make 20 units; the second added to the third make 30 units and the third added to the first make 40 units.

Solution. Let the sum of the three numbers be x. Then the first number is $(x-30)$, the second $(x-40)$, the third $(x-20)$ and

$$(x-30)+(x-40)+(x-20)=x$$

So, $x=45$ and the required numbers are 15, 5, and 25.

Modern Solution. Let the required numbers be y, z, and w. Then

$$\begin{aligned} y+z &= 20 \\ z+w &= 30 \\ w+y &= 40 \end{aligned}$$

We solve the above system and see that $y=15$, $z=5$, and $w=25$.

General Problem and Solution.

Given numbers: a, b, c

Required numbers: y, z, w. Then

$$\begin{aligned} y+z &= a \\ z+w &= b \\ w+y &= c \end{aligned}$$

We solve the above system and we get:

$$\begin{aligned} y &= \frac{a-b+c}{2} \\ z &= \frac{a+b-c}{2} \\ w &= \frac{-a+b+c}{2} \end{aligned}$$

The necessary condition is *that half of the sum of the given numbers be greater than each of these numbers* and guarantees positive solutions. Indeed, if the given numbers are

a, b, c and the required numbers x, y, z we have:

$$\frac{a+b+c}{2} > a, \text{ or } \frac{a+b+c}{2} - a = \frac{-a+b+c}{2} = w > 0$$

$$\frac{a+b+c}{2} > b, \text{ or } \frac{a+b+c}{2} - b = \frac{a-b+c}{2} = y > 0$$

$$\frac{a+b+c}{2} > c, \text{ or } \frac{a+b+c}{2} - c = \frac{a+b-c}{2} = z > 0$$

Book I:17. *To find four numbers which added three by three produce numbers that are given.*

It is necessary, however, that one third of the sum of the four numbers must be greater than each of the numbers.

Let it be required that the sum of the first three numbers be 20 units; the sum of second, third and fourth numbers be 22 units, the sum of third fourth and first be 24 units, and the sum of fourth, first and second be 27 units.

Solution. Let the sum of the four numbers be x. Then the first number is $(x-22)$, the second $(x-24)$, the third $(x-27)$, the fourth $(x-20)$ and

$$(x-22)+(x-24)+(x-27)+(x-20) = x$$

$x = 31$ and the required numbers are 9, 7, 4 and 11.

Modern Solution. Let the required numbers be s, t, u and v. Then

$$\begin{aligned} s+t+u &= 20 \\ t+u+v &= 22 \\ u+v+s &= 24 \\ v+s+t &= 27 \end{aligned}$$

We solve the above system and see that $s = 9$, $t = 7$, $u = 4$, and $v = 11$.

Book I:18. *To find three numbers such that the sum of any two exceeds the third by a given number.*

Let it be required that the sum of first two numbers exceeds the third by 20 units; the sum of second and third exceeds the first by 30 units, and the sum of third and first exceeds the second by 40 units.

Solution. Let the sum of the three numbers be $2x$. Since $(1) + (2) = (3) + 20$ if we add the third number (3) to both sides we will have:

$$(1) + (2) + (3) = 2(3) + 20$$

$$2x = 2(3) + 20$$

Then the third number (3) is $(x - 10)$. Similarly, the first number (1) is $(x - 15)$ and the second number (2) is $(x - 20)$. From

$$(x - 15) + (x - 20) + (x - 10) = 2x$$

we have $x = 45$ and the required numbers are 30, 25, 35.

Modern Solution. Let the required numbers be y, z and w. Then

$$\begin{aligned} y + z &= w + 20 \\ z + w &= y + 30 \\ w + y &= z + 40 \end{aligned}$$

and the solutions are $y = 30$, $z = 25$ and $w = 35$.

Book I:19. *To find four numbers such that the sum of any tree exceeds the fourth by a given number*

It is necessary that half the sum of the four given differences must be greater than any one of them.

Let it be required that the sum of first three numbers exceeds the fourth by 20 units; the sum of second, third and fourth number exceeds the first by 30 units, the sum of third, *fourth and first number exceeds the second by 40 units and finally, the sum of fourth, first and second number exceeds the third by 50 units.*

Solution. Let the sum of the four numbers be $2x$, i. e.

$$(1) + (2) + (3) + (4) = 2x.$$

Since

$$(1) + (2) + (3) = (4) + 20,$$

if we add the fourth number (4) to both sides we will have

$$\begin{aligned} (1) + (2) + (3) + (4) &= 2(4) + 20 \\ 2x &= 2(4) + 20 \end{aligned}$$

Then the fourth number (4) is $(x - 10)$. Similarly, the first number (1) is $(x - 15)$, the second number (2) is $(x - 20)$ and the third number (3) is $(x - 25)$. From

$$(x - 15) + (x - 20) + (x - 25) + (x - 10) = 2x$$

we have $x = 35$ and the required numbers are 20, 15, 10, 25.

Modern Solution. Let the required numbers be s, t, u and v. Then

$$\begin{aligned} s + t + u &= v + 20 \\ t + u + v &= s + 30 \\ s + u + v &= t + 40 \\ s + t + v &= u + 50 \end{aligned}$$

and the solutions are $s = 20$, $t = 15$, $u = 10$ and $v = 25$.

Book I:20. *To divide a given number into three numbers is such a way that that the sum of each extreme and the mean has to the other extreme a given ratio.*

Let it be required to divide 100 into three numbers in such a way that the sum of the first two numbers must be three times the third and the sum of the second and third number must be four times the first.

Solution. Let the third number be x, i. e. $(3) = x$. Since $(1) + (2) = 3x$ if we add the third number (3) to both sides we will have

$$(1) + (2) + (3) = 3x + (3),$$

or $100 = 4x$, $x = 25$ and the third number (3) is 25.

Then $(1) + (2) = 75$ and $(2) + (3) = 4(1)$.

Let $(1) = y$. Then $(2) + (3) = 4y$. Add (1) to both sides.

$$(1) + (2) + (3) = 4y + (1)$$

$100 = 5y$ and $y = 20$. Therefore $(1) = 20$.

$(2) + (3) = 80$. Therefore $(2) = 55$ and the required numbers are 20, 55, 25.

Modern Solution. Let the required numbers be s, t and u. Then

$$\begin{aligned} s + t + u &= 100 \\ s + t &= 3u \\ t + u &= 4s \end{aligned}$$

The solutions are $s = 20$, $t = 55$ and $u = 25$.

Book I:21. *To find three number sin such a way that the greatest exceeds the middle number by a given fraction of the least, the middle exceeds the least by a given fraction of the greatest, and the least exceeds a given number by a given fraction of the middle.*

It is necessary that the middle number exceeds the least by such a fraction of the greatest that if its denominator be multiplied by the excess of the middle number over the least, the coefficient of the arithmos in the product is greater than the coefficient of the arithmos in the expression for the middle number resulting from the assumption made.

Suppose that the greatest number exceeds the middle by one third of the lest, the middle exceeds the lest by one third of the greatest and the least exceeds 10 units by one third of the middle.

<u>Solution.</u> Let the least number be $(x+10)$. Then the middle number will be $3x$ and the greatest number $(6x-30)$. Also, since the greatest number exceeds the middle by one third of the least we have that the least is $(9x-90)$. Therefore $x+10=9x-90$ and $x=12\frac{1}{2}$. The required numbers are $22\frac{1}{2}$, $37\frac{1}{2}$ and 45.

<u>Modern Solution.</u> Let the required numbers be y, z and w. Then

$$\begin{aligned} w-z &= \frac{1}{3}y \\ z-y &= \frac{1}{3}w \\ y-10 &= \frac{1}{3}z \end{aligned}$$

And the solutions are $y=22\frac{1}{2}$, $z=37\frac{1}{2}$, $w=45$.

<u>General Problem and Solution.</u>

Given fractions: $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$

Given number: a

Required numbers: y, z, w.

Then

$$\begin{aligned} w-z &= \frac{1}{p}y \\ z-y &= \frac{1}{q}w \\ y-a &= \frac{1}{r}z \end{aligned}$$

We simplify the above system to:

$$\begin{aligned} y+pz-pw &= 0 \\ qy-qz+w &= 0 \\ ry-z &= ar \end{aligned}$$

And the solutions are:

$$
\begin{aligned}
y &= \frac{apr(q-1)}{pqr - pr - pq - 1} \\
z &= \frac{ar(pq+1)}{pqr - pr - pq - 1} \\
w &= \frac{aqr(1+p)}{pqr - pr - pq - 1}
\end{aligned}
$$

The necessary condition is *that the middle number exceeds the least by such a fraction of the greatest that if its denominator be multiplied by the excess of the middle number over the least, the coefficient of the arithmos in the product is greater than the coefficient of the arithmos in the expression for the middle number resulting from the assumption made,* and guarantees positive solutions. Indeed, from the above general solution we see that in order to get positive solutions for y, z, w we must require that $pqr-pr-pq-1 > 0$ or $qr > r+q$.

3.3 Determinate systems of equations reducible to first degree

Book I:26. *Given two numbers to find a number which when multiplied by the given numbers respectively, make one product a square and the other a side of that square.*

Let the two given numbers be 200 and 5.

Solution. Let the required number be x and the side of a square be t. Then

$$
\begin{aligned}
200x &= t^2 \\
5x &= t
\end{aligned}
$$

Square both sides of the second equations and we have:

$$
\begin{aligned}
25x^2 &= 200x \\
x &= 8.
\end{aligned}
$$

General Problem and Solution.

Given numbers: a, b

Required number: x

$$\begin{aligned} ax &= t^2 \\ bx &= t \end{aligned}$$

The solution is $x = \dfrac{a}{b^2}$.

Book I:29. *To find two numbers such that their sum and the difference of their squares are given numbers.*

Let it be required that the sum of the numbers is 20 units and the difference of their squares is 80 units.

<u>Solution.</u> Let the required numbers be $(x + 10)$ and $(x - 10)$. Then

$$(x + 10)^2 - (x - 10)^2 = 40x$$

So, $x = 2$ and the required numbers are 12 and 8.

<u>Modern Solution.</u> Let the required numbers be y and z. Then

$$\begin{aligned} y + z &= 20 \\ y^2 - z^2 &= 80 \end{aligned}$$

The solutions are $y = 12$ and $z = 8$.

<u>General Problem and Solution.</u>

Given numbers: a, b

Required numbers: y, z

$$\begin{aligned} y + z &= a \\ y^2 - z^2 &= b \end{aligned}$$

The solutions are $y = \dfrac{a^2 + b}{2a}$ and $z = \dfrac{a^2 - b}{2a}$.

Book I:31. *To find two numbers in a given ratio and such that the sum of their squares has to their sum a given ratio.*

So, let it be required that the greater number be three times the lesser and the sum of their squares be five times their sum.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$x^2 + (3x)^2 = 5(x + 3x)$$

So, $x = 2$ and the required numbers are 2 and 6.

Modern Solution. Let the required numbers be x and y, $(x < y)$. Then

$$\begin{aligned} \frac{y}{x} &= 3 \\ \frac{x^2 + y^2}{x + y} &= 5 \end{aligned}$$

The solutions are $x = 2$ and $y = 6$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} \frac{y}{x} &= p \\ \frac{x^2 + y^2}{x + y} &= q \end{aligned}$$

The solutions are $x = \dfrac{(p+1)q}{p^2+1}$ and $y = \dfrac{(p+1)pq}{p^2+1}$.

Book I:32. *To find two numbers in a given ratio and such that the sum of their squares has to their difference a given ratio.*

So, let it be required that the greater number be three times the lesser and the sum of their squares be ten times their difference.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$\frac{x^2 + 9x^2}{2x} = 10$$

$x = 2$ and the required numbers are 2 and 6.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} \frac{y}{x} &= 3 \\ \frac{x^2 + y^2}{y - x} &= 10 \end{aligned}$$

The solutions are $x = 2$ and $y = 6$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} \frac{y}{x} &= p \\ \frac{x^2 + y^2}{y - x} &= q \end{aligned}$$

The solutions are $x = \dfrac{q(p-1)}{p^2+1}$ and $y = \dfrac{pq(p-1)}{p^2+1}$.

Book I:33. *To find two numbers in a given ratio and such that the difference of their squares has to their sum a given ratio.*

Let it be required that the greater number be three times the lesser and the difference of the squares of the numbers be six times the sum of the numbers.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$\frac{(3x)^2 - x^2}{4x} = 6$$

$x = 3$ and the required numbers are 3 and 9.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} \frac{y}{x} &= 3 \\ \frac{y^2 - x^2}{x + y} &= 6 \end{aligned}$$

The solutions are $x = 3$ and $y = 9$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} \frac{y}{x} &= p \\ \frac{y^2 - x^2}{x + y} &= q \end{aligned}$$

The solutions are $x = \dfrac{q(p+1)}{p^2 - 1}$ and $y = \dfrac{pq(p+1)}{p^2 - 1}$.

Book I:34. *To find two numbers in a given ratio and such that the difference of their squares has to their difference a given ratio.*

Let it be required that the greater number be three times the lesser and the difference of squares of the numbers be twelve times the difference of these numbers.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$\frac{(3x)^2 - x^2}{2x} = 12$$

$x = 3$ and the required numbers are 3 and 9.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} \frac{y}{x} &= 3 \\ \frac{y^2 - x^2}{y - x} &= 12 \end{aligned}$$

The solutions are $x = 3$ and $y = 9$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} \frac{y}{x} &= p \\ \frac{y^2 - x^2}{y - x} &= q \end{aligned}$$

The solutions are $x = \dfrac{q}{p+1}$ and $y = \dfrac{pq}{p+1}$.

Book I:35. *To find two numbers in a given ratio and such that the square of the lesser number has to the greater a given ratio.*

Let it be required that the greater number be three times the lesser and the square of the lesser be six times the greater number.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$x^2 = 18x$$

$x = 18$ and the required numbers are 18 and 54.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} y &= 3x \\ x^2 &= 6y \end{aligned}$$

The solutions are $x = 18$ and $y = 54$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} y &= px \\ x^2 &= qy \end{aligned}$$

The solutions are $x = pq$ and $y = p^2q$.

Book I:36. *To find two numbers in a given ratio and such that the square of the lesser has to the lesser itself a given ratio.*

Let it be required that the greater number be three times the lesser and the square of the lesser be six times the lesser itself.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$x^2 = 6x$$

$x = 6$ and the required numbers are 6 and 18.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} y &= px \\ x^2 &= qx \end{aligned}$$

The solutions are $x = q$ and $y = pq$.

Book I:37. *To find two numbers in a given ratio and such that the square of the lesser has to the sum of both numbers a given ratio.*

Let it be required that the greater number be three times the lesser and the square of the lesser be twice the sum of the two numbers.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$x^2 = 2(4x)$$

$x = 8$ and the required numbers are 8 and 24.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} y &= 3x \\ x^2 &= 2(x + y) \end{aligned}$$

The solutions are $x = 8$ and $y = 24$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} y &= px \\ x^2 &= q(x+y) \end{aligned}$$

The solutions are $x = q(p+1)$ and $y = pq(p+1)$.

Book I:38. *To find two numbers in a given ratio and such that the square of the lesser has to the difference of the numbers a given ratio.*

Let it be required that the greater number be three times the lesser and the square of the lesser be six times the difference of the numbers.

Solution. Let the lesser number be x, then the greater number will be $3x$.

$$x^2 = 6(2x)$$

$x = 12$ and the required numbers are 12 and 36.

Modern Solution. Let the required numbers be x and y $(x < y)$. Then

$$\begin{aligned} y &= 3x \\ x^2 &= 6(y-x) \end{aligned}$$

The solutions are $x = 12$ and $y = 36$.

General Problem and Solution.

Given ratios: p, q

Required numbers: x, y $(x < y)$

$$\begin{aligned} y &= px \\ x^2 &= q(y-x) \end{aligned}$$

The solutions are $x = q(p-1)$ and $y = pq(p-1)$.

Book IV:36. *To find three numbers such that the product of any two has to the sum of those a given ratio.*

Let it be required that the product of the first two numbers be three times their sum; the product of second and third numbers be four times their sum, and the product of first and third numbers be five times their sum.

Solution. Let the second number be x, then the first number will be $\dfrac{3x}{x-3}$ and the third number $\dfrac{4x}{x-4}$.

Since the product of first and third number is 5 times their sum, we have:

$$\begin{aligned} \frac{3x}{x-3} \cdot \frac{4x}{x-4} &= 5\left[\frac{3x}{x-3} + \frac{4x}{x-4}\right] \\ \frac{12x^2}{x^2-7x+12} &= 5 \cdot \frac{7x^2-24}{x^2-7x+12} \end{aligned}$$

So, $x = \dfrac{120}{23}$ and the required numbers are $\dfrac{360}{51}$, $\dfrac{120}{23}$ and $\dfrac{480}{28}$.

Modern Solution. Let the required numbers be y, z, and w. Then

$$\begin{aligned} yz &= 3(y+z) \\ zw &= 4(z+w) \\ yw &= 5(y+w) \end{aligned}$$

From the first equation we solve for y in terms of z and substitute it in the third equation.

$$\frac{3zw}{z-3} = 5\left[\frac{3z}{z-3} + w\right]$$

After we simplify the above we get:

$$2zw = 15w - 15z.$$

But from the second equation we have

$$zw = 4(z+w),$$

and so after we substitute zw we obtain:

$$w = \frac{23z}{7}.$$

Now, combine this with the second equation and find $z = \dfrac{120}{23}$.
Similarly, we find that $y = \dfrac{360}{51}$ and $w = \dfrac{480}{28}$.

3.4 Systems of equations apparently indeterminate but really reduced, by arbitrary assumptions, to determinate equations of first degree

Book I:14. *To find two numbers such that their product has to their sum a given ratio.*

It is necessary, however, that the assumed value of one of the two numbers must be greater than the number representing the given ratio.

Let it be required that the product is three times their sum.

Let one of the numbers be 1 arithmos and the other, greater than 3 units (which follows from the proceeding conditions) be 12 arithmos. From then on, the product of the numbers is 12 arithmos and their sum is 1 arithmos plus 12 units.

<u>Solution.</u> Let the lesser number be x and the greater is given to be 12.

$$\frac{12x}{x+12} = 3$$

$x = 4$ and the required numbers are 4 and 12.

<u>General Problem and Solution.</u>

Given ratios: p

Required numbers: x, y $(x < y)$

$$\frac{xy}{x+y} = p$$

The solutions are $x = \dfrac{pk}{k-p}$ and $y = k$, where k is an arbitrary real number.

The necessary condition is *that the assumed value of one of the two numbers must be greater than the number representing the given ratio*, or in our notation $y > p$, guarantees a positive value for the first number (x).

Book I:22. *To find three numbers such that if each give to the next following a given fraction of itself, in order, the results after each had given and taken may be equal.*

Let it be required that the first number gives a third of itself to second, the second number gives a fourth of itself to third, and the third gives a fifth of itself to first number and after the mutual transfer we obtain equal numbers.

Solution. Let the first number be $3x$ and the second 4.

The second number after giving and receiving becomes $(x+3)$, so all numbers after giving and receiving must be equal to $(x+3)$. The first number after giving a third of itself becomes $2x$. So $2x+? = x+3$ and we find that the part which the 1st number receives is $(3-x)$, So, $\frac{1}{5}$ of the 3rd number is $(3-x)$ and the 3rd number is $(15-5x)$. We now have that

$$15 - 5x - (3-x) + 1 = x + 3$$

and $x = 2$.

The required numbers are 6, 4, 5.

Modern Solution. Let the required numbers be x, y, z. Then

$$\frac{2}{3}x + \frac{1}{5}z = \frac{3}{4}y + \frac{1}{3}x = \frac{4}{5}z + \frac{1}{4}y$$

If we solve in terms of y we get the triple $(\frac{3}{2}y, y, \frac{5}{4}y)$.

And for $y = 4$ we get Diophantus' solution triple 6, 4, 5.

General Problem and Solution.

Given fractions: $\frac{1}{m}, \frac{1}{n}, \frac{1}{p}$

Required numbers: x, y, z

$$x - \frac{1}{m}x + \frac{1}{p}z = y - \frac{1}{n}y + \frac{1}{m}x = z - \frac{1}{p}z + \frac{1}{n}y$$

The solution is:

$x = \frac{m}{n}\left[\frac{(n-1)(p-2)+1}{(m-2)(p-2)+m-1}\right]k$, $y = k$, $z = \frac{p}{n}\left[\frac{(m-1)(n-1)-m+2}{(m-2)(p-2)+m-1}\right]k$, where k is an arbitrary real number.

Book I:23. *To find four numbers such that if each give to the next following a given fraction of itself, the results may all be equal.*

Let it be required that the first number gives a third of itself to second; the second gives a fourth of itself to third; the third gives a fifth of itself to fourth; the fourth gives a sixth of itself to first, and after the mutual transfer we obtain equal numbers.

Solution. Let the first number be $3x$ and the second 4.

The second number after giving and receiving becomes $(x+3)$, so all numbers after giving and receiving must become equal to $(x+3)$.

The 1st number after giving x of itself and receiving $\frac{1}{6}$ of the 4th becomes also $(x+3)$.

$3x - x + \frac{1}{6}$ of the 4th number$= x+3$.

So the 4th number is $(18-6x)$.

The 4th number after giving and receiving must also equal to $(x+3)$. So,

$18-6x-\frac{1}{6}(18-6x)+\frac{1}{5}$ of the 3rd number$= x+3$ and the 3rd number is $(30x-60)$.

Finally, the 3rd number after giving and receiving must equal to $(x+3)$. Therefore,

$$30x-60-\frac{1}{5}(30x-60)+1 = x+3$$

and $x=\frac{50}{23}$.

The required numbers are $\frac{150}{23}$,4, $\frac{120}{23}$, $\frac{114}{23}$, or after multiplying by the common denominator, 150, 92, 120, 114.

Modern Solution. Let the required numbers be $3x, 4, 5z, 6w$. Then

$$2x+w = 3+x = 4z+1 = 5w+z,$$

from which we get $x=\frac{50}{23}$, $w=\frac{19}{23}$, $z=\frac{24}{23}$.

The required numbers are $\frac{150}{23}$,4, $\frac{120}{23}$, $\frac{114}{23}$.

Book I:24. *To find three numbers such that if each receives a given fraction of the sum of the other two, the results are all equal.*

Let it be required that the first number receives a third of the sum of the two remaining numbers, the second receives a fourth of the sum of the two remaining numbers, the third receives a fifth of the sum of the two remaining numbers and we obtain equal numbers.

Solution. Let the first number be x and the sum of 2nd and 3rd numbers be 3. Then the sum of the three numbers will be $(x+3)$.

The 1st number after receiving a third of the sum of the other two becomes $(x+1)$ and therefore the other numbers after receiving must also become equal to $(x+1)$.

The 2nd number plus $\frac{1}{4}$ of the 1st and 3rd number is equal to $(x+1)$, so the 2nd number is $x+\frac{1}{3}$.

Similarly, the 3rd number plus $\frac{1}{5}$ of the 1st and 2nd number is equal to $(x+1)$, so the 3rd number is $x+\frac{1}{2}$

And since the sum of all three numbers is $(x+3)$ we have:

$$x+x+\frac{1}{3}+x+\frac{1}{2}=x+3$$

and $x=\frac{13}{12}$.

The numbers after multiplying by the common denominator are 13, 17, 19.

Modern Solution. Let the required numbers be x, y, z. Then

$$x+\frac{1}{3}(y+z)=y+\frac{1}{4}(x+z)=z+\frac{1}{5}(x+y)$$

From this indeterminate system we get:

$$\begin{aligned} 9x-8y+z &= 0 \\ x+16y-15z &= 0 \end{aligned}$$

And the solutions are $x=k$, $y=\frac{17}{13}k$ and $z=\frac{19}{13}k$, where k is an arbitrary real number.

We see that for $k = 13$, we get Diophantus' solution triple 13, 17, 19.

Book I:25. *To find four numbers such that, if each receives a given fraction of the sum of the remaining three, the four resulting numbers are equal.*

Let it be required that the first number receives a third of the sum of the remaining three numbers; the second receives a quarter of the sum of the remaining three numbers, in the same way the third also receives a fifth and finally the fourth number receives a sixth, and the resulting numbers are equal.

Solution. Let the first number be x and the sum of the remaining three numbers be 3. Then the sum of the four numbers will be $(x + 3)$.

The 1st number after receiving a third of the sum of the other three becomes $x + 1$ and therefore the other numbers after receiving must also become equal to $(x + 1)$.

As in the previous number we see that the 2nd number is $x + \frac{1}{3}$.

Similarly, the 3rd number is $x + \frac{1}{2}$ and the 4th number $x + \frac{3}{5}$

And since the sum of all four numbers is $(x + 3)$ we have:

$$x + x + \frac{1}{3} + x + \frac{1}{2} + x + \frac{3}{5} = x + 3$$

and $x = \frac{47}{90}$.

The numbers, after multiplying by the common denominator, are 47, 77, 92, 101.

Modern Solution. Let the required numbers be x, y, z, w. Then

$$x + \frac{1}{3}(y + z + w) = y + \frac{1}{4}(x + z + w) = z + \frac{1}{5}(x + y + w) = w + \frac{1}{6}(x + y + z)$$

From this indeterminate system we get:

$$\begin{aligned} 9x - 8y + z + w &= 0 \\ x + 16y - 15z + w &= 0 \\ x + y + 25z - 24w &= 0 \end{aligned}$$

And the solutions are $x = k$, $y = \frac{77}{47}k$, $z = \frac{92}{47}k$ and $w = \frac{101}{47}k$, where k is an arbitrary real number.

We see that for $k = 47$, we get Diophantus' solution 47, 77, 92, 101.

Book IV:33. *To find two numbers such that each after receiving from the other the same part or parts, has to the remainder of the giving number a given ratio.*

Let it be required that the first number, having received a part or parts of the second, be three times the remainder [of the second number] [2]*, and the second number having received the same part or parts of the first number, be five times the remainder [of the first number].*[3]

Solution. Let the second number be $(x+1)$ where 1 is the part it will give to the 1st number. Then the 1st number will be $(3x-1)$ and the sum of the two numbers is $4x$.

After the 2nd number receives a part from the first, it becomes 5 times the remainder of the 1st. Therefore, the remainder of the 1st is $\frac{1}{6}$ of $4x$, which is equal to $\frac{2}{3}x$. So,

$$3x - 1 - \frac{2}{3}x = \frac{7}{3}x - 1$$

Now, the 2nd number, which is $(x+1)$, after receiving $\frac{7}{3}x - 1$ from the 1st, becomes 5 times the remainder of the 1st.

Since $\frac{7}{3}x - 1$ is part of $(3x-1)$ as 1 is part of $(x+1)$, we have:

$$\frac{\frac{7}{3}x - 1}{3x - 1} = \frac{1}{x+1}$$

$x = \frac{5}{7}$ and the numbers are $\frac{8}{7}$ and $\frac{5}{7}$.

Note that 1 is $\frac{7}{12}$ of the 2nd number. If we multiply both numbers by 7 we get 8 and 12 for the required numbers and $\frac{7}{12}$ for the fraction. Since 8 is not divisible by 12, we multiply by 3 and the pair $(24, 36)$ is our solution.

[2] A phrase filled in by Bachet.[16]

[3] A phrase filled in by Bachet.[16]

Modern Solution. Let the required numbers be x and y and the fraction be z. Then

$$\begin{aligned} x + zy &= 3(1-z)y \\ y + zx &= 5(1-z)x. \end{aligned}$$

We simplify the above system and get:

$$\frac{x}{y} = 3 - 4z$$
$$\frac{y}{x} = 5 - 6z,$$

from where we can write an equation in terms of z.

$$(5-6z)(3-4z) = 1.$$

Now, $z = 1$ (trivial) and $z = \frac{7}{12}$. Then $x = k$ and $y = \frac{3}{2}k$, where k is an arbitrary real number, are the required numbers.

3.5 Indeterminate equations of first degree

Lemma to IV:34. *To find two numbers indeterminately such that their product together with their sum is a given number.*

Let the given number be 8 units. Assume that the first number is 1 arithmos and the second number is 3 units.

Solution. Let the first number be x. Since the 2nd number is 3, we have:

$$3x + x + 3 = 8$$

and $x = \frac{5}{4}$.

Since $\frac{5}{4} = \frac{8-3}{3+1}$, if we assume the 2nd number to be $(x-1)$ we have for the 1st number:

$$\frac{8-(x+1)}{x-1+1} = \frac{9}{x} - 1$$

The required numbers are x and $\frac{9}{x} - 1$.

<u>General Solution.</u>

Given number: a

Required numbers: x, y

Then

$$\begin{aligned} xy + x + y &= a \\ y &= \frac{a-x}{1+x} \end{aligned}$$

and the required numbers are k and $\frac{a-k}{1+k}$, where k is a real number.

Lemma to IV:35. *To find two numbers indeterminately such that their product minus their sum is a given number.*

Let the given number be 8. Let it be supposed that the first number is 1 arithmos and the second is 3 units.

<u>Solution.</u> Let the first number be x. Since the 2nd number is 3, we have:

$$3x - (x+3) = 8$$

And so, $x = \frac{11}{2}$.

Now, since $\frac{11}{2} = \frac{8+3}{3-1}$, if we assume the 2nd number to be $(x+1)$ we have for the 1st number:

$$\frac{8+(x+1)}{x+1-1} = 1 + \frac{9}{x}$$

The required numbers are x and $1 + \frac{9}{x}$.

<u>General Solution.</u>

Given number: a

Required numbers: x, y

Then:

$$\begin{aligned} xy - x - y &= a \\ y &= \frac{x+a}{x-1} \end{aligned}$$

The required numbers are k and $\dfrac{k+a}{k-1}$, where k is a real number.

Lemma to IV:36. *To find two numbers indeterminately such that their product has their sum is a given ratio.*

Let it be required that the product of the numbers be three times their sum. Let the first number be 1 arithmos and the second number be 5 units.

Solution. Let the first number be x. Since the 2nd number is 5, we have:

$$5x = 3(x+5)$$

and $x = \dfrac{15}{2}$.

Since $\dfrac{15}{2} = \dfrac{35}{5-3}$, if we assume the 2nd number to be x we have for the 1st number $\dfrac{3x}{x-3}$.

The required numbers are x and $\dfrac{3x}{x-3}$.

General Solution.

Given ratio: p

Required numbers: x, y

Then

$$xy = p(x+y)$$

and $y = \dfrac{px}{x-p}$.

The required numbers are k and $\dfrac{pk}{k-p}$, where k is a real number.

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APPENDICES

Appendix A

Selected problems of second degree from Arithmetica

Book II:9.[1] To divide a given number which is the sum of two squares into two other squares.

Given number $13 = 2^2 + 3^2$. As the roots of these squares are 2, 3, take $(x+2)^2$ as the first square and $(mx-3)^2$ as the second (where m is an integer), say $(2x-3)^2$. Therefore $(x^2+4x+4)+(4x^2+9-12x) = 13$, or $5x^2+13-8x = 13$. Therefore $x = \frac{8}{5}$, and the required squares are $\frac{324}{25}, \frac{1}{25}$.

Book II:10. To find two square numbers having a given difference.

Given difference 60. Side of one number x, side of the other x plus any number the square of which is not greater than 60, say 3.
Therefore

$$(x+3)^2 - x^2 = 60,$$

$x = 8\frac{1}{2}$ and the required squares are $72\frac{1}{4}$, $132\frac{1}{4}$.

Book II:19. To find three squares such that the difference between the greatest and the middle has to the difference between the middle and the least a given ratio.

Given ratio 3:1. Assume the least square $= x^2$, the middle $= x^2+2x+1$. Therefore the greatest $= x^2+8x+4 =$square$= (x+3)^2$, say. Thus $x = 2\frac{1}{2}$, and the squares are $30\frac{1}{4}$, $12\frac{1}{4}$, $6\frac{1}{4}$.

[1] These problems are in modern notation as published by Heath in [6].

Appendix B

Selected problems from the Arabic Books

Arabic Book IV:3.[1] We wish to find two square numbers the sum of which is a cubic number.

We put x^2 as the smaller square and $4x^2$ as the greater square. The sum of the two squares is $5x^2$, and this must be equal to a cubic number. Let us make its side any number of x's we please, say x again,[2] so that the cube is x^3. Therefore, $5x^2$ is equal to x^3. As the side which contains the x^2's is the lesser in degree, we divide the whole by x^2; hence x is equal to 5. Then, since we assumed the smaller square to be x^2, and since x^2 arises from the multiplication of x- which we found to be 5-by itself, x^2 is 25. And, since we put for the greater square $4x^2$, it is 100. The sum of the two squares is 125, which is a cubic number with 5 as its side.

Therefore, we have found two square numbers the sum of which is a cubic number, namely 125[3] This is what we intended to find.

Arabic Book VI:11. We wish to find a cubic number such that if we add it to its square, the result is a square number.

We put x as the side of the cubic number, so that the cubic number is x^3. Adding x^3 to its square, that is, (to) x^6, we obtain $x^6 + x^3$, which must be a square. Let us put for its side a number [of x^3's such that, when we subtract from their square x^6 the remainder is a cube; such is] [4] $3x^3$: when we subtract x^6 from the square of $3x^3$, we obtain $8x^6$, which is a cubic number. Hence, if we equate $8x^6$ with a cubic number, the problem will be soluble and

[1] These problems are reproduced from [12].

[2] as the side of the smaller square

[3] With a slight change in the text, we have the usual statement of the required magnitudes, i. e.., "and these are 100 and 25." The text's reading could well be a scribe's mistake.

[4] We assume that there is a gap in the text here.

the treatment will not be impossible. Let us multiply the $3x^3$ by themselves, so we obtain $9x^6$, which then equals $x^6 + x^3$. We remove the x^6 which is common, so $8x^6$ equals x^3. The division of the two sides by x^3 gives $8x^3$ equals to 1; hence x^3 is $\frac{1}{8}$, or one part of 8. If we increase this by its square, that is, (by) one part of 64 parts of the unit, the result is 9 parts of 64 parts of the unit, which is a square number with 3 parts of 8 as its side.

Therefore, we have found a number fulfilling the condition imposed upon us, and this is one part of 8 parts of the unit. This is what we intended to find.

Appendix C

Selection from the Greek Anthology

Book XIV:49.[1] Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and much-wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, how much brass, how much tin, and how much iron, so as to make the whole crown weigh sixty minae.

Solution. Let the weights of the gold, brass, tin and iron be respectively g, b, t and i minae. Then

$$g + b + t + i = 60$$

$$g + b = 40$$

$$g + t = 45$$

$$g + i = 36$$

If we add the last three equations we get

$$2g + (g + b + t + i) = 121$$

$$2g = 61$$

Therefore $g = 30\frac{1}{2}$, $b = 9\frac{1}{2}$, $t = 14\frac{1}{2}$, $i = 5\frac{1}{2}$.

Book XIV:51 A. I have what the second has and the third of what the third has. B. I have what the third has and the third of what the first has. C. And I have ten minae and the third of what the second has.

[1] The text for these problems are reproduced from [9], but the solutions are original.

<u>Solution.</u> Let the required numbers be A, B and C. Then

$$A = B + \frac{1}{3}C$$

$$B = C + \frac{1}{3}A$$

$$C = 10 + \frac{1}{3}B$$

and we obtain the following solutions: $A = 45$, $B = 37\frac{1}{2}$, $C = 22\frac{1}{2}$.

Book XIV:128. What violence my brother has done me, dividing our father's fortune of five talents unjustly! Poor tearful I have this fifth part of the seventh-elevenths of my brother's share. Zeus, thou sleepest sound.

<u>Solution.</u> Let x and y be the shares and $x > y$. Then

$$x + y = 5$$

$$y = \frac{1}{5} \cdot \frac{7}{11}x$$

And so, $x = 4\frac{27}{62}$ and $y = \frac{35}{62}$.

Mterodorus, who collected the arithmetical epigrams, notes that *the solution offered is that the one brother had* $4\frac{4}{11}$ *of a talent, the other* $\frac{7}{11}$*, but I can not work it out.*[] It is obvious that the offered solution is incorrect.