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Zeta function of self-adjoint operators on surfaces of revolution

Tianshi Lu[1,3], Thalia Jeffres[1] and Klaus Kirsten[2]

[1] Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033, USA
[2] GCAP-CASPER, Mathematics Department, Baylor University, Waco, TX 76798-7328, USA

E-mail: lu@math.wichita.edu

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Abstract
In this article we analyze the zeta function for the Laplace operator on a surface of revolution. A variety of boundary conditions, separated and unseparated, are considered. Formulas for several residues and values of the zeta function as well as for the determinant of the Laplacian are obtained. The analysis is based upon contour integration techniques in combination with a WKB analysis of solutions of related initial value problems.

Keywords: zeta function, self-adjoint, contour integral, WKB

1. Introduction

Spectral zeta functions of typically Laplace-type operators are directly related to topics such as analytic torsion [21], the heat kernel [13, 22], Casimir energies [4, 8, 20] and effective actions [5, 6, 9]. It is therefore very desirable to have effective analytical tools available to understand specific properties of zeta functions. Whereas in one-dimension closed answers are relatively easily obtained for quantities like the functional determinant, see, e.g., [7, 10, 18], in higher dimensions the situation is much more involved. However, a contour integral approach established in [2, 16] has been shown to be very useful as long as the Laplace-type operator separates in a suitable fashion. This approach has been used in a variety of configurations like the generalized cone [3], the spherical suspension [11], warped product manifolds [12] and surfaces of revolution [15].

In some detail, in [15] the Laplacian on a surface of revolution was considered with Dirichlet boundary conditions imposed. Properties of the zeta function like residues, values and its derivative at zero were analyzed. Given that the strictly positive function

[3] Author to whom any correspondence should be addressed.

$f \in C^2(x_0, x_1)$ used to generate the surface of revolution is kept general, the analysis is not based upon known eigenfunctions or known solutions of ordinary differential equations, but instead on the asymptotic analysis of solutions of an initial value problem related to the boundary conditions imposed. As the boundary conditions change the relevant initial value problem changes and so does the pertinent asymptotic analysis. These changes capture how spectral zeta functions depend on the boundary conditions. This is the main subject of the current article, furthermore we consider the influence of kinks on the surface of revolution on spectral properties.

The article is organized as follows. In section 2 we introduce the Laplacian on a surface of revolution and find implicit eigenvalue equations when separated or unseparated boundary conditions are imposed. Furthermore, using the WKB method [1, 19], asymptotic properties of solutions of relevant initial value problems are determined. In section 3 we use these properties to analyze the spectral zeta function for a variety of separated boundary conditions, whereas in section 4 unseparated boundary conditions are considered. A particular case are periodic boundary conditions, where as long as the function *f* and its derivative agree at the endpoints the surface of revolution can be thought of as a smooth torus. However, if the derivative does not agree this introduces a kink point on the torus. This leads to the discussion about non-smooth surfaces in section 5. The conclusions point to the most important results of the article. In the appendix we give an independent proof that the implicit eigenvalue equations do not only capture the value of eigenvalues correctly but also their degeneracies.

2. Spectrum of the self-adjoint Sturm–Liouville equation

Let $f \in C^2(x_0, x_1)$ be a strictly positive function from $[x_0, x_1]$ to $\mathbb{R}$. We consider the Laplacian on the surface of revolution that is generated by revolving the graph of *f* around the *x*-axis. Using separation of variables, the resulting eigenvalue equation for the Laplacian on this surface of revolution is [15]

$$-(pu')' + \frac{k^2}{p}u = \lambda r u, \quad p(x) = \frac{f}{\sqrt{1 + f'^2}}, \; r(x) = f\sqrt{1 + f'^2}, \tag{1}$$

where $k \in \mathbb{Z}$ is the separation constant entering from the cross-section S^1. In rewriting equation (1) as a system of first order differential equations, the quantity $v = pu'$ is convenient. The equivalent form of equation (1) then is

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{p} \\ \frac{k^2}{p} - \lambda r & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2}$$

We denote the fundamental solution of (2) as

$$E_k(\lambda; x) = \begin{pmatrix} u_k^{\mathrm{N}}(\lambda; x) & u_k^{\mathrm{D}}(\lambda; x) \\ v_k^{\mathrm{N}}(\lambda; x) & v_k^{\mathrm{D}}(\lambda; x) \end{pmatrix}, \tag{3}$$

where the superscripts *N* and *D* stand for solutions of the initial value problem $u_k^{\mathrm{D}}(\lambda; x_0) = 0$, $v_k^{\mathrm{D}}(\lambda; x_0) = 1$ and $u_k^{\mathrm{N}}(\lambda; x_0) = 1$, $v_k^{\mathrm{N}}(\lambda; x_0) = 0$. In this way $E_k(\lambda; x_0) = I$, furthermore $\det\, E_k(\lambda; x) = \det\, E_k(\lambda; x_0) = 1$.

To guarantee the operator is self-adjoint, the boundary condition must be in one of two categories [23]. The first category is the separated boundary condition

$$au(x_0) + bv(x_0) = 0, \quad cu(x_1) + \mathrm{d}v(x_1) = 0, \tag{4}$$

where $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$. Following [17], the corresponding eigenvalues are the zeros of the following function of λ,

$$F_k(\lambda) = \det\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} E_k(\lambda; x_1)\right) = \begin{pmatrix} c & d \end{pmatrix} E_k(\lambda; x_1)\begin{pmatrix} -b \\ a \end{pmatrix}. \tag{5}$$

The second category is the unseparated boundary condition

$$\begin{pmatrix} u(x_1) \\ v(x_1) \end{pmatrix} = M\begin{pmatrix} u(x_0) \\ v(x_0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u(x_0) \\ v(x_0) \end{pmatrix}, \tag{6}$$

with $\det\ M = 1$. Rewriting

$$\begin{pmatrix} u(x_1) \\ v(x_1) \end{pmatrix} = E_k(\lambda; x_1)\begin{pmatrix} u(x_0) \\ v(x_0) \end{pmatrix},$$

the corresponding eigenvalues are seen to be zeroes of

$$\begin{aligned} F_k(\lambda) = \det\left(E_k(\lambda; x_1) - M\right) = 2 - du_k^{\mathrm{N}}(\lambda; x_1) \\ - av_k^{\mathrm{D}}(\lambda; x_1) + cu_k^{\mathrm{D}}(\lambda; x_1) + bv_k^{\mathrm{N}}(\lambda; x_1). \end{aligned} \tag{7}$$

We prove in the appendix that each zero of $F_k(\lambda)$ has the same multiplicity as that of the corresponding eigenvalue. In general, the fundamental solution (3) will not be given in terms of known special functions. In order to find certain properties of the zeta function associated with eigenvalue problems on surfaces of revolution it will turn out to be sufficient to have a knowledge of the large-λ uniform asymptotic expansion of (3). To this end we need to analyze $u_k^{\mathrm{N}}, v_k^{\mathrm{N}}$ and $u_k^{\mathrm{D}}, v_k^{\mathrm{D}}$. In order to prepare for the application of the WKB approximation [1, 19], substituting the ansatz

$$u_k^{\pm}(\lambda; x) = \exp\left[\pm\int_{x_0}^{x} \frac{T_k^{\pm}(\lambda; y)}{p(y)}\mathrm{d}y\right],$$

into equation (1), we get

$$\left(T_k^{\pm}\right)^2(\lambda; x) \pm p(x)\left(T_k^{\pm}\right)'(\lambda; x) = k^2 - \lambda f^2(x). \tag{8}$$

A suitable quantity is

$$T_k(\lambda; x) = \frac{T_k^{+}(\lambda; x) + T_k^{-}(\lambda; x)}{2},$$

and we have the identity

$$\begin{aligned} \left(T_k^{+}\right)^2 + p\left(T_k^{+}\right)' = \left(T_k^{-}\right)^2 - p\left(T_k^{-}\right)' \Rightarrow \int_{x_0}^{x_1} \frac{T_k^{+}(\lambda; x)}{p(x)}\mathrm{d}x \\ = \int_{x_0}^{x_1} \frac{T_k(\lambda; x)}{p(x)}\mathrm{d}x - \left[\frac{\ln T_k(\lambda; x)}{2}\right]_{x_0}^{x_1}. \end{aligned} \tag{9}$$

Imposing the initial conditions as indicated below equation (3), we can write the elements of $E_k(\lambda; x_1)$ as

$$u_k^{\mathrm{D}}(\lambda;\, x_1) = \frac{u_k^+(\lambda;\, x_1) - u_k^-(\lambda;\, x_1)}{2T_k(\lambda;\, x_0)}, \tag{10}$$

$$u_k^{\mathrm{N}}(\lambda;\, x_1) = \frac{T_k^-(\lambda;\, x_0)u_k^+(\lambda;\, x_1) + T_k^+(\lambda;\, x_0)u_k^-(\lambda;\, x_1)}{2T_k(\lambda;\, x_0)}, \tag{11}$$

$$v_k^{\mathrm{D}}(\lambda;\, x_1) = \frac{T_k^+(\lambda;\, x_1)u_k^+(\lambda;\, x_1) + T_k^-(\lambda;\, x_1)u_k^-(\lambda;\, x_1)}{2T_k(\lambda;\, x_0)}, \tag{12}$$

$$v_k^{\mathrm{N}}(\lambda;\, x_1) = \frac{T_k^-(\lambda;\, x_0)T_k^+(\lambda;\, x_1)u_k^+(\lambda;\, x_1) - T_k^+(\lambda;\, x_0)T_k^-(\lambda;\, x_1)u_k^-(\lambda;\, x_1)}{2T_k(\lambda;\, x_0)}. \tag{13}$$

The left-hand sides of equations (10)–(13) will be the needed input for the contour integral formulation of the zeta function. The right-hand sides will be the starting point for the computation of the relevant uniform asymptotic expansion.

Next we make some general statements about the zeta function associated with equation (1) supplemented by any choice of boundary conditions. Following [15], we first compute

$$D_k(\lambda) = \prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_{k,n}}\right),$$

where $\lambda_{k,n}$ is the nth positive eigenvalue of equation (1) under a certain boundary condition. For $F_k(0) \neq 0$,

$$D_k(\lambda) = F_k(\lambda)/F_k(0). \tag{14}$$

If $F_k(\lambda)$ has a first order zero instead

$$D_k(\lambda) = F_k(\lambda)\big/\big(F_k'(0)\lambda\big). \tag{15}$$

The zeta function can be represented as

$$\zeta(s) = \zeta_1(s) + \zeta_2(s) = \sum_{n=1}^{\infty}\lambda_{0,n}^{-s} + 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{k,n}^{-s},$$

where, using the contour integral representation

$$\zeta_1(s) = \frac{\sin \pi s}{\pi}\int_0^{\infty} z^{-2s}\frac{\mathrm{d}}{\mathrm{d}z}\ln D_0\left(-z^2\right)\mathrm{d}z,$$

$$\zeta_2(s) = 2\frac{\sin \pi s}{\pi}\sum_{k=1}^{\infty}k^{-2s}\int_0^{\infty} z^{-s}\frac{\mathrm{d}}{\mathrm{d}z}\ln D_k\left(-k^2 z\right)\mathrm{d}z.$$

The functions $\zeta_1(s)$ and $\zeta_2(s)$ will be analyzed further by subtracting and adding back leading terms in a suitable asymptotic expansion. Following [15], splitting the asymptotic expansion into relevant and irrelevant pieces for the computation of the values and residues of $\zeta_1(s)$, we write $D_0(-z^2)$ as

$$D_0\left(-z^2\right) = \left(L_0\left(-z^2\right) + R_0\left(-z^2\right)\right)\big/C_0, \tag{16}$$

where $L_0(-z^2)$ and $R_0(-z^2)$ are the leading term and the remainder respectively, such that $R_0(-z^2)$ is exponentially small for large z; it therefore will not contribute to $\mathrm{Res}\,\zeta_1(1/2 - n)$ or $\zeta_1(-n)$ for $n \geqslant 0$. For $\zeta_1'(0)$, in addition to $\ln L_0(-z^2)$, the contribution from other terms is

$-\ln C_0$. Similarly, we write $D_k(-k^2 z)$ as

$$D_k\left(-k^2 z\right) = \left(L_k\left(-k^2 z\right) + R_k\left(-k^2 z\right)\right) \Big/ \left(L_k(0) + R_k(0)\right), \tag{17}$$

where $L_k(-k^2 z)$ and $R_k(-k^2 z)$ are the leading term and the remainder respectively, and $L_k(0) = 1$. As before, $R_k(-k^2 z)$ is exponentially small for large z and it does not contribute to Res $\zeta_2(1)$, Res $\zeta_2(1/2 - n)$ or $\zeta_2(-n)$ for $n \geqslant 0$. In order to compute $\zeta_2'(0)$, the relevant splitting will be

$$\ln D_k\left(-k^2 z\right) = \ln L_k\left(-k^2 z\right) + \ln\left(1 + R_k\left(-k^2 z\right) \Big/ L_k\left(-k^2 z\right)\right) - \ln\left(1 + R_k(0)\right),$$

and following [15], in addition to $\ln L_k(-k^2 z)$, the contribution from other terms to $\zeta_2'(0)$ is $-2\sum_{k=1}^{\infty} \ln(1 + R_k(0))$.

After this outline of the computation for the general case, let us next consider specific separated and unseparated boundary conditions.

3. Separated boundary conditions

For separated boundary conditions, we consider the following four cases:Dirichlet–Dirichlet (DD), Neumann–Dirichlet (ND), Dirichlet–Neumann (DN), and Neumann–Neumann (NN). The relevant choices for a, b, c, d in equation (4) are $a = c = 1$, $b = d = 0$ for DD, $c = -b = 1$, $a = d = 0$ for ND, $a = d = 1$, $b = c = 0$ for DN, and finally $d = -b = 1$, $a = c = 0$ for NN. By equation (5), we then find

$$\begin{aligned} &F_k^{\mathrm{DD}}(\lambda) = u_k^{\mathrm{D}}\left(\lambda; x_1\right), \quad F_k^{\mathrm{ND}}(\lambda) = u_k^{\mathrm{N}}\left(\lambda; x_1\right), \\ &F_k^{\mathrm{DN}}(\lambda) = v_k^{\mathrm{D}}\left(\lambda; x_1\right), \quad F_k^{\mathrm{NN}}(\lambda) = v_k^{\mathrm{N}}\left(\lambda; x_1\right). \end{aligned} \tag{18}$$

The relevant aysmptotic terms in equation (18) are found from equations (10)–(13), once the $T_k^{\pm}$ have been expanded. A WKB expansion starting with equation (8) shows (following [15])

$$T_0^{\pm}\left(-z^2; x\right) = zf \mp \frac{f'}{2\sqrt{1 + f'^2}} + \frac{1}{4zf}\left[-\frac{f'^2}{2\left(1 + f'^2\right)} + \frac{f''f}{\left(1 + f'^2\right)^2}\right] + O\left(z^{-2}\right), \tag{19}$$

which gives

$$T_0\left(-z^2; x\right) = zf + \frac{1}{4zf}\left[-\frac{f'^2}{2\left(1 + f'^2\right)} + \frac{f''f}{\left(1 + f'^2\right)^2}\right] + O\left(z^{-3}\right). \tag{20}$$

Furthermore

$$\begin{aligned} T_k^{\pm}\left(-k^2 z; x\right) = {} & k\sqrt{t + 1} \mp \frac{f'}{2\sqrt{1 + f'^2}} \frac{t}{t + 1} \\ & + \frac{t}{4k(t + 1)^{3/2}}\left[-\frac{f'^2}{2\left(1 + f'^2\right)} \frac{t - 4}{t + 1} + \frac{f''f}{\left(1 + f'^2\right)^2}\right] + O\left(k^{-2}\right), \end{aligned} \tag{21}$$

where $t = zf^2(x)$. This implies

$$T_k\left(-k^2z;\, x\right) = k\sqrt{t+1} + \frac{t}{4k(t+1)^{3/2}}\left[-\frac{f'^2}{2\left(1+f'^2\right)}\frac{t-4}{t+1} + \frac{f''f}{\left(1+f'^2\right)^2}\right] + O\left(k^{-3}\right). \tag{22}$$

We next apply these expansions to (10)–(13) to deal with the various boundary conditions.

3.1. Dirichlet boundary condition

This case was studied in [15], but for convenience we include the results of [15] here. For Dirichlet boundary condition, it is easy to show that

$$u_0^{\rm D}\left(0;\, x_1\right) = A, \quad u_k^{\rm D}\left(0;\, x_1\right) = \sinh(kA)/k,\ k \neq 0,$$

where

$$A = \int_{x_0}^{x_1} \frac{\sqrt{1+f'^2}}{f}\mathrm{d}x. \tag{23}$$

Substituted into equation (14) for $F_k^{\rm DD}(\lambda)$, this shows

$$D_0^{\rm DD}\left(-z^2\right) = \frac{u_0^{\rm D}\left(-z^2;\, x_1\right)}{A}, \quad D_k^{\rm DD}\left(-k^2z\right) = \frac{u_k^{\rm D}\left(-k^2z;\, x_1\right)}{\sinh(kA)/k}.$$

Noting that upper indices + and − correspond to exponentially growing and decaying terms, substituting the WKB expansion of $u_0^{\rm D}(-z^2;\, x_1)$ obtained from equation (10) into $D_0^{\rm DD}(-z^2)$, and following equation (16), we obtain

$$L_0^{\rm DD}\left(-z^2\right) = \frac{u_0^+\left(-z^2;\, x_1\right)}{2T_0\left(-z^2;\, x_0\right)}, \quad R_0^{\rm DD}\left(-z^2\right) = -\frac{u_0^-\left(-z^2;\, x_1\right)}{2T_0\left(-z^2;\, x_0\right)}, \quad C_0^{\rm DD} = A.$$

Taking the logarithm, and using equation (9), shows

$$\ln L_0^{\rm DD}\left(-z^2\right) = \int_{x_0}^{x_1} \frac{T_0\left(-z^2;\, x\right)}{p(x)}\mathrm{d}x - \frac{\ln T_0\left(-z^2;\, x_0\right) + \ln T_0\left(-z^2;\, x_1\right)}{2} - \ln 2.$$

Making the asymptotic terms explicit, by equation (20)

$$\int_{x_0}^{x_1} \frac{T_0\left(-z^2;\, x\right)}{p(x)}\mathrm{d}x = \int_{x_0}^{x_1}\left[z\sqrt{1+f'^2} + \frac{f'^2}{8zf^2\sqrt{1+f'^2}}\right]\mathrm{d}x + \left[\frac{f'}{4zf\sqrt{1+f'^2}}\right]_{x_0}^{x_1} + O\left(z^{-3}\right), \tag{24}$$

$$\ln T_0\left(-z^2;\, x\right) = \ln z + \ln f(x) + \frac{1}{4z^2f^2}\left[-\frac{f'^2}{2\left(1+f'^2\right)} + \frac{f''f}{\left(1+f'^2\right)^2}\right] + O\left(z^{-4}\right). \tag{25}$$

The information gathered so far is sufficient to obtain the following properties of the function $\zeta_1(s)$ associated with $k = 0$ (see [15])

$$\mathrm{Res}\,\zeta_1^{\mathrm{DD}}\left(\frac{1}{2}\right) = \frac{1}{2\pi}\int_{x_0}^{x_1}\sqrt{1+f'(x)^2}\,\mathrm{d}x,$$

$$\zeta_1^{\mathrm{DD}}(0) = -\frac{1}{2},$$

$$\mathrm{Res}\,\zeta_1^{\mathrm{DD}}\left(-\frac{1}{2}\right) = \frac{1}{2\pi}\left[\frac{1}{8}\int_{x_0}^{x_1}\frac{f'^2\mathrm{d}x}{f^2\sqrt{1+f'^2}} + \frac{f'(x_1)}{4f(x_1)\sqrt{1+f'^2(x_1)}} - \frac{f'(x_0)}{4f(x_0)\sqrt{1+f'^2(x_0)}}\right],$$

$$\zeta_1^{\mathrm{DD}\,\prime}(0) = -\frac{\ln f(x_0) + \ln f(x_1)}{2} - \ln(2A).$$

For $k \neq 0$, substituting $u_k^{\mathrm{D}}(-k^2z;\, x_1)$ from equation (10) into $D_k^{\mathrm{DD}}(-k^2z)$, and following equation (17), the splitting reads

$$L_k^{\mathrm{DD}}\left(-k^2z\right) = \frac{u_k^+\left(-k^2z;\, x_1\right)\mathrm{e}^{-kA}}{T_k\left(-k^2z;\, x_0\right)/k},$$

$$R_k^{\mathrm{DD}}\left(-k^2z\right) = -\frac{u_k^-\left(-k^2z;\, x_1\right)\mathrm{e}^{-kA}}{T_k\left(-k^2z;\, x_0\right)/k}, \quad R_k^{\mathrm{DD}}(0) = -\mathrm{e}^{-2\,kA}.$$

Taking the logarithm, and using equation (9)

$$\ln L_k^{\mathrm{DD}}\left(-k^2z\right) = \int_{x_0}^{x_1}\frac{T_k\left(-k^2z;\, x\right) - k}{p(x)}\mathrm{d}x - \frac{\ln\left(T_k\left(-k^2z;\, x_0\right)/k\right) + \ln\left(T_k\left(-k^2z;\, x_1\right)/k\right)}{2},$$

asymptotic terms are found from equation (22)

$$\int_{x_0}^{x_1}\frac{T_k\left(-k^2z;\, x\right) - k}{p(x)}\mathrm{d}x = \int_{x_0}^{x_1}\left[\frac{k}{p}\left(\sqrt{t+1} - 1\right) + \frac{t^2}{8k(t+1)^{5/2}}\frac{f'^2}{f\sqrt{1+f'^2}}\right]\mathrm{d}x + \left[\frac{t}{4k(t+1)^{3/2}}\frac{f'}{\sqrt{1+f'^2}}\right]_{x_0}^{x_1} + O\left(k^{-3}\right), \tag{26}$$

$$\ln\frac{T_k\left(-k^2z;\, x\right)}{k} = \ln\sqrt{t+1} + \frac{t}{4k^2(t+1)^2} \times\left[\frac{4-t}{2(t+1)}\frac{f'^2}{1+f'^2} + \frac{f''f}{\left(1+f'^2\right)^2}\right] + O\left(k^{-4}\right). \tag{27}$$

From here, one can show

$$\mathrm{Res}\,\zeta_2^{\mathrm{DD}}(1) = \frac{1}{2}\int_{x_0}^{x_1} f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x,$$

$$\operatorname{Res} \zeta_2^{\mathrm{DD}}\left(\frac{1}{2}\right) = -\frac{1}{2\pi}\int_{x_0}^{x_1} \sqrt{1 + f'(x)^2}\,\mathrm{d}x - \frac{1}{4}\left[f(x_0) + f(x_1)\right],$$

$$\operatorname{Res} \zeta_2^{\mathrm{DD}}(0) = \frac{1}{2},$$

$$\begin{aligned}
\operatorname{Res} \zeta_2^{\mathrm{DD}}\left(-\frac{1}{2}\right) = &-\frac{1}{16\pi}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)^2\sqrt{1 + f'(x)^2}}\mathrm{d}x \\
&- \frac{1}{8\pi}\frac{f'(x_1)}{f(x_1)\sqrt{1 + f'(x_1)^2}} + \frac{1}{8\pi}\frac{f'(x_0)}{f(x_0)\sqrt{1 + f'(x_0)^2}} \\
&- \frac{1}{256}\frac{f'(x_0)^2}{f(x_0)\left(1 + f'(x_0)^2\right)} - \frac{1}{32}\frac{f''(x_0)}{\left(1 + f'(x_0)^2\right)^2} \\
&- \frac{1}{8\pi}\frac{f'(x_0)}{f(x_0)\sqrt{1 + f'(x_0)^2}} + \frac{1}{8\pi}\frac{f'(x_1)}{f(x_1)\sqrt{1 + f'(x_1)^2}} \\
&- \frac{1}{256}\frac{f'(x_1)^2}{f(x_1)\left(1 + f'(x_1)^2\right)} - \frac{1}{32}\frac{f''(x_1)}{\left(1 + f'(x_1)^2\right)^2},
\end{aligned}$$

$$\begin{aligned}
\zeta_2^{\mathrm{DD}\prime}(0) = &-2\ln\phi\left(\mathrm{e}^{-2A}\right) + \frac{A}{6} + \ln(2\pi) + \frac{1}{2}\left(\ln f(x_0) + \ln f(x_1)\right) \\
&+ \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x + \frac{1}{2}\int_{x_0}^{x_1} \frac{f''(x)}{\left(1 + f'(x)^2\right)^{3/2}}\mathrm{d}x,
\end{aligned}$$

where ϕ is the Euler function, $\phi(q) = \prod_{k=1}^{\infty}(1 - q^k)$. Adding up, $\zeta(s) = \zeta_1(s) + \zeta_2(s)$, we confirm the result in [15]

$$\operatorname{Res} \zeta^{\mathrm{DD}}(1) = \frac{1}{2}\int_{x_0}^{x_1} f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x, \tag{28}$$

$$\operatorname{Res} \zeta^{\mathrm{DD}}\left(\frac{1}{2}\right) = -\frac{f(x_0)}{4} - \frac{f(x_1)}{4}, \tag{29}$$

$$\zeta^{\mathrm{DD}}(0) = 0, \tag{30}$$

$$\begin{aligned}
\operatorname{Res} \zeta^{\mathrm{DD}}\left(-\frac{1}{2}\right) = &-\frac{f'^2(x_0)}{256 f(x_0)\left(1 + f'^2(x_0)\right)} - \frac{f''(x_0)}{32\left(1 + f'^2(x_0)\right)^2} \\
&- \frac{f'^2(x_1)}{256 f(x_1)\left(1 + f'^2(x_1)\right)} - \frac{f''(x_1)}{32\left(1 + f'^2(x_1)\right)^2},
\end{aligned} \tag{31}$$

$$\begin{aligned}
\zeta^{\mathrm{DD}\prime}(0) = &-2\ln\phi\left(\mathrm{e}^{-2A}\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x \\
&- \frac{f'(x_0)}{2\sqrt{1 + f'(x_0)^2}} + \frac{f'(x_1)}{2\sqrt{1 + f'(x_1)^2}} + \ln\pi - \ln A.
\end{aligned}$$

The residues and the value at $s = 0$ can be verified from known heat kernel asymptotics [13, 16]. Let us denote the surface of revolution by M and its boundary by ∂M. The flat space $\mathbb{R}^3$ induces a metric tensor on the surface of revolution, which is given by

$$g = \begin{pmatrix} 1 + f'(x)^2 & 0 \\ 0 & f^2(x) \end{pmatrix}.$$

We then have

$$\text{Res } \zeta^{\text{DD}}(1) = \frac{1}{4\pi}\text{vol}(M),$$

which agrees with (28) as the Riemannian volume element of M is $|g|^{1/2} = f(x)\sqrt{1 + f'(x)^2}$. Also,

$$\text{Res } \zeta^{\text{DD}}\left(\frac{1}{2}\right) = -\frac{1}{8\pi}\text{vol}(\partial M),$$

which is seen to agree with (29). To compare $\zeta^{\text{DD}}(0)$ and Res $\zeta^{\text{DD}}(-1/2)$ with the results known from the heat kernel coefficients we need some curvature tensors of the surface of revolution. In particular, the Riemann scalar reads

$$R = -\frac{2f''(x)}{f(x)\left(1 + f'(x)^2\right)^2}, \tag{32}$$

and the second fundamental form for the boundary at x_1, respectively x_0, is

$$K_{x_1} = \frac{f'(x_1)}{f(x_1)\sqrt{1 + f'(x_1)^2}}, \quad K_{x_0} = -\frac{f'(x_0)}{f(x_0)\sqrt{1 + f'(x_0)^2}}. \tag{33}$$

Use of these in the local formula for $\zeta^{\text{DD}}(0)$ shows (30)

$$\zeta^{\text{DD}}(0) = \frac{1}{4\pi}\cdot\frac{1}{6}\left\{\int_M R\,|g|^{1/2}\text{d}x\text{d}\theta + 2\int_{\partial M} K\,|h|^{1/2}\text{d}\theta\right\} = 0,$$

once the induced Riemannian volume element on the boundary is realized as $|h|^{1/2} = f$. Finally, to verify Res $\zeta^{\text{DD}}(-1/2)$, note that the normal component of the Ricci tensor is $R_{\text{mm}} = 1/2R$, and so

$$\text{Res } \zeta^{\text{DD}}\left(-\frac{1}{2}\right) = \frac{1}{1536\pi}\int_{\partial M}\left(12R - 3K^2\right)|h|^{1/2}\text{d}\theta,$$

in complete agreement with (31), once the expressions (32) and (33) have been substituted.

Note, that also for other boundary conditions we will find Res $\zeta(1)$ = Res $\zeta^{\text{DD}}(1)$, as this residue is proportional to the volume of the surface, so it is independent of boundary conditions.

The structure of the computation for the other boundary conditions is as just presented. The numerical coefficients in front of most terms will be different, but the strategy outlined works equally well.

3.2. Dirichlet–Neumann boundary conditions

Next we consider the Neumann condition at x_0 and Dirichlet condition at x_1 (ND). In this case

$$u_0^{\text{N}}\left(0; x_1\right) = 1, \quad u_k^{\text{N}}\left(0; x_1\right) = \cosh(kA),\ k \neq 0.$$

Substituted into equation (14) for $F_k^{\mathrm{ND}}(\lambda)$,

$$D_0^{\mathrm{ND}}\left(-z^2\right) = u_0^{\mathrm{N}}\left(-z^2;\, x_1\right), \quad D_k^{\mathrm{ND}}\left(-k^2 z\right) = \frac{u_k^{\mathrm{N}}\left(-k^2 z;\, x_1\right)}{\cosh(kA)}.$$

Substituting $u_0^{\mathrm{N}}(-z^2; x_1)$ from equation (11) into $D_0^{\mathrm{ND}}(-z^2)$, and following equation (16), shows

$$\ln L_0^{\mathrm{ND}}\left(-z^2\right) = \ln L_0^{\mathrm{DD}}\left(-z^2\right) + \ln\left(T_0^{-}\left(-z^2;\, x_0\right)\right), \quad C_0^{\mathrm{ND}} = 1.$$

For $k \neq 0$, substituting $u_k^{\mathrm{N}}(-k^2 z; x_1)$ from equation (11) into $D_k^{\mathrm{ND}}(-k^2 z)$, and following equation (17), gives

$$\ln L_k^{\mathrm{ND}}\left(-k^2 z\right) = \ln L_k^{\mathrm{DD}}\left(-k^2 z\right) + \ln\left(T_k^{-}\left(-k^2 z;\, x_0\right)/k\right), \quad R_k^{\mathrm{ND}}(0) = \mathrm{e}^{-2\,kA}.$$

This reduces the calculation to the *DD* case, except for two terms, which follow from equations (19) and (21), namely

$$\ln T_0^{\pm}\left(-z^2;\, x\right) = \ln z + \ln f(x) \mp \frac{f'}{2zf\sqrt{1+f'^2}} + \frac{1}{4z^2 f^2}\left[-\frac{f'^2}{1+f'^2} + \frac{f''f}{\left(1+f'^2\right)^2}\right] + O\left(z^{-3}\right),$$

$$\ln \frac{T_k^{\pm}\left(-k^2 z;\, x\right)}{k} = \ln\sqrt{t+1} \mp \frac{f'}{2k\sqrt{1+f'^2}}\frac{t}{(t+1)^{3/2}} + \frac{t}{4k^2(t+1)^2}\left[\frac{2-t}{t+1}\frac{f'^2}{1+f'^2} + \frac{f''f}{\left(1+f'^2\right)^2}\right] + O\left(k^{-3}\right).$$

Using the above expressions, we then find

$$\operatorname{Res}\zeta^{\mathrm{ND}}\left(\frac{1}{2}\right) = \frac{f(x_0)}{4} - \frac{f(x_1)}{4},$$

$$\zeta^{\mathrm{ND}}(0) = 0,$$

$$\operatorname{Res}\zeta^{\mathrm{ND}}\left(-\frac{1}{2}\right) = -\frac{5f'^2(x_0)}{256 f(x_0)\left(1+f'^2(x_0)\right)} + \frac{f''(x_0)}{32\left(1+f'^2(x_0)\right)^2} - \frac{f'^2(x_1)}{256 f(x_1)\left(1+f'^2(x_1)\right)} - \frac{f''(x_1)}{32\left(1+f'^2(x_1)\right)^2},$$

$$\zeta^{\mathrm{ND}\prime}(0) = -2\left(\ln\phi\left(\mathrm{e}^{-4A}\right) - \ln\phi\left(\mathrm{e}^{-2A}\right)\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1}\frac{f'(x)^2}{f(x)\sqrt{1+f'(x)^2}}\mathrm{d}x$$

$$+\frac{f'(x_0)}{2\sqrt{1+f'(x_0)^2}}+\frac{f'(x_1)}{2\sqrt{1+f'(x_1)^2}}-\ln 2,$$

where the residues at $s = 1/2$, $s = -1/2$, and the value at $s = 0$ compare favorably with the known results as they follow from known heat kernel coefficients [13, 16].

Similarly, for Dirichlet condition at x_0 and Neumann condition at x_1 (DN), one has

$$v_0^{\rm D}(0; x_1) = 1, \quad v_k^{\rm D}(0; x_1) = \cosh(kA),\ k \neq 0.$$

Substituted into equation (14) for $F_k^{\rm DN}(\lambda)$, gives

$$D_0^{\rm DN}\left(-z^2\right) = v_0^{\rm D}\left(-z^2; x_1\right), \quad D_k^{\rm DN}\left(-k^2 z\right) = \frac{v_k^{\rm D}\left(-k^2 z; x_1\right)}{\cosh(kA)}.$$

Substituting in $v_k^{\rm D}(y; x_1)$ from equation (12), and following equations (16) and (17), one verifies

$$\ln L_0^{\rm DN}\left(-z^2\right) = \ln L_0^{\rm DD}\left(-z^2\right) + \ln T_0^+\left(-z^2; x_1\right), \quad C_0^{\rm DN} = 1,$$

$$\ln L_k^{\rm DN}\left(-k^2 z\right) = \ln L_k^{\rm DD}\left(-k^2 z\right) + \ln\left(T_k^+\left(-k^2 z; x_1\right)\Big/k\right), \quad R_k^{\rm DN}(0) = {\rm e}^{-2kA}.$$

Again skipping to write out answers for ζ_1 and ζ_2, the results are

$$\mathrm{Res}\ \zeta^{\rm DN}\left(\frac{1}{2}\right) = -\frac{f(x_0)}{4} + \frac{f(x_1)}{4},$$

$$\zeta^{\rm DN}(0) = 0,$$

$$\mathrm{Res}\ \zeta^{\rm DN}\left(-\frac{1}{2}\right) = -\frac{f'^2(x_0)}{256 f(x_0)\left(1+f'^2(x_0)\right)} - \frac{f''(x_0)}{32\left(1+f'^2(x_0)\right)^2} - \frac{5f'^2(x_1)}{256 f(x_1)\left(1+f'^2(x_1)\right)} + \frac{f''(x_1)}{32\left(1+f'^2(x_1)\right)^2},$$

$$\zeta^{\rm DN\,\prime}(0) = -2\left(\ln\phi\left({\rm e}^{-4A}\right) - \ln\phi\left({\rm e}^{-2A}\right)\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1}\frac{f'(x)^2}{f(x)\sqrt{1+f'(x)^2}}{\rm d}x - \frac{f'(x_0)}{2\sqrt{1+f'(x_0)^2}} - \frac{f'(x_1)}{2\sqrt{1+f'(x_1)^2}} - \ln 2.$$

The value and residue are again in agreement with expectations from the heat kernel coefficients. Also, $\zeta^{\rm ND}(s)$ and $\zeta^{\rm DN}(s)$ are symmetric as expected.

3.3. Neumann boundary conditions

Unlike in previous cases, zero is an eigenvalue for the Neumann boundary condition, which makes certain modifications necessary. First we note that

$$v_0^{\rm N}(0; x_1) = 0, \quad \frac{\rm d}{{\rm d}y}v_0^{\rm N}(0; x_1) = -B,$$

where

$$B = \int_{x_0}^{x_1} f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x. \tag{34}$$

Substituted into equation (15) for $F_0^{\mathrm{NN}}(y)$, this yields

$$D_0^{\mathrm{NN}}\left(-z^2\right) = \frac{v_0^{\mathrm{N}}\left(-z^2; x_1\right)}{z^2 B}.$$

For $k \neq 0$, instead

$$v_k^{\mathrm{N}}\left(0; x_1\right) = k \sinh(kA),$$

and substituted into equation (14) for $F_k^{\mathrm{NN}}(y)$ shows

$$D_k^{\mathrm{NN}}\left(-k^2 z\right) = \frac{v_k^{\mathrm{N}}\left(-k^2 z; x_1\right)}{k \sinh(kA)}.$$

Using the WKB expansion of $v_k^{\mathrm{N}}(y; x_1)$ in equation (13), and following equations (16) and (17), the relevant pieces are

$$\ln L_0^{\mathrm{NN}}\left(-z^2\right) = \ln L_0^{\mathrm{DD}}\left(-z^2\right) + \ln T_0^-\left(-z^2; x_0\right) + \ln T_0^+\left(-z^2; x_1\right), \quad C_0^{\mathrm{NN}} = z^2 B,$$

$$\ln L_k^{\mathrm{NN}}\left(-k^2 z\right) = \ln L_k^{\mathrm{DD}}\left(-k^2 z\right) + \ln\left(T_k^-\left(-k^2 z; x_0\right)\big/k\right) + \ln\left(T_k^+\left(-k^2 z; x_1\right)\big/k\right), \quad R_k^{\mathrm{NN}}(0) = -\mathrm{e}^{-2kA}.$$

From here, one verifies the final answers are

$$\operatorname{Res} \zeta^{\mathrm{NN}}\left(\frac{1}{2}\right) = \frac{f(x_0)}{4} + \frac{f(x_1)}{4},$$

$$\zeta^{\mathrm{NN}}(0) = -1,$$

$$\operatorname{Res} \zeta^{\mathrm{NN}}\left(-\frac{1}{2}\right) = -\frac{5f'^2(x_0)}{256 f(x_0)\left(1 + f'^2(x_0)\right)} + \frac{f''(x_0)}{32\left(1 + f'^2(x_0)\right)^2} - \frac{5f'^2(x_1)}{256 f(x_1)\left(1 + f'^2(x_1)\right)} + \frac{f''(x_1)}{32\left(1 + f'^2(x_1)\right)^2},$$

$$\zeta_2^{\mathrm{NN}\,\prime}(0) = -2\ln\phi\left(\mathrm{e}^{-2A}\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x + \frac{f'(x_0)}{2\sqrt{1 + f'(x_0)^2}} - \frac{f'(x_1)}{2\sqrt{1 + f'(x_1)^2}} - \ln(4\pi) - \ln B,$$

and the same remarks as for the other boundary conditions hold.

4. Unseparated boundary conditions

For unseparated boundary conditions, we first consider two special cases, namely, periodic boundary conditions (P)

$$u(x_1) = u(x_0), \quad v(x_1) = v(x_0),$$

and anti-periodic boundary conditions (AP)

$$u(x_1) = -u(x_0), \quad v(x_1) = -v(x_0).$$

We assume $f(x_0) = f(x_1)$ for both conditions. The periodic boundary condition represents a torus. The associated functions are from (7)

$$F_k^{\mathrm{P}}(y) = 2 - u_k^{\mathrm{N}}(y; x_1) - v_k^{\mathrm{D}}(y; x_1), \quad F_k^{\mathrm{AP}}(y) = 2 + u_k^{\mathrm{N}}(y; x_1) + v_k^{\mathrm{D}}(y; x_1). \tag{35}$$

4.1. Periodic and antiperiodic conditions

For periodic boundary conditions, zero is an eigenvalue.The relevant information needed is

$$F_0^{\mathrm{P}}(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}y} F_0^{\mathrm{P}}(0) = AB,$$

where A and B are defined in (23) and (34). Substituted into equation (15) for $F_0^{\mathrm{P}}(y)$,

$$D_0^{\mathrm{P}}\left(-z^2\right) = \frac{u_0^{\mathrm{N}}\left(-z^2; x_1\right) + v_0^{\mathrm{D}}\left(-z^2; x_1\right) - 2}{z^2 AB}.$$

For $k \neq 0$,

$$F_k^{\mathrm{P}}(0) = 2(1 - \cosh(kA)).$$

Substituted into equation (14) for $F_k^{\mathrm{P}}(y)$,

$$D_k^{\mathrm{P}}\left(-k^2 z\right) = \frac{u_k^{\mathrm{N}}\left(-k^2 z; x_1\right) + v_k^{\mathrm{D}}\left(-k^2 z; x_1\right) - 2}{2(\cosh(kA) - 1)}.$$

Using the WKB expansions of $u_k^{\mathrm{N}}(y; x_1)$ and $v_k^{\mathrm{D}}(y; x_1)$ in $D_k^{\mathrm{P}}(y)$, and following equations (16) and (17), one obtains

$$\ln L_0^{\mathrm{P}}\left(-z^2\right) = \ln L_0^{\mathrm{DD}}\left(-z^2\right) + \ln\left[T_0^-\left(-z^2; x_0\right) + T_0^+\left(-z^2; x_1\right)\right], \quad C_0^{\mathrm{P}} = z^2 AB,$$

$$\ln L_k^{\mathrm{P}}\left(-k^2 z\right) = \ln L_k^{\mathrm{DD}}\left(-k^2 z\right) + \ln \frac{T_k^-\left(-k^2 z; x_0\right) + T_k^+\left(-k^2 z; x_1\right)}{2k},$$

$$L_k^{\mathrm{P}}(0) + R_k^{\mathrm{P}}(0) = \left(1 - \mathrm{e}^{-kA}\right)^2.$$

Using the assumption that $f(x_0) = f(x_1)$, and writing $t(x_0)$ as t_0, we have

$$\ln L_0^{\mathrm{P}}\left(-z^2\right) = \int_{x_0}^{x_1}\left[z\sqrt{1 + f'^2} + \frac{f'^2}{8zf^2\sqrt{1 + f'^2}}\right]\mathrm{d}x - \frac{\delta_0^2}{32z^2 f^2(x_0)} + O\left(\frac{1}{z^3}\right),$$

$$\begin{aligned}\ln L_k^{\mathrm{P}}\left(-k^2 z\right) = &\int_{x_0}^{x_1}\left[\frac{k}{p}\left(\sqrt{t + 1} - 1\right) + \frac{t^2}{8k(t + 1)^{5/2}} \frac{f'^2}{f\sqrt{1 + f'^2}}\right]\mathrm{d}x \\ &- \frac{t_0^2 \delta_0^2}{32k^2\left(t_0 + 1\right)^3} + O\left(\frac{1}{k^3}\right),\end{aligned}$$

where

$$\delta_0 = \frac{f'(x_0)}{\sqrt{1 + f'(x_0)^2}} - \frac{f'(x_1)}{\sqrt{1 + f'(x_1)^2}}. \tag{36}$$

From the expansion we obtain for the $k = 0$ mode

$$\text{Res}\, \zeta_1^{\text{P}}\left(\frac{1}{2}\right) = \frac{1}{2}\int_{x_0}^{x_1} \sqrt{1 + f'(x)^2}\, \mathrm{d}x, \tag{37}$$

$$\zeta_1^{\text{P}}(0) = -1, \tag{38}$$

$$\text{Res}\, \zeta_1^{\text{P}}\left(-\frac{1}{2}\right) = \frac{1}{16\pi}\int_{x_0}^{x_1} \frac{f'(x)^2}{f^2(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x, \tag{39}$$

$$\zeta_1^{\text{P}}{}'(0) = -\ln A - \ln B, \tag{40}$$

and for the $k \neq 0$ modes

$$\text{Res}\, \zeta_2^{\text{P}}\left(\frac{1}{2}\right) = -\frac{1}{2}\int_{x_0}^{x_1} \sqrt{1 + f'(x)^2}\, \mathrm{d}x, \tag{41}$$

$$\zeta_2^{\text{P}}(0) = 0, \tag{42}$$

$$\text{Res}\, \zeta_2^{\text{P}}\left(-\frac{1}{2}\right) = -\frac{1}{16\pi}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)^2\sqrt{1 + f'(x)^2}}\mathrm{d}x - \frac{3}{512f(x_0)}\delta_0^2, \tag{43}$$

$$\zeta_2^{\text{P}}{}'(0) = -4\ln\phi\left(\mathrm{e}^{-A}\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x. \tag{44}$$

Adding up, $\zeta(s) = \zeta_1(s) + \zeta_2(s)$, we get

$$\text{Res}\, \zeta^{\text{P}}\left(\frac{1}{2}\right) = 0,$$

$$\zeta^{\text{P}}(0) = -1,$$

$$\text{Res}\, \zeta^{\text{P}}\left(-\frac{1}{2}\right) = -\frac{3}{512f(x_0)}\delta_0^2,$$

and

$$\zeta^{\text{P}\prime}(0) = -4\ln\phi\left(\mathrm{e}^{-A}\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x - \ln A - \ln B.$$

The first of the above equations reflects that the torus does not have a boundary. The second equation says that its Euler characteristic is zero, taking into account that the one zero mode is not included in the zeta function.

For antiperiodic boundary condition, the corresponding equations are

$$D_0^{\mathrm{AP}}\left(-z^2\right) = \frac{u_0^{\mathrm{N}}\left(-z^2; x_1\right) + v_0^{\mathrm{D}}\left(-z^2; x_1\right) + 2}{4},$$
$$D_k^{\mathrm{AP}}\left(-k^2 z\right) = \frac{u_k^{\mathrm{N}}\left(-k^2 z; x_1\right) + v_k^{\mathrm{D}}\left(-k^2 z; x_1\right) + 2}{2(\cosh(kA) + 1)},$$

which imply that

$$\ln L_0^{\mathrm{AP}}\left(-z^2\right) = \ln L_0^{\mathrm{P}}\left(-z^2\right), \quad C_0^{\mathrm{AP}} = 4,$$

$$\ln L_k^{\mathrm{AP}}\left(-k^2 z\right) = \ln L_k^{\mathrm{P}}\left(-k^2 z\right), \quad L_k^{\mathrm{AP}}(0) + R_k^{\mathrm{AP}}\left(-k^2 z\right) = \left(1 + \mathrm{e}^{-kA}\right)^2.$$

The resulting residues of $\zeta_1(s)$ and $\zeta_2(s)$ are the same as those for periodic boundary conditions. Their values and derivatives at $s = 0$ are

$$\zeta_1^{\mathrm{AP}}(0) = 0, \tag{45}$$

$$\zeta_1^{\mathrm{AP}\,\prime}(0) = -\ln 4, \tag{46}$$

$$\zeta_2^{\mathrm{AP}}(0) = 0, \tag{47}$$

$$\zeta_2^{\mathrm{AP}\prime}(0) = -4\left(\ln \phi\left(\mathrm{e}^{-2A}\right) - \ln \phi\left(\mathrm{e}^{-A}\right)\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x. \tag{48}$$

They lead to

$$\mathrm{Res}\, \zeta^{\mathrm{AP}}\left(\frac{1}{2}\right) = 0,$$

$$\zeta^{\mathrm{AP}}(0) = 0,$$

$$\mathrm{Res}\, \zeta^{\mathrm{AP}}\left(-\frac{1}{2}\right) = -\frac{3}{512 f(x_0)}\delta_0^2,$$

and

$$\zeta^{\mathrm{AP}\prime}(0) = -4\left(\ln \phi\left(\mathrm{e}^{-2A}\right) - \ln \phi\left(\mathrm{e}^{-A}\right)\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1} \frac{f'(x)^2}{f(x)\sqrt{1 + f'(x)^2}}\mathrm{d}x - \ln 4.$$

Similar remarks as those made above for periodic boundary conditions apply.

4.2. Klein bottle

We consider a special unseparated boundary condition that involves the azimuthal angle θ,

$$u\left(x_1, \theta\right) = u\left(x_0, -\theta\right), \quad v\left(x_1, \theta\right) = v\left(x_0, -\theta\right).$$

We assume $f(x_0) = f(x_1)$. The boundary condition represents a Klein bottle (K). For $k = 0$, it is the same as periodic boundary condition. Therefore

$$\zeta_1^K(s) = \zeta_1^{\mathrm{P}}(s).$$

For $k \neq 0$, the eigenfunction are no longer in the form of $\phi(x)\exp(\mathrm{i}k\theta)$. Instead, the eigenfunction is either

$$u(x,\,\theta) = \phi(x)\cos(k\theta),$$

with periodic $\phi(x)$, or

$$u(x,\,\theta) = \phi(x)\sin(k\theta),$$

with antiperiodic $\phi(x)$. As a result

$$\zeta_2^K(s) = \frac{\zeta_2^{\mathrm{P}}(s) + \zeta_2^{\mathrm{AP}}(s)}{2},$$

and

$$\mathrm{Res}\,\zeta^K\left(\frac{1}{2}\right) = 0,$$

$$\zeta^K(0) = -1,$$

$$\mathrm{Res}\,\zeta^K\left(-\frac{1}{2}\right) = -\frac{3}{512 f(x_0)}\delta_0^2,$$

$$\zeta^{K'}(0) = -2\ln\phi\left(\mathrm{e}^{-2A}\right) + \frac{A}{6} + \frac{1}{6}\int_{x_0}^{x_1}\frac{f'(x)^2}{f(x)\sqrt{1+f'(x)^2}}\mathrm{d}x - \ln A - \ln B.$$

Once again, observations made earlier regarding the first two equations are valid also for the Klein bottle.

5. Nonsmooth surfaces

For the examples with unseparated boundary conditions, a smooth surface requires that $f'(x_0) = f'(x_1)$, which implies that $\mathrm{Res}\,\zeta^{\mathrm{P}}(-1/2) = 0$. On the other hand, if $f'(x_0) \neq f'(x_1)$, the kink points on the torus would generate a nonzero residue of the zeta function at $-1/2$. In this section we will study the effect of kink points in $f(x)$ inside the interval $[x_0,\,x_1]$ on the zeta function for various boundary conditions. For clarity we assume $f(x)$ has only one kink point at x_K. Let

$$f(x) = f_1(x),\ x_0 \leqslant x \leqslant x_K, \qquad \text{and} \qquad f(x) = f_2(x),\ x_K < x \leqslant x_1,$$

where $f_1(x)$ and $f_2(x)$ are smooth, and

$$f_1(x_K) = f_2(x_K), \quad f_1'(x_K) \neq f_2'(x_K).$$

The WKB method cannot be applied to nonsmooth $f(x)$ directly, but it can be applied to f_1 and f_2 respectively. Introducing the fundamental solution for the intervals $[x_0,\,x_K]$ and $[x_K,\,x_1]$ as

$$E_{k,1}(\lambda;\,x) = \begin{pmatrix} u_{k,1}^{\mathrm{N}}(\lambda;\,x) & u_{k,1}^{\mathrm{D}}(\lambda;\,x) \\ v_{k,1}^{\mathrm{N}}(\lambda;\,x) & v_{k,1}^{\mathrm{D}}(\lambda;\,x) \end{pmatrix} \quad E_{k,2}(\lambda;\,x) = \begin{pmatrix} u_{k,2}^{\mathrm{N}}(\lambda;\,x) & u_{k,2}^{\mathrm{D}}(\lambda;\,x) \\ v_{k,2}^{\mathrm{N}}(\lambda;\,x) & v_{k,2}^{\mathrm{D}}(\lambda;\,x) \end{pmatrix},$$

with the conditions $E_{k,1}(\lambda;\,x_0) = I$ and $E_{k,2}(\lambda;\,x_K) = I$, the fundamental solution from x_0 to x_1 is

$$E_k(y;\,x_1) = E_{k,2}(y;\,x_1)E_{k,1}(y;\,x_K). \tag{49}$$

First consider Dirichlet boundary condition. By equation (49)

$$u^{\mathrm{D}}(x_1) = u_2^{\mathrm{N}}(x_1)u_1^{\mathrm{D}}(x_K) + u_2^{\mathrm{D}}(x_1)v_1^{\mathrm{D}}(x_K).$$

Following equation (16)

$$L^{\mathrm{DD}} = \frac{u_1^+(x_K)u_2^+(x_1)}{2T_1(x_0)2T_2(x_K)}\left[T_1^+(x_K) + T_2^-(x_K)\right].$$

For $k = 0$, taking the logarithm

$$\ln L_0^{\mathrm{DD}}\left(-z^2\right) = \int_{x_0}^{x_K} \frac{T_0\left(-z^2; x\right)}{p(x)}\mathrm{d}x + \int_{x_K}^{x_1} \frac{T_0\left(-z^2; x\right)}{p(x)}\mathrm{d}x - \frac{\ln T_0\left(-z^2; x_0\right) + \ln T_0\left(-z^2; x_1\right)}{2} - \ln 2 + K(z),$$

where

$$K(z) = \ln \frac{T_{0,2}^-\left(-z^2; x_K\right) + T_{0,1}^+\left(-z^2; x_K\right)}{2} - \frac{\ln T_{0,1}\left(-z^2; x_K\right) + \ln T_{0,2}\left(-z^2; x_K\right)}{2}.$$

Note, that $K(z)$ would vanish if $f_1'(x_K) = f_2'(x_K)$. For $f_1'(x_K) \neq f_2'(x_K)$,

$$K(z) = \frac{\delta_K}{4zf(x_K)} - \frac{\delta_K^2}{32z^2f^2(x_K)} + O\left(z^{-3}\right),$$

where

$$\delta_K = \frac{f_2'(x_K)}{\sqrt{1 + f_2'(x_K)^2}} - \frac{f_1'(x_K)}{\sqrt{1 + f_1'(x_K)^2}}. \tag{50}$$

Using equation (24)

$$\ln L_0^{\mathrm{DD}}\left(-z^2\right) = \int_{x_0}^{x_1}\left[z\sqrt{1 + f'^2} + \frac{f'^2}{8zf^2\sqrt{1 + f'^2}}\right]\mathrm{d}x + \left[\frac{f'}{4zf\sqrt{1 + f'^2}}\right]_{x_0}^{x_1} - \frac{\ln T_0\left(-z^2; x_0\right) + \ln T_0\left(-z^2; x_1\right)}{2} - \ln 2 - \frac{\delta_K^2}{32z^2f^2(x_K)} + O\left(z^{-3}\right).$$

The formula for Res $\zeta_1^{\mathrm{DD}}(1/2)$, $\zeta_1^{\mathrm{DD}}(0)$, Res $\zeta_1^{\mathrm{DD}}(-1/2)$ and $\zeta_1^{\mathrm{DD}\,\prime}(0)$ are not affected, though $\zeta_1^{\mathrm{DD}}(-1)$ changes. Similarly

$$\ln L_k^{\mathrm{DD}}\left(-k^2z\right) = \int_{x_0}^{x_K} \frac{T_k\left(-k^2z; x\right) - k}{p(x)}\mathrm{d}x + \int_{x_K}^{x_1} \frac{T_k\left(-k^2z; x\right) - k}{p(x)}\mathrm{d}x - \frac{\ln\left(T_k\left(-k^2z; x_0\right)/k\right) + \ln\left(T_k\left(-k^2z; x_1\right)/k\right)}{2} + K(k, z),$$

where

$$K(k, z) = \frac{t_K \delta_K}{4k\left(t_K^2 + 1\right)^{3/2}} - \frac{t_K^2 \delta_K^2}{32k^2\left(t_K + 1\right)^3} + O\left(k^{-3}\right),$$

in which $t_K = zf^2(x_K)$. With equation (26), this shows

$$\begin{aligned}\ln L_k^{\mathrm{DD}}\left(-k^2 z\right) = &\int_{x_0}^{x_1} \left[\frac{k}{p}\left(\sqrt{t+1} - 1\right) + \frac{t^2}{8k(t+1)^{5/2}} \frac{f'^2}{f\sqrt{1+f'^2}}\right] \mathrm{d}x \\ &+ \left[\frac{t}{4k(t+1)^{3/2}} \frac{f'}{\sqrt{1+f'^2}}\right]_{x_0}^{x_1} \\ &- \frac{\ln\left(T_k\left(-k^2 z; x_0\right)\big/k\right) + \ln\left(T_k\left(-k^2 z; x_1\right)\big/k\right)}{2} \\ &- \frac{t_K^2 \delta_K^2}{32k^2\left(t_K + 1\right)^3} + O\left(k^{-3}\right).\end{aligned}$$

The formula for $\mathrm{Res}\,\zeta_2^{\mathrm{DD}}(1/2)$, $\zeta_2^{\mathrm{DD}}(0)$ and $\zeta_2^{\mathrm{DD}\prime}(0)$ are not affected. However, $\mathrm{Res}\,\zeta_2^{\mathrm{DD}}(-1/2)$ changes, and we find

$$\begin{aligned}\mathrm{Res}\,\zeta^{\mathrm{DD}}\left(-\frac{1}{2}\right) = &-\frac{f'^2(x_0)}{256f(x_0)\left(1 + f'^2(x_0)\right)} - \frac{f''(x_0)}{32\left(1 + f'^2(x_0)\right)^2} \\ &- \frac{f'^2(x_2)}{256f(x_2)\left(1 + f'^2(x_2)\right)} - \frac{f''(x_2)}{32\left(1 + f'^2(x_2)\right)^2} \\ &- \frac{3\delta_K^2}{512f(x_K)}.\end{aligned} \tag{51}$$

If there is more than one kink point, $\mathrm{Res}\,\zeta^{\mathrm{DD}}(-1/2)$ will have an extra term $-3\delta_K^2\big/(512f(x_K))$ for each kink point. We will prove that the effect on the zeta function is the same for other boundary conditions, by showing that the ratio between $L^{\mathrm{DD}}(y)$ and the $L(y)$ for a given boundary condition is unaffected by kink points. For clarity we drop the dependence on k and y in the derivation and denote the leading term of $u^{\mathrm{D}}(x)$ by $\hat{u}^{\mathrm{D}}(x)$, etc. Then for a kink point in $f(x)$ at x_K,

$$\hat{E}_1(x_K) = \begin{pmatrix} 1 \\ T_1^+(x_K) \end{pmatrix} \left(T_1^-(x_0) \quad 1\right) \hat{u}_1^{\mathrm{D}}(x_K),$$

$$\hat{E}_2(x_1) = \begin{pmatrix} 1 \\ T_2^+(x_1) \end{pmatrix} \left(T_2^-(x_K) \quad 1\right) \hat{u}_2^{\mathrm{D}}(x_1),$$

which gives

$$\hat{E}(x_1) = \hat{E}_2(x_1)\hat{E}_1(x_K) = \begin{pmatrix} 1 \\ T_2^+(x_1) \end{pmatrix} \left(T_1^-(x_0) \quad 1\right) \hat{u}^{\mathrm{D}}(x_1),$$

where

$$\hat{u}^{\mathrm{D}}(x_1) = \hat{u}_2^{\mathrm{D}}(x_1)\hat{u}_1^{\mathrm{D}}(x_K)\Big[T_2^-(x_K) + T_1^+(x_K)\Big].$$

It indicates that the ratios between $\hat{u}^{\mathrm{N}}(x_1)$, $\hat{v}^{\mathrm{D}}(x_1)$, $\hat{v}^{\mathrm{N}}(x_1)$ and $\hat{u}^{\mathrm{D}}(x_1)$ are unaffected by the kink point, therefore the corresponding zeta functions will change as much as $\zeta^{\mathrm{DD}}(s)$.

6. Conclusions

This paper provides the analysis of the spectral zeta function for the Laplacian on a surface of revolution with a variety of boundary conditions imposed. Explicit results for several residues and values of the zeta function are given; all are in agreement with results known for more general geometries [13]. Furthermore, surprisingly simple results for the determinant are found. Our analysis allowed for the introduction of kink points such that the effect of non-smoothness could be studied. Additional contributions to some properties due to the kink point were found as was expected from a general perspective [14]. In some detail, denoting by Σ the circle of the surface located at x_K, in the notation of [14] our continuity assumptions on the eigenfunctions along Σ imply $U = 0$. Then, $\zeta^{\mathrm{DD}}(0)$ and Res $\zeta^{\mathrm{DD}}(-1/2)$ obtain additional contributions due to the fact that the surface is not smooth, namely (see theorem 2.3 in [14] restricted to the surface of revolution)

$$\zeta^{\mathrm{DD}}(0) = \frac{1}{4\pi}\cdot\frac{1}{6}\left\{\int_M R\left|g\right|^{1/2}\mathrm{d}x\mathrm{d}\theta + 2\int_{\partial M} K\,|h|^{1/2}\mathrm{d}\theta + 2\int_\Sigma\left(K^+ + K^-\right)\left|h_\Sigma\right|^{1/2}\mathrm{d}\theta\right\},$$

$$\mathrm{Res}\,\zeta^{\mathrm{DD}}\left(-\frac{1}{2}\right) = \frac{1}{1536\pi}\left\{\int_{\partial M}\left(12R - 3K^2\right)|h|^{1/2}\mathrm{d}\theta + \frac{9}{2}\int_\Sigma\left(K^+ + K^-\right)^2\left|h_\Sigma\right|^{1/2}\mathrm{d}\theta\right\},$$

where K^+, respectively K^-, are the second fundamental forms as induced from the surface to the left, respectively to the right, of x_K, and $|h_\Sigma|^{1/2}$ is the Riemannian volume element of the circle at x_K, namely $|h_\Sigma|^{1/2} = f(x_K)$. The additional contribution along Σ in $\zeta^{\mathrm{DD}}(0)$ guarantees that still $\zeta^{\mathrm{DD}}(0) = 0$, as this piece is needed to compensate a contribution coming from the integral along M because of the non-smoothness at x_K.

The contribution along Σ in Res $\zeta^{\mathrm{DD}}(-1/2)$ generates exactly the last term in (51), so that our result is in agreement with [14]. This was not completely clear as in [14] continuity of the metric is assumed which here is not given. Also in agreement with our findings, [14] predicts that these additional contributionas are independent of the boundary conditions imposed at ∂M.

Appendix. Spectral function of Sturm–Liouville equation

In this appendix we give an independent proof that equations (5) and (7) not only determine the eigenvalues but also the degeneracy correctly.

Consider the Sturm–Liouville problem.

$$\mathcal{L}(u) = -(Pu')' + Qu = \lambda Ru, \quad 0 \leqslant t \leqslant 1,$$

where $P > 0$, Q, $R > 0$, and u are functions of t, and for simplicity we have chosen the interval [0, 1]. With $v = Pu'$, the equation can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & P^{-1} \\ Q - \lambda R & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

$P(t)$, $Q(t)$, and $R(t)$ are not necessarily continuous. We write the fundamental solution $E(\lambda;\, t)$ as in equation (3). As described in section 2, to guarantee that $\mathcal{L}$ is self-adjoint, the boundary condition can be chosen as separated, equation (4), or unseparated ones, equation (6). For separated boundary conditions, the corresponding eigenvalues are the zeros of the following function of λ, see equation (5)

$$F(\lambda) = \det\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} E(\lambda;\, 1) \right) = \begin{pmatrix} c & d \end{pmatrix} E(\lambda;\, 1) \begin{pmatrix} -b \\ a \end{pmatrix}, \tag{52}$$

whereas for unseparated conditions the corresponding eigenvalues are the zeros of, see equation (7),

$$F(\lambda) = \det\left(E(\lambda;\, 1) - M\right) = 2 - du^{\mathrm{N}}(\lambda;\, 1) - av^{\mathrm{N}}(\lambda;\, 1) + cu^{\mathrm{D}}(\lambda;\, 1) + bv^{\mathrm{N}}(\lambda;\, 1). \tag{53}$$

For the separated boundary condition, each eigenvalue is simple [23]. We will prove that the corresponding $F(\lambda)$ also has only simple zeros. For the unseparated boundary condition, the eigenvalues can be simple or double. For example, for $P = 1$, $Q = 0$, $R = 1$, and the periodic boundary condition, all eigenvalues are double except for $\lambda = 0$. We will prove that each zero of $F(\lambda)$ has the same multiplicity as that of the corresponding eigenvalue.

Taking the derivative with respect to λ on both sides of the following equation,

$$\mathcal{L}\left(u^{\mathrm{N}}\right) = \lambda R u^{\mathrm{N}},$$

we have

$$\mathcal{L}\left(\frac{\partial u^{\mathrm{N}}}{\partial \lambda}\right) = \lambda R \frac{\partial u^{\mathrm{N}}}{\partial \lambda} + R u^{\mathrm{N}}.$$

The solution is

$$\frac{\partial u^{\mathrm{N}}(\lambda;\, t)}{\partial \lambda} = u^{\mathrm{N}}(\lambda;\, t) \int_0^t R(\tau) u^{\mathrm{N}}(\lambda;\, \tau) u^{\mathrm{D}}(\lambda;\, \tau) \mathrm{d}\tau - u^{\mathrm{D}}(\lambda;\, t) \int_0^t R(\tau) \left(u^{\mathrm{N}}(\lambda;\, \tau)\right)^2 \mathrm{d}\tau.$$

Similarly

$$\frac{\partial u^{\mathrm{D}}(\lambda;\, t)}{\partial \lambda} = u^{\mathrm{N}}(\lambda;\, t) \int_0^t R(\tau) \left(u^{\mathrm{D}}(\lambda;\, \tau)\right)^2 \mathrm{d}\tau - u^{\mathrm{D}}(\lambda;\, t) \int_0^t R(\tau) u^{\mathrm{N}}(\lambda;\, \tau) u^{\mathrm{D}}(\lambda;\, \tau) \mathrm{d}\tau.$$

Setting $t = 1$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda} \begin{pmatrix} u^{\mathrm{N}}(\lambda;\, 1) \\ u^{\mathrm{D}}(\lambda;\, 1) \end{pmatrix} &= K(\lambda) \begin{pmatrix} u^{\mathrm{N}}(\lambda;\, 1) \\ u^{\mathrm{D}}(\lambda;\, 1) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^1 R(t) u^{\mathrm{N}}(\lambda;\, t) u^{\mathrm{D}}(\lambda;\, t) \mathrm{d}t & -\int_0^1 R(t) \left(u^{\mathrm{N}}(\lambda;\, t)\right)^2 \mathrm{d}t \\ \int_0^1 R(t) \left(u^{\mathrm{D}}(\lambda;\, t)\right)^2 \mathrm{d}t & -\int_0^1 R(t) u^{\mathrm{N}}(\lambda;\, t) u^{\mathrm{D}}(\lambda;\, t) \mathrm{d}t \end{pmatrix} \\ &\quad \times \begin{pmatrix} u^{\mathrm{N}}(\lambda;\, 1) \\ u^{\mathrm{D}}(\lambda;\, 1) \end{pmatrix}. \end{aligned} \tag{54}$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\begin{pmatrix} v^{\mathrm{N}}(\lambda;\,1) \\ v^{\mathrm{D}}(\lambda;\,1) \end{pmatrix} = K(\lambda)\begin{pmatrix} v^{\mathrm{N}}(\lambda;\,1) \\ v^{\mathrm{D}}(\lambda;\,1) \end{pmatrix},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}E(\lambda;\,1) = E(\lambda;\,1)K^T(\lambda).$$

For separated the boundary condition, if λ is a zero of F, by equation (52) there exists $k \neq 0$ such that

$$\begin{pmatrix} c & d \end{pmatrix}E(\lambda;\,1) = k\begin{pmatrix} a & b \end{pmatrix},$$

$k \neq 0$, because $(c, d) \neq (0, 0)$ and $E(\lambda;\,1)$ is nonsingular. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda) &= \begin{pmatrix} c & d \end{pmatrix}E(\lambda;\,1)K^T(\lambda)\begin{pmatrix} -b \\ a \end{pmatrix} = k\begin{pmatrix} a & b \end{pmatrix}K^T(\lambda)\begin{pmatrix} -b \\ a \end{pmatrix} \\ &= k\int_0^1 R(t)\Big[au^{\mathrm{D}}(\lambda;\,t) - bu^{\mathrm{N}}(\lambda;\,t)\Big]^2\mathrm{d}t,\end{aligned}$$

which is nonzero since $u^{\mathrm{N}}(\lambda;\,t)$ and $u^{\mathrm{D}}(\lambda;\,t)$ are linearly independent. Therefore λ is a simple zero of F.

For the unseparated boundary condition, if λ is a double eigenvalue, $E(\lambda;\,1) = M$, and so λ is a zero of each element of $E(\lambda;\,1) - M$. By equation (53) λ is a zero of $F(\lambda)$ with multiplicity at least 2. We prove that the multiplicity is indeed 2 by noticing

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}F(\lambda) = \det\frac{\mathrm{d}E(\lambda;\,1)}{\mathrm{d}\lambda} = \det E(\lambda;\,1)\det K^T(\lambda) = \det K^T(\lambda),$$

which is positive by the Cauchy–Schwarz inequality and because $u^{\mathrm{N}}(\lambda;\,t)$ and $u^{\mathrm{D}}(\lambda;\,t)$ are linearly independent. Finally, to prove that a single eigenvalue of L must be a single zero of F, we will show that if λ is a zero of F with multiplicity more than 1, $E(\lambda;\,1) = M$ must hold. Indeed, if λ is a zero of F with multiplicity at least 2

$$\det\left(E(\lambda;\,1) - M\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det\Big(E(\lambda;\,1)\big(I + \epsilon K^T(\lambda)\big) - M\Big)\Big|_{\epsilon=0} = 0.$$

Let $A = I - \mathrm{e}^{-1}(\lambda;\,1)M$, we have

$$\det A = 0, \quad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det\Big(A + \epsilon K^T(\lambda)\Big)\Big|_{\epsilon=0} = 0.$$

Combining $\det A = 0$ with $\det(I - A) = \det M = 1$, we have $\operatorname{tr} A = 0$. Notice that we also have $\operatorname{tr} K^T(\lambda) = 0$. Denoting the elements of A and K by a_{ij} and k_{ij}, we have

$$a_{11}k_{22} + a_{22}k_{11} = 2a_{11}k_{22} = a_{12}k_{12} + a_{21}k_{21}. \tag{55}$$

On the other hand, $\det A = 0$ implies

$$-a_{12}a_{21} = a_{11}^2,$$

and $\det K^T > 0$ implies

$$-k_{12}k_{21} > k_{22}^2.$$

Consequently

$$\left|a_{12}k_{12} + a_{21}k_{21}\right| \geqslant 2\sqrt{a_{12}k_{12}a_{21}k_{21}} \geqslant 2\left|a_{11}k_{22}\right|.$$

Therefore equation (55) only holds when $a_{11} = 0$, which implies $A = 0$, or $E(\lambda; 1) = M$.

References

[1] Bender C and Orszag S 2010 *Advanced Mathematical Methods for Scientists and Engineers: I. Asymptotic Methods and Perturbation Theory* (New York: Springer)

[2] Bordag M, Elizalde E and Kirsten K 1996 Heat kernel coefficients of the Laplace operator on the D-dimensional ball *J. Math. Phys.* **37** 895–916

[3] Bordag M, Kirsten K and Dowker J S 1996 Heat kernels and functional determinants on the generalized cone *Commun. Math. Phys.* **182** 371–94

[4] Bordag M, Mohideen U and Mostepanenko V M 2001 New developments in the Casimir effect *Phys. Rep.* **353** 1–205

[5] Buchbinder I L, Odintsov S D and Shapiro I L 1992 *Effective Action in Quantum Gravity* (Bristol: Hilger)

[6] Bytsenko A A, Cognola G, Vanzo L and Zerbini S 1996 Quantum fields and extended objects in space-times with constant curvature spatial section *Phys. Rep.* **266** 1–126

[7] Dreyfuss T and Dym H 1978 Product formulas for the eigenvalues of a class of boundary value problems *Duke Math. J.* **45** 15–37

[8] Elizalde E 1995 Ten physical applications of spectral zeta functions *Lecture Notes in Physics m35* (Berlin: Springer)

[9] Esposito G, Kamenshchik A Y and Pollifrone G 1997 Euclidean quantum gravity on manifolds with boundary *Fundamental Theories of Physics* vol 85 (Dordrecht: Kluwer)

[10] Forman R 1987 Functional determinants and geometry *Inventory Math.* **88** 447–93
Forman R 1992 Functional determinants and geometry *Inventory Math.* **108** 453–4 (erratum)

[11] Fucci G and Kirsten K 2012 Heat kernel coefficients for laplace operators on the spherical suspension *Commun. Math. Phys.* **314** 483–507

[12] Fucci G and Kirsten K 2013 The spectral zeta function for laplace operators on warped manifolds of the type $I \times_f N$ *Commun. Math. Phys.* **317** 635–65

[13] Gilkey P B 1995 *Invariance Theory, the Heat Equation and the Atiyah–Singer Index Theorem* (Boca Raton, FL: CRC Press)

[14] Gilkey P B, Kirsten K and Vassilevich D V 2001 Heat trace asymptotics with transmittal boundary conditions and quantum brane world scenario *Nucl. Phys.* B **601** 125–48

[15] Jeffres T D, Kirsten K and Lu T 2012 Zeta function on surfaces of revolution *J. Phys. A: Math. Theor.* **45** 345201

[16] Kirsten K 2002 *Spectral Functions in Mathematics and Physics* (London, Boca Raton, FL: Chapman & Hall, CRC Press)

[17] Kirsten K and McKane A J 2004 Functional determinants for general Sturm-Liouville problems *J. Phys. A: Math. Gen.* **37** 4649–70

[18] Levit S and Smilansky U 1977 A theorem on infinite products of eigenvalues of Sturm-Liouville type operators *Proc. Am. Math. Soc.* **65** 299–302

[19] Miller P D 2006 *Applied Asymptotic Analysis* (Providence, RI: American Mathematical Society)

[20] Milton K A 2001 *The Casimir Effect: Physical Manifestations of Zero-Point Energy* (River Edge, USA: World Scientific)

[21] Ray D B and Singer I M 1971 R-torsion and the Laplacian on Riemannian manifolds *Adv. Math.* **7** 145–210

[22] Vassilevich D V 2003 Heat kernel expansion: user's manual *Phys. Rep.* **388** 279–360

[23] Zettl A 2005 *Sturm–Liouville Theory, Mathematical Surveys and Monographs* (Providence, RI: American Mathematical Society)