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Stochastic comparison for elliptically contoured random fields

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ABSTRACT

This paper presents necessary and sufficient conditions for the peakedness comparison and convex ordering between two elliptically contoured random fields about their centers. A somewhat surprising finding is that the peakedness comparison for the infinite dimensional case differs from the finite dimensional case. For example, a Student's t distribution is known to be more heavy-tailed than a normal distribution, but a Student's t random field and a Gaussian random field are not comparable in terms of the peakedness. In particular, the peakedness comparison and convex ordering are made for isotropic elliptically contoured random fields on compact two-point homogeneous spaces.

1. Introduction

The peakedness comparison and convex ordering between two elliptically contoured random fields about their centers are in essence stochastic comparisons in the geometric aspect and the analytic aspect, respectively, as is seen from Wang and Ma (2018), Chen et al. (2021), and in the present paper. An elliptically contoured (or spherically invariant) random field is a scale mixture of Gaussian random fields, and its finite-dimensional distributions are symmetric about the center (Huang and Cambanis, 1979; Ma, 2011). More precisely, $\{Z(x), x \in \mathbb{D}\}$ is called an elliptically contoured random field, if it can be expressed as

$$Z(x) = UZ_0(x) + \mu(x), \qquad x \in \mathbb{D}, \tag{1.1}$$

where $\{Z_0(x), x \in \mathbb{D}\}$ is a Gaussian random field with mean function identical to 0, the mixing random variable U takes merely nonnegative values and is independent with $\{Z_0(x), x \in \mathbb{D}\}$, $\mu(x)$ is a (non-random) function, and $\mathbb{D}$ is a temporal, spatial, or spatio-temporal index domain. When $\mu(x) \equiv 0$, $x \in \mathbb{D}$, $\{Z(x), x \in \mathbb{D}\}$ is said to be centered. Examples of elliptically contoured random fields include, but are not limited to, Gaussian, Student's t, Cauchy, hyperbolic, hyperbolic cosine ratio, hyperbolic sine ratio, hyperbolic secant, Laplace, logistic, variance Gamma, normal inverse Gaussian, K-differenced, K-combined, stable, Linnik, and Mittag-Leffler ones. An elliptically contoured random field may or may not have first-order moments, but its finite-dimensional distributions are symmetric about its center. Among all second-order random fields, the class of second-order elliptically contoured random fields is one of the largest, if not the largest, classes that allow for any given correlation structure.

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For two real-valued random fields $\{Z_k(x), x \in \mathbb{D}\}$ whose finite-dimensional distributions are symmetric about $\mu_k(x)$ ($k = 1, 2$), we say that $\{Z_1(x), x \in \mathbb{D}\}$ is more peaked about $\mu_1(x)$ than $\{Z_2(x), x \in \mathbb{D}\}$ about $\mu_2(x)$ (Wang and Ma, 2018), and denote it by $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$, if

$$\begin{aligned} & P((Z_1(x_1) - \mu_1(x_1), \dots, Z_1(x_n) - \mu_1(x_n))' \in A_n) \\ \geq\, & P((Z_2(x_1) - \mu_2(x_1), \dots, Z_2(x_n) - \mu_2(x_n))' \in A_n) \end{aligned} \tag{1.2}$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ ($k = 1, \dots, n$), and any $A_n \in \mathscr{A}_n$, where $\mathbb{N}$ is the set of positive integers, and $\mathscr{A}_n$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^n$. A particular example of $A_n \in \mathscr{A}_n$ is $A_n = [-z_1, z_1] \times \cdots \times [-z_n, z_n]$, and inequality (1.2) reads

$$\begin{aligned} & P(|Z_1(x_1) - \mu_1(x_1)| \leq z_1, \dots, |Z_1(x_n) - \mu_1(x_n)| \leq z_n) \\ \geq\ & P(|Z_2(x_1) - \mu_2(x_1)| \leq z_1, \dots, |Z_2(x_n) - \mu_2(x_n)| \leq z_n), \qquad z_1, \dots, z_n \geq 0. \end{aligned} \tag{1.3}$$

More specifically, for $n = 1$, (1.3) means that $|Z_1(x) - \mu_1(x)|$ is smaller than $|Z_2(x) - \mu_2(x)|$ in the usual stochastic order. By definition, a random variable Z_1 is said to be smaller than Z_2 in the usual stochastic order (denoted by $Z_1 \preceq_{st} Z_2$), if

$$P(Z_1 > x) \leq P(Z_2 > x), \quad \text{or equivalently,} \quad P(Z_1 \leq x) \geq P(Z_2 \leq x), \qquad x \in \mathbb{R}.$$

A random field $\{Z_1(x), x \in \mathbb{D}\}$ is said to be smaller than another random field $\{Z_2(x), x \in \mathbb{D}\}$ in the convex order, denoted by $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$, if the inequality

$$\mathrm{E}g(Z_1(x_1), \dots, Z_1(x_n)) \leq \mathrm{E}g(Z_2(x_1), \dots, Z_2(x_n)) \tag{1.4}$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ ($k = 1, 2, \dots, n$), and any convex function $g(\mathbf{z})$ such that the expected values on both sides of (1.4) exist.

Two pairs of elliptically contoured random fields are compared in Sections 2 and 3, with respect to the peakedness and convex orderings, respectively, with necessary and sufficient conditions derived. In one pair, two elliptically contoured random fields share the same mixing random variable U but have different Gaussian random fields in (1.1). In the other pair, two elliptically contoured random fields share the same Gaussian random field but have different mixing random variables. The peakedness comparison and convex ordering are conducted in Section 4 for isotropic elliptically contoured random fields on compact two-point homogeneous spaces. For the investigation on isotropic random fields on $\mathbb{M}^d$, we refer the reader to Gangolli (1967), Askey and Bingham (1976), Lu and Ma (2020), Ma and Malyarenko (2020), and Lu et al. (2021). Proofs of theorems are given in Section 5.

2. Peakedness comparison

By definition, the peakedness comparison is the comparison between finite-dimensional distributions only. To address our findings neater in this section, however, we simply make the statements in terms of stochastic representation pairs. The pair of elliptically contoured random fields in Theorem 2.1 share the same Gaussian random field but have distinct mixing random variables. The finding in Theorem 2.1 or Corollary 2.1.1 is somewhat surprising, and reveals a fact that the peakedness comparison for the infinite dimensional case differs from the finite dimensional case. Another pair of elliptically contoured random fields in Theorem 2.2 share the same mixing random variable but have different Gaussian random field, which includes Theorem 5 (i) of Wang and Ma (2018) as a special case where the moment existence is assumed for the mixing variable.

By a positive random variable U, we mean $P(U < 0) = 0$ and $P(U = 0) < 1$. In this paper "positive definite" and "nonnegative definite" are synonyms, and in a strict sense, the term "strictly positive definite" is adopted.

Theorem 2.1. *Let two elliptically contoured random fields be defined by*

$$Z_k(x) = U_k Z_0(x) + \mu_k(x), \qquad x \in \mathbb{D}, \quad k = 1, 2,$$

where U_k ($k = 1, 2$) are two positive random variables and are independent of a zero-mean Gaussian random field $\{Z_0(x), x \in \mathbb{D}\}$ whose covariance function is $C(x_1, x_2)$. Under the assumptions that $C(x_1, x_2)$ is strictly positive definite, a necessary and sufficient condition for $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ is $U_1 \preceq_{st} U_2$.

In particular, if $P(U_2 = 1) = 1$, then $\{Z_2(x), x \in \mathbb{D}\}$ reduces to a Gaussian random field, and Theorem 2.1 results in the following corollary.

Corollary 2.1.1. *For a Gaussian random field $\{Z_0(x), x \in \mathbb{D}\}$ and an elliptically contoured random field $\{UZ_0(x), x \in \mathbb{D}\}$, $\{UZ_0(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_0(x), x \in \mathbb{D}\}$ if and only if $P(U > 1) = 0$.*

Consider a Student's t random field $\{Z(x), x \in \mathbb{D}\}$ (Ma, 2013; Røislien and Omre, 2006) defined by

$$Z(x) = c_0 V^{\frac{1}{2}} Z_0(x), \qquad x \in \mathbb{D},$$

where $c_0 = I_{[0,1)}(\lambda) + \left(\frac{\lambda-1}{\beta}\right)^{\frac{1}{2}} I_{(1,\infty)}(\lambda)$, β are λ are positive constants, V is an inverse Gamma random variable with density function $\frac{\beta^\lambda}{\Gamma(\lambda)} u^{-\lambda-1} \exp\left(-\frac{\beta}{u}\right) I_{(0,\infty)}(u)$, and $I_A(x)$ denotes the indicator function of a set A. By Corollary 2.1.1, this Student's t random field is not comparable in the peakedness with the Gaussian random field $\{Z_0(x), x \in \mathbb{D}\}$. In contrast, a Student's t distribution is known to be more heavy-tailed than a normal distribution. As another application of Theorem 2.1, two Student's t random fields are not comparable in the peakedness, which agrees with the finite dimensional cases observed by Dunn (1965) that the univariate Student's t distribution of a higher degree of freedom is more peaked, while the order of peakedness is reversed in sufficiently high dimensions.

A necessary and sufficient condition is presented in Theorem 2.2, for the peakedness comparison between two elliptically contoured random fields who share the same mixing random variable U and have distinct Gaussian random fields $\{Y_k(x), x \in \mathbb{D}\}$ $(k = 1, 2)$ as ingredients of format (1.1). It contains Theorem 5 (i) of Wang and Ma (2018) as a special case, where an additional moment assumption is made on the mixing variable.

Theorem 2.2. *Suppose that two elliptically contoured random fields are defined by*

$$Z_k(x) = UY_k(x) + \mu_k(x), \qquad x \in \mathbb{D}, \quad k = 1, 2,$$

where U is a positive random variable and is independent of the Gaussian random field $\{Y_k(x), x \in \mathbb{D}\}$ whose mean function is identical to 0 and whose covariance function is $C_k(x_1, x_2)$. Then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ if and only if $C_2(x_1, x_2) - C_1(x_1, x_2)$ is a positive definite function on $\mathbb{D}$.

Corollary 2.2.1. *$\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ and $\{Z_2(x) - \mu_2(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_1(x) - \mu_1(x), x \in \mathbb{D}\}$ if and only if $C_2(x_1, x_2) \equiv C_1(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$.*

Example 2.3. Let $\mathbb{D}$ be one of the three metric spaces, $\mathbb{R}^d$, a unit sphere $\mathbb{S}^d = \left\{\mathbf{x} \in \mathbb{R}^{d+1} : \ \|\mathbf{x}\| = 1\right\}$, and a hyperbolic space $\mathbb{H}^d$, and denote by $\rho(\mathbf{x}_1, \mathbf{x}_2)$ the metric over $\mathbb{D}$. Since $\rho(\mathbf{x}_1, \mathbf{x}_2)$ is conditionally negative definite (Gangolli, 1967) over $\mathbb{D}$, there exists a fractional Brownian motion $\{Y_1(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ with covariance function (Istas, 2005; Cohen and Lifshits, 2012; Ma, 2015)

$$C(\mathbf{x}_1, \mathbf{x}_2) = \rho^\nu(\mathbf{x}_1, \mathbf{x}_0) + \rho^\nu(\mathbf{x}_2, \mathbf{x}_0) - \rho^\nu(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}, \tag{2.1}$$

where the legitimate range of the parameter ν varies with respect to $\mathbb{D}$, $\nu \in (0, 2]$ if $\mathbb{D} = \mathbb{R}^d$, $\nu \in (0, 1]$ for $\mathbb{D} = \mathbb{S}^d$, $\nu \in (0, 1]$ for $\mathbb{D} = \mathbb{H}^d$, and $\mathbf{x}_0 \in \mathbb{D}$ is a fixed point. Given a positive random variable U with Laplace transform $\ell_U(\omega) = \mathrm{E}\exp(-U\omega)$, $\omega \ge 0$, an elliptically contoured random field

$$Z_1(\mathbf{x}) = \sqrt{U} Y_1(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{D},$$

has stationary increments, in the sense that the distribution of every increment $\frac{Z_1(\mathbf{x}_1) - Z_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}$ does not depend on either $\mathbf{x}_1$ or $\mathbf{x}_2$, for distinct $\mathbf{x}_1$ and $\mathbf{x}_2$. In fact, the characteristic function of $\frac{Z_1(\mathbf{x}_1) - Z_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}$ is

$$\begin{aligned}
\mathrm{E}\exp\left(\imath\omega \frac{Z_1(\mathbf{x}_1) - Z_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}\right) &= \mathrm{E}\exp\left(\imath\omega\sqrt{U} \frac{Y_1(\mathbf{x}_1) - Y_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}\right) \\
&= \int_0^\infty \mathrm{E}\exp\left(\imath\omega\sqrt{u} \frac{Y_1(\mathbf{x}_1) - Y_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}\right) dF_U(u) \\
&= \int_0^\infty \exp\left(-\frac{\omega^2 u}{2} \operatorname{var}\left(\frac{Y_1(\mathbf{x}_1) - Y_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}\right)\right) dF_U(u) \\
&= \ell_U\left(\omega^2\right), \qquad \omega \in \mathbb{R},
\end{aligned}$$

where $\imath$ is the imaginary unit, and $F_U(u)$ denotes U's distribution function. In particular, when U is a positive stable random variable with Laplace transform $\mathrm{E}\exp(-\omega U) = \exp(-\omega^\kappa)$, $\omega \ge 0$, $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ is a stable random field, and $\frac{Z_1(\mathbf{x}_1) - Z_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}$ is a stable random variable with characteristic function

$$\mathrm{E}\exp\left(\imath\omega \frac{Z_1(\mathbf{x}_1) - Z_1(\mathbf{x}_2)}{\rho^{\frac{\nu}{2}}(\mathbf{x}_1, \mathbf{x}_2)}\right) = \exp\left(-|\omega|^{2\kappa}\right), \qquad \omega \in \mathbb{R},$$

where $0 < \kappa \le 1$. If $0 < \kappa \le \frac{1}{2}$, then $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$ does not have a finite first-order moment. It reduces to a Gaussian random field in case U is a degenerate random variable with $P(U = 1) = 1$, and reduces to a Cauchy random field when $\kappa = \frac{1}{2}$.

Over $\mathbb{D} = \mathbb{S}^d$, $\rho(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1'\mathbf{x}_2)$ is the spherical (angular, or geodesic) distance between $\mathbf{x}_1$ and $\mathbf{x}_2$ on the largest circle on $\mathbb{S}^d$ that passes through them, where $\mathbf{x}_1'\mathbf{x}_2$ is the inner product between $\mathbf{x}_1$ and $\mathbf{x}_2$. For $\nu \in (0, 1]$, a bifractional Brownian motion $\{Y_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ is a Gaussian random field with covariance function

$$C(\mathbf{x}_1, \mathbf{x}_2) = (\rho(\mathbf{x}_1, \mathbf{x}_0) + \rho(\mathbf{x}_2, \mathbf{x}_0))^\nu - \rho^\nu(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d, \tag{2.2}$$

and a trifractional Brownian motion $\{Y_3(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ is a Gaussian random field (Ma, 2015) with covariance function

$$C(\mathbf{x}_1, \mathbf{x}_2) = \rho^\nu(\mathbf{x}_1, \mathbf{x}_0) + \rho^\nu(\mathbf{x}_2, \mathbf{x}_0) - (\rho(\mathbf{x}_1, \mathbf{x}_0) + \rho(\mathbf{x}_2, \mathbf{x}_0))^\nu, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d. \tag{2.3}$$

Since (2.1) is a sum of (2.2) and (2.3), it follows from Theorem 2.1 that

$$\{\sqrt{U} Y_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\} \overset{p}{\succeq} \{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}, \quad \text{and} \quad \{\sqrt{U} Y_3(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\} \overset{p}{\succeq} \{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}.$$

A similar peakedness comparison can be made over $\mathbb{H}^d$, after verifying both (2.2) and (2.3) are positive definite on $\mathbb{H}^d$ via Theorems 2 and 3 of Ma (2015).

As the definition suggested, the peakedness is compared about the finite-dimensional distributions of two elliptically contoured random fields around their centers, which may be or may be not on the same probability space. The following theorem makes the peakedness comparison when they are defined on the same probability space, and, significantly, its novel part is that neither the mixing random variables nor the associated Gaussian random fields are specified.

Theorem 2.4. *Suppose that $\{Z_1(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\}$ are two centered elliptically contoured random fields on the same probability space. If they are independent each other, then*

$$\{Z_k(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_1(x) + Z_2(x), x \in \mathbb{D}\}, \qquad k = 1, 2.$$

3. Convex ordering

Two pairs of elliptically contoured random fields are compared in terms of convex ordering in this section. As is conjectured in Wang and Ma (2018), the existence of the expected value of the mixing random variable is crucial for convex ordering between two elliptically contoured random fields who share the same mixing random variable, with a necessary and sufficient condition given in Theorem 5 (ii) of Wang and Ma (2018) under the assumption of the mixing random variable U that $\mathrm{E}U^\tau < \infty$ for a constant $\tau \geq 1$, which is questioned whether such moment condition could be dropped there. Interestingly, the first-order moment assumption is actually sharp for the convex ordering, as the following theorem states.

Theorem 3.1. *Two elliptically contoured random fields $\{Z_k(x), x \in \mathbb{D}\}$ are defined by*

$$Z_k(x) = U Y_k(x), \qquad x \in \mathbb{D}, \quad k = 1, 2,$$

where U is a positive random variable and is independent with the Gaussian random field $\{Y_k(x), x \in \mathbb{D}\}$ whose mean function is identical to 0 and whose covariance function is $C_k(x_1, x_2)$.

(i) *Under the assumption that $\mathrm{E}U < \infty$, $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$ if and only if $C_2(x_1, x_2) - C_1(x_1, x_2)$ is positive definite.*

(ii) *If $\mathrm{E}U = \infty$, then $\mathrm{E}g(Z_k(x_1), \ldots, Z_k(x_n)) = g(\mathbf{0})$ whenever it exists, $k = 1, 2$.*

The conclusion of Theorem 3.1(ii) is somewhat beyond our expectations, since it states that two elliptically contoured random fields are "equal" in the sense of the convex order, whenever $\mathrm{E}U = \infty$; in other words, it makes no sense to compare them via the convex order.

Corollary 3.1.1. *Under the assumption that $\mathrm{E}U < \infty$, $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_1(x), x \in \mathbb{D}\}$ if and only if $C_2(x_1, x_2) \equiv C_1(x_1, x_2)$, $x_1, x_2 \in \mathbb{D}$, or equivalently, $\{Z_1(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\}$ has the same finite-dimensional distributions.*

For two random variables U_1 and U_2, U_1 is said to be smaller than U_2 in the increasing convex, denoted by $U_1 \preceq_{icx} U_2$, if

$$\mathrm{E}g(U_1) \leq \mathrm{E}g(U_2)$$

holds for all increasing convex functions $g(x)$ on $\mathbb{R}$, provided that the expectations exist. The last inequality is equivalent to

$$\mathrm{E}(U_1 - x)_+ \leq \mathrm{E}(U_2 - x)_+, \qquad x \in \mathbb{R},$$

where $(x)_+ = \max(x, 0)$, $x \in \mathbb{R}$; see, for example, Section 4.A of Shaked and Shanthikumar (2007).

Theorem 3.2. *Let two elliptically contoured random fields $\{Z_k(x), x \in \mathbb{D}\}$ defined by*

$$Z_k(x) = U_k Y(x), \qquad x \in \mathbb{D}, \quad k = 1, 2,$$

where U_k $(k = 1, 2)$ are positive random variables and are independent with the Gaussian random field $\{Y(x), x \in \mathbb{D}\}$ whose mean function is identical to 0 and whose covariance function is $C(x_1, x_2)$. Under the assumptions that $C(x_1, x_2)$ is strictly positive definite, Then $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$ if and only if either $\mathrm{E}U_1$ or $\mathrm{E}U_2$ does not exist, or $U_1 \preceq_{icx} U_2$.

The following corollary follows as a consequence of Theorems 2.1 and 3.2, since $U_1 \preceq_{st} U_2$ implies $U_1 \preceq_{icx} U_2$.

Corollary 3.2.1. *If* $\{Z_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x), x \in \mathbb{D}\}$, *then* $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$.

4. Peakedness and convex ordering on compact two-point homogeneous spaces

In this section we apply the results obtained in the last two sections to compare elliptically contoured random fields on a d-dimensional compact two-point homogeneous space $\mathbb{M}^d$, which is a compact Riemannian symmetric space of rank one and belongs to one of the five families described in Wang (1952). The distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$ between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\mathbb{M}^d$ is defined in such a way (Ma and Malyarenko, 2020) that the length of any geodesic line on all $\mathbb{M}^d$ is equal to 2π, or the distance between any two points is bounded between 0 and π, i.e., $0 \le \rho(\mathbf{x}_1, \mathbf{x}_2) \le \pi$. Over the unit sphere $\mathbb{S}^d$, for instance, $\rho(\mathbf{x}_1, \mathbf{x}_2)$ is defined by $\rho(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1'\mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d$.

A second-order random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is called a stationary (homogeneous) and isotropic random field, if its mean function $\mathrm{E}Z(\mathbf{x})$ does not depend on $\mathbf{x}$, and its covariance function $\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2))$ depends only on the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$ between $\mathbf{x}_1$ and $\mathbf{x}_2$. Such a covariance function is denoted by $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d$, and is called an isotropic covariance function on $\mathbb{M}^d$. An isotropic random field on $\mathbb{S}^d$ enjoys a simple orthogonal decomposition (4.1), due to the facts that $\mathbf{x} \in \mathbb{S}^d$ if and only if $-\mathbf{x} \in \mathbb{S}^d$ and

$$\rho(-\mathbf{x}_1, \mathbf{x}_2) = \pi - \rho(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d.$$

Similar properties seem not to hold on other compact two-point homogeneous spaces.

Theorem 4.1. *For an isotropic random field* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *with covariance function* $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$, *define*

$$Z_1(\mathbf{x}) = \frac{Z(\mathbf{x}) + Z(-\mathbf{x})}{2}, \quad \text{and} \quad Z_2(\mathbf{x}) = \frac{Z(\mathbf{x}) - Z(-\mathbf{x})}{2}, \qquad \mathbf{x} \in \mathbb{S}^d.$$

(i) $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *possesses an orthogonal decomposition*

$$Z(x) = Z_1(\mathbf{x}) + Z_2(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d, \tag{4.1}$$

in the sense that $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *and* $\{Z_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *are uncorrelated, i.e.,*

$$\mathrm{cov}(Z_1(\mathbf{x}_1), Z_2(\mathbf{x}_2)) = 0, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d.$$

(ii) *The covariance function of* $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *is* $\frac{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) + C(\pi - \rho(\mathbf{x}_1,\mathbf{x}_2))}{2}$.

(iii) *The covariance function of* $\{Z_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *is* $\frac{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) - C(\pi - \rho(\mathbf{x}_1,\mathbf{x}_2))}{2}$.

(iv) *As an additional assumption, suppose that* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ *is an isotropic elliptically contoured random field, and*

$$Z(\mathbf{x}) = UY(\mathbf{x}) + \mu(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{S}^d,$$

where U *is a second-order positive random variable and is independent with a zero-mean Gaussian random field* $\{Y(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$. *Then*
$\{Z_k(\mathbf{x}) - \mathrm{E}Z_k(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\} \overset{p}{\succeq} \{Z(\mathbf{x}) - \mathrm{E}Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$, $k = 1, 2$.

(v) *Under the assumption of Part* (iv),
$\{Z_k(\mathbf{x}) - \mathrm{E}Z_k(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\} \preceq_{cx} \{Z(\mathbf{x}) - \mathrm{E}Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$, $k = 1, 2$.

According to Theorem 2.1 of Schoenberg (1942), the covariance function $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$ of an isotropic and mean square continuous random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ takes the form

$$C(\rho(\mathbf{x}_1, \mathbf{x}_2)) = \sum_{n=0}^{\infty} b_n P_n^{\left(\frac{d-1}{2}\right)}(\cos \rho(\mathbf{x}_1, \mathbf{x}_2)), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d,$$

where $\{b_n, n \in \mathbb{N}_0\}$ is a sequence of nonnegative constants, the series $\sum_{n=0}^{\infty} b_n P_n^{\left(\frac{d-1}{2}\right)}(1)$ converges, and $P_n^{\left(\frac{d-1}{2}\right)}(x)$ $(n \in \mathbb{N}_0)$ are ultraspherical or Gegenbauer's polynomials (Szegö, 1975). They are special cases of Jacobi polynomials, whose exact expressions are

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha + n + 1)}{n!\Gamma(\alpha + \beta + n + 1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha + \beta + n + k + 1)}{\Gamma(\alpha + k + 1)} \left(\frac{x-1}{2}\right)^k, \ x \in \mathbb{R}, n \in \mathbb{N}_0.$$

Associated with the orthogonal decomposition (4.1), $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ possesses the covariance function

$$\mathrm{cov}(Z_1(\mathbf{x}_1), Z_1(\mathbf{x}_2)) = \textstyle\sum_{n=0}^{\infty} b_{2n} P_{2n}^{\left(\frac{d-1}{2}\right)}(\cos \rho(\mathbf{x}_1, \mathbf{x}_2)), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d,$$

and $\{Z_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ possesses the covariance function

$$\mathrm{cov}(Z_2(\mathbf{x}_1), Z_2(\mathbf{x}_2)) = \sum_{n=0}^{\infty} b_{2n+1} P_{2n+1}^{\left(\frac{d-1}{2}\right)}(\cos\rho(\mathbf{x}_1, \mathbf{x}_2)), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d,$$

noticing that $P_n^{\left(\frac{d-1}{2}\right)}(x)$ is an even function on $\mathbb{R}$ if n is even, and is an odd function if n is odd.

Theorem 4.2. *Suppose that two second-order elliptically contoured random fields are defined by*

$$Z_k(\mathbf{x}) = UY_k(\mathbf{x}) + \mu_k(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{M}^d, \quad k = 1, 2,$$

where U is a positive random variable and is independent with the Gaussian random field $\{Y_k(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ whose mean function is identical to 0 and whose covariance function is

$$C_k(\rho(\mathbf{x}_1, \mathbf{x}_2)) = \sum_{n=0}^{\infty} b_n^{(k)} P_n^{(\alpha,\beta)}\left(\cos\rho(\mathbf{x}_1, \mathbf{x}_2)\right), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d. \tag{4.2}$$

Then $\{Z_1(\mathbf{x}) - \mu_1(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\} \overset{p}{\succeq} \{Z_2(\mathbf{x}) - \mu_2(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ if and only if the inequality $b_n^{(2)} \geq b_n^{(1)}$ holds for each $n \in \mathbb{N}_0$.

Corollary 4.2.1. *If $\{Z_1(\mathbf{x}) - \mu_1(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\} \overset{p}{\succeq} \{Z_2(\mathbf{x}) - \mu_2(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$, then the inequality*

$$\int_0^{\pi} (C_2(\vartheta) - C_1(\vartheta)) \cos(n\vartheta) d\vartheta \geq 0 \tag{4.3}$$

holds for every $n \in \mathbb{N}_0$.

Theorem 4.3. *In addition to the assumption of Theorem 4.2, let $\mathrm{E}U < \infty$. Then $\{Z_1(\mathbf{x}) - \mu_1(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\} \preceq_{cx} \{Z_2(\mathbf{x}) - \mu_2(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ if and only if the inequality $b_n^{(2)} \geq b_n^{(1)}$ holds for each $n \in \mathbb{N}_0$.*

5. Proofs

In the proof of Theorems 2.1 and 3.2, we need the following lemma, which is of interest in its own right. For an $n \times n$ nonsingular matrix Σ, denote an ellipsoid by

$$\mathbb{B}_\alpha = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}'\Sigma^{-1}\mathbf{x} \leq \alpha^2\}, \qquad \alpha \geq 0,$$

and write $x\mathbb{B}_\alpha = \mathbb{B}_{x\alpha}, x \geq 0$.

Lemma 5.1. *If $\mathbf{Y}$ is an n-variate normal random vector with mean $\mathbf{0}$ and a nonsingular variance–covariance matrix Σ, then*

(i) *$\mathbf{Y}$ has the same distribution as $W\mathbf{U}$, where $\mathbf{U}$ is an n-variate random vector uniformly distributed on a unit ellipsoid $\mathbb{B}_1$, and is independent with a positive random variable W that has a density function*

$$f_W(w) = \frac{w^{n+1}}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)} \exp\left(-\frac{w^2}{2}\right) I_{(0,\infty)}(w);$$

(ii) *the probability of $\mathbf{Y}$ over an ellipsoid $\mathbb{B}_x$ is*

$$P\left(\mathbf{Y} \in \mathbb{B}_x\right) = P\left(W_n^2 \leq x^2\right), \qquad x > 0, \tag{5.1}$$

where W_n^2 is a χ^2 random variable with n degrees of freedom;

(iii)

$$\lim_{n\to\infty} P\left(u\mathbf{Y} \in \mathbb{B}_{\sqrt{n}x}\right) = I_{[0,x]}(u), \qquad x > 0, u \geq 0, x \neq u; \tag{5.2}$$

(iv)

$$\lim_{n\to\infty} \int_x^{\infty} P\left(u\mathbf{Y} \notin \mathbb{B}_{\sqrt{n}y}\right) dy = (u - x)_+, \qquad x > 0, u \geq 0. \tag{5.3}$$

5.1. Proof of Theorem 2.1

The sufficiency is proved in Wang and Ma (2018). To verify $U_1 \preceq_{st} U_2$ is a necessary condition, let $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$. It implies that, for an arbitrary $n \in \mathbb{N}$,

$$(U_1 Z_{10}(x_1), \ldots, U_1 Z_{n0}(x_n))' \overset{p}{\succeq} (U_2 Z_{10}(x_1), \ldots, U_2 Z_{n0}(x_n))', \quad x_k \in \mathbb{D}, k = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are assumed to be distinct so that the variance matrix Σ of the normal random vector $(Z_{10}(x_1), \ldots, Z_{n0}(x_n))'$ is nonsingular. We have

$$P((U_1 Z_{10}(x_1), \ldots, U_1 Z_{n0}(x_n))' \in \mathbb{B}_{\sqrt{n}x}) \geq P((U_2 Z_{10}(x_1), \ldots, U_2 Z_{n0}(x_n))' \in \mathbb{B}_{\sqrt{n}x}), \quad x \geq 0,$$

and

$$\begin{aligned} & \int_0^\infty P(u(Z_{10}(x_1), \ldots, Z_{n0}(x_n))' \in \mathbb{B}_{\sqrt{n}x}) dF_{U_1}(u) \\ \geq \; & \int_0^\infty P(u(Z_{10}(x_1), \ldots, Z_{n0}(x_n))' \in \mathbb{B}_{\sqrt{n}x}) dF_{U_2}(u), \quad x \geq 0. \end{aligned}$$

It follows from the proof of Lemma 5.1 (iii) and the dominated convergence theorem that $P(U_1 < x) + P(U_1 = x)/2 \geq P(U_2 < x) + P(U_2 = x)/2$, as $n \to \infty$. Finally we have $U_1 \preceq_{st} U_2$ by noting that $P(U_k \leq x) = \inf_{y > x}(P(U_k < y) + P(U_k = y)/2)$, $k = 1, 2$.

5.2. Proof of Theorem 2.2

It suffices to prove the "only if" part, while the "if" part is established in Wang and Ma (2018).

Suppose that $\{Z_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x), x \in \mathbb{D}\}$, but $C_2(x_1, x_2) - C_1(x_1, x_2)$ is not positive definite. Then there exists n points $x_k \in \mathbb{D}$ and $a_k \in \mathbb{R}$ $(k = 1, \ldots, n)$ such that

$$\sum_{i=1}^n \sum_{j=1}^n a_i [C_2(x_i, x_j) - C_1(x_i, x_j)] a_j < 0.$$

In other words, $\text{var}(Y_{2n}) < \text{var}(Y_{1n})$, where Y_{kn} is a normal random variable with mean 0 and

$$Y_{kn} = \sum_{i=1}^n a_i (Y_k(x_i) - \mu_k(x_i)), \qquad k = 1, 2.$$

It implies that $Y_{2n} \overset{p}{\succeq} Y_{1n}$, and, moreover, $UY_{2n} \overset{p}{\succeq} UY_{1n}$, which contradicts the assumption that $\{Z_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x), x \in \mathbb{D}\}$.

5.3. Proof of Theorem 2.4

Suppose that $Z_1(x) = UY_1(x), x \in \mathbb{D}$, where U is a positive random variable and is independent with a zero-mean Gaussian random field $\{Y_1(x), x \in \mathbb{D}\}$. For every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, and any $A_n \in \mathscr{A}_n$, we have

$$P((Y_1(x_1), \ldots, Y_1(x_n))' + \mathbf{y}_n \in A_n) \leq P((Y_1(x_1), \ldots, Y_1(x_n))' \in A_n),$$

for an arbitrary $\mathbf{y}_n \in \mathbb{R}^n$, according to Theorem 4.2.4 of Tong (1990). Furthermore, it follows from the independent assumption between $\{Z_1(x), x \in \mathbb{D}\}$ and $\{Z_2(x), x \in \mathbb{D}\}$ that

$$\begin{aligned} & P((Z_1(x_1) + Z_2(x_1), \ldots, Z_1(x_n) + Z_2(x_n))' \in A_n) \\ = \; & \int_{\mathbf{y}_n \in \mathbb{R}^n} P((Z_1(x_1), \ldots, Z_1(x_n))' + \mathbf{y}_n \in A_n) dF_{(Z_2(x_1), \ldots, Z_2(x_n))'}(\mathbf{y}_n) \\ = \; & \int_{\mathbf{y}_n \in \mathbb{R}^n} P(U_1(Y_1(x_1), \ldots, Y_1(x_n))' + \mathbf{y}_n \in A_n) dF_{(Z_2(x_1), \ldots, Z_2(x_n))'}(\mathbf{y}_n) \\ = \; & \int_0^\infty \int_{\mathbf{y}_n \in \mathbb{R}^n} P(u_1(Y_1(x_1), \ldots, Y_1(x_n))' + \mathbf{y}_n \in A_n) dF_{(Z_2(x_1), \ldots, Z_2(x_n))'}(\mathbf{y}_n) dF_{U_1}(u_1) \\ \leq \; & \int_0^\infty \int_{\mathbf{y}_n \in \mathbb{R}^n} P(u_1(Y_1(x_1), \ldots, Y_1(x_n))' \in A_n) dF_{(Z_2(x_1), \ldots, Z_2(x_n))'}(\mathbf{y}_n) dF_{U_1}(u_1) \\ = \; & \int_0^\infty P(u_1(Y_1(x_1), \ldots, Y_1(x_n))' \in A_n) dF_{U_1}(u_1) \\ = \; & P((Z_1(x_1), \ldots, Z_1(x_n))' \in A_n); \end{aligned}$$

that is, $\{Z_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_1(x) + Z_2(x), x \in \mathbb{D}\}$.

5.4. Proof of Theorem 3.1

Part (i) is proved in Wang and Ma (2018). For Part (ii), denote by Σ the variance–covariance matrix of an n-variate normal random vector $(Y_1(x_1), \ldots, Y_1(x_n))'$, where $x_k \in \mathbb{D}, k = 1, \ldots, n$. Let the rank of Σ be $r = rank(\Sigma) \geq 1$, and write its eigenvalue decomposition as

$$\Sigma = \mathbf{Q}'\mathbf{D}\mathbf{Q},$$

where $\mathbf{Q}$ is an $n \times n$ orthogonal matrix, and $\mathbf{D}$ is an $n \times n$ diagonal matrix whose diagonal entries are Σ's eigenvalues in the descending order, $\lambda_1 \geq \cdots \geq \lambda_r > 0$, $\lambda_{r+1} = \cdots = 0$.

Given a convex function $g(\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^n$, define $h(\mathbf{z}) = g(\mathbf{Q}'\mathbf{z})$, $\mathbf{z} \in \mathbb{R}^n$. Clearly, the convexity of $g(\mathbf{z})$ over $\mathbb{R}^n$ implies that of $h(\mathbf{z})$.

Let $(W_1, \ldots, W_n)' = \mathbf{Q}(Y_1(x_1), \ldots, Y_1(x_n))'$. It is an n-variate normal random vector with mean $\mathbf{0}$ and variance matrix $\mathbf{D}$, and

$$\begin{aligned} \mathrm{E}g(Z_1(x_1), \ldots, Z_1(x_n)) &= \mathrm{E}g(U(Y_1(x_1), \ldots, Y_1(x_n))) \\ &= \mathrm{E}h(U(W_1, \ldots, W_n)) \\ &= \mathrm{E}h(U(W_1, \ldots, W_r, 0, \ldots, 0)), \end{aligned}$$

where the last equality follows since $(W_1, \ldots, W_n)'$ and $(W_1, \ldots, W_r, 0, \ldots, 0)$ have the same distribution.

If the function $h(\mathbf{z})$ is identical to a constant, then this constant has to be $h(\mathbf{0})$ and is the same as $g(\mathbf{0})$. As a result, we have

$$\mathrm{E}g(Z_1(x_1), \ldots, Z_1(x_n)) = h(\mathbf{0}) = g(\mathbf{0}).$$

Now suppose that the function $h(\mathbf{z})$ is not identical to $h(\mathbf{0})$. Then, by its convexity, there exists a nonzero $z_{0k} \in \mathbb{R}$ $(k = 1, \ldots, r)$ such that $h(z_{01}, \ldots, z_{0r}, 0, \ldots, 0) > h(\mathbf{0})$, and, by the Geometric Hahn–Banach Theorem (Rudin (1991)), there exists a linear function $l(z_1, \ldots, z_r) = a_0 + \sum_{i=1}^r a_i z_i$ such that

$$l(z_{01}, \ldots, z_{0r}) = h(z_{01}, \ldots, z_{0r}, 0, \ldots, 0), \quad \text{and} \quad h(z_{01}, \ldots, z_{0r}, 0, \ldots, 0) \geq l(z_1, \ldots, z_r).$$

Here the vector $\mathbf{a} = (a_1, \ldots, a_r)'$ is nonzero, since $h(z_{01}, \ldots, z_{0r}, 0, \ldots, 0) > h(\mathbf{0})$. Denoting $(W_1, \ldots, W_r)$ by $\mathbf{W}$ and its distribution function by $F_{\mathbf{W}}(\mathbf{w})$, we obtain

$$\begin{aligned} \mathrm{E}g(Z_1(x_1), \ldots, Z_1(x_n)) &= \mathrm{E}h(U(W_1, \ldots, W_r, 0, \ldots, 0)) \\ &\geq \int \int_{\mathbf{a}'\mathbf{w}u + a_0 \geq 0} (\mathbf{a}'\mathbf{w} + a_0) dF_U(u) dF_{\mathbf{W}}(\mathbf{w}) \\ &\geq \int_{u \geq 1} \int_{\mathbf{a}'\mathbf{w} \geq \max(-a_0, 0)} \mathbf{a}'\mathbf{w} u dF_U(u) dF_{\mathbf{W}}(\mathbf{w}) + \min(a_0, 0) \\ &= \int_{u \geq 1} u dF_U(u) \int_{\mathbf{a}'\mathbf{w} \geq \max(-a_0, 0)} \mathbf{a}'\mathbf{w} dF_{\mathbf{W}}(\mathbf{w}) + \min(a_0, 0), \end{aligned}$$

which diverges provided that $\mathrm{E}U = \infty$.

5.5. Proof of Theorem 3.2

For an arbitrary $n \in \mathbb{N}$, if $g(\mathbf{x})$ is a convex function in $\mathbb{R}^n$, then $\mathrm{E}g(uY(x_1), \ldots, uY(x_n))$ is a convex function of $u \in \mathbb{R}$ for $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, whenever the expectation exists. Observing that

$$\mathrm{E}g(-uY(x_1), \ldots, -uY(x_n)) = \mathrm{E}g(uY(x_1), \ldots, uY(x_n)), \qquad u \in \mathbb{R},$$

$\mathrm{E}g(uY(x_1), \ldots, uY(x_n))$ is also an increasing function of $u \in [0, \infty)$. If $U_1 \preceq_{icx} U_2$, then

$$\mathrm{E}g(U_1 Y(x_1), \ldots, U_1 Y(x_n)) \leq \mathrm{E}g(U_2 Y(x_1), \ldots, U_2 Y(x_n)),$$

which implies $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), x \in \mathbb{D}\}$.

Conversely, suppose that $\{Z_1(x), x \in \mathbb{D}\} \preceq_{cx} \{Z_2(x), \mathbf{x} \in \mathbb{D}\}$. For an arbitrary $n \in \mathbb{N}$ and distinct $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, the variance matrix Σ of $(Y(x_1), \ldots, Y(x_n))$ is nonsingular. For nonnegative constants α and β, write

$$g_{\alpha,\beta}(\mathbf{y}) = (\inf\{r \geq 0 : \mathbf{y} \in r\mathbb{B}_\alpha\} - \beta)_+, \quad \mathbf{y} \in \mathbb{R}^n.$$

It is easy to verify that $g_{\alpha,\beta}(\mathbf{y})$ is a convex function in $\mathbb{R}^n$. Thus,

$$\mathrm{E}g_{\alpha,\beta}(U_1 Y(x_1), \ldots, U_1 Y(x_n)) \leq \mathrm{E}g_{\alpha,\beta}(U_2 Y(x_1), \ldots, U_2 Y(x_n)),$$

or

$$\int_0^\infty \mathrm{E}g_{\alpha,\beta}(uY(x_1), \ldots, uY(x_n)) dF_{U_1}(u) \leq \int_0^\infty \mathrm{E}g_{\alpha,\beta}(uY(x_1), \ldots, uY(x_n)) dF_{U_2}(u).$$

In particular, taking $\alpha = \sqrt{n}x$ and $\beta = x \geq 0$ yields

$$\begin{aligned} & \int_0^\infty \int_x^\infty P(u(Y(x_1), \ldots, Y(x_n))' \notin \mathbb{B}_{\sqrt{n}y}) dy dF_{U_1}(u) \\ \leq\ & \int_0^\infty \int_x^\infty P(u(Y(x_1), \ldots, Y(x_n))' \notin \mathbb{B}_{\sqrt{n}y}) dy dF_{U_2}(u). \end{aligned}$$

Letting $n \to \infty$, it follows from Lemma 5.1 (iv) and the dominated convergence theorem that $\mathrm{E}(U_1 - x)_+ \leq \mathrm{E}(U_2 - x)_+$, $x \geq 0$; that is $U_1 \preceq_{icx} U_2$.

5.6. Proof of Theorem 4.1

(i) Both $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ and $\{Z_2(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ are isotropic random fields, and they are uncorrelated, since

$$\begin{aligned}
\operatorname{cov}(Z_1(\mathbf{x}_1), Z_2(\mathbf{x}_2)) &= \frac{1}{4}\operatorname{cov}(Z(\mathbf{x}_1)+Z(-\mathbf{x}_1), Z(\mathbf{x}_2)-Z(-\mathbf{x}_2)) \\
&= \frac{1}{4}\{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) - C(\rho(\mathbf{x}_1,-\mathbf{x}_2)) + C(\rho(-\mathbf{x}_1,\mathbf{x}_2)) - C(\rho(-\mathbf{x}_1,-\mathbf{x}_2))\} \\
&= 0, \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}^d,
\end{aligned}$$

where the last equality is due to the fact that

$$\rho(-\mathbf{x}_1,\mathbf{x}_2) = \arccos((-\mathbf{x}_1)'\mathbf{x}_2) = \pi - \arccos(\mathbf{x}_1'\mathbf{x}_2) = \pi - \rho(\mathbf{x}_1,\mathbf{x}_2), \quad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}^d.$$

(ii) The covariance function of $\{Z_1(\mathbf{x}), \mathbf{x} \in \mathbb{S}^d\}$ is

$$\begin{aligned}
\operatorname{cov}(Z_1(\mathbf{x}_1), Z_1(\mathbf{x}_2)) &= \frac{1}{4}\operatorname{cov}(Z(\mathbf{x}_1)+Z(-\mathbf{x}_1), Z(\mathbf{x}_2)+Z(-\mathbf{x}_2)) \\
&= \frac{1}{4}\{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) + C(\rho(\mathbf{x}_1,-\mathbf{x}_2)) + C(\rho(-\mathbf{x}_1,\mathbf{x}_2)) + C(\rho(-\mathbf{x}_1,-\mathbf{x}_2))\} \\
&= \frac{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) + C(\pi-\rho(\mathbf{x}_1,\mathbf{x}_2))}{2}, \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}^d.
\end{aligned}$$

(iii) This is derived in the similar way as Part (ii).

(iv) It follows directly from Theorem 2.1, due to a simple decomposition of $C(\rho(\mathbf{x}_1,\mathbf{x}_2))$,

$$C(\rho(\mathbf{x}_1,\mathbf{x}_2)) = \frac{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) + C(\pi-\rho(\mathbf{x}_1,\mathbf{x}_2))}{2} + \frac{C(\rho(\mathbf{x}_1,\mathbf{x}_2)) - C(\pi-\rho(\mathbf{x}_1,\mathbf{x}_2))}{2}, \quad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{S}^d.$$

(v) It follows from Theorem 2.4, similar to Part (iv).

5.7. Proof of theorem Theorems 4.2 and 4.3

It follows from (4.2) that

$$C_2(\rho(\mathbf{x}_1,\mathbf{x}_2)) - C_1(\rho(\mathbf{x}_1,\mathbf{x}_2)) = \sum_{n=0}^{\infty}\left(b_n^{(2)} - b_n^{(1)}\right) P_n^{(\alpha,\beta)}(\cos\rho(\mathbf{x}_1,\mathbf{x}_2)), \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathbb{M}^d.$$

By Theorem 2.1, $\{Z_1(\mathbf{x}) - \mu_1(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\} \overset{p}{\succeq} \{Z_2(\mathbf{x}) - \mu_2(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ if and only if $C_2(\rho(\mathbf{x}_1,\mathbf{x}_2)) - C_1(\rho(\mathbf{x}_1,\mathbf{x}_2)), \mathbf{x}_1 \in \mathbb{M}^d, \mathbf{x}_2 \in \mathbb{M}^d$, is positive definite, or, equivalently, all its coefficients, $b_n^{(2)} - b_n^{(1)}$, $n \in \mathbb{N}_0$, are nonnegative, according to Theorem 2 of Ma and Malyarenko (2020). The proof of Theorem 4.3 is similar to that of Theorem 4.2.

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