THEORY PROBAB. APPL.
Vol. 68, No. 1, pp. xx–xx

SERIES EXPANSIONS OF FRACTIONAL BROWNIAN MOTIONS AND STRONG LOCAL NONDETERMINISM OF BIFRACTIONAL BROWNIAN MOTIONS ON BALLS AND SPHERES*

T. LU†, C. MA†, AND F. WANG‡

Abstract. This paper provides series expansions for fractional Brownian motions on the unit ball and the unit sphere by means of ultraspherical polynomials and spherical harmonics. It establishes the property of strong local nondeterminism of isotropic Gaussian random fields on the unit sphere and that of fractional and bifractional Brownian motions on the unit ball and the unit sphere.

Key words. conditionally negative definiteness, distance function on the ball, spherical harmonics, trifractional Brownian motion, ultraspherical polynomial

DOI. 10.1137/S0040585X97T991301

1. Introduction. The objectives of this paper are to introduce the definitions and to investigate the properties of the fractional Brownian motion and the related Gaussian random fields on the unit ball $\mathbf{B}^d = \{\mathbf{x} \in \mathbf{R}^d \colon \|\mathbf{x}\| \leqslant 1\}$, whose covariance functions depend only on a distance function on $\mathbf{B}^d$, where $\|\mathbf{x}\|$ is the Euclidean norm in $\mathbf{R}^d$. As observed in [20], few theoretical or computational results in the literature deal with a scalar or vector random field on $\mathbf{B}^d$, which may have potential applications in many areas, such as medical imaging, atmospheric sciences, geophysics, and solar physics. In the recent work [20], a class of second-order vector random fields on $\mathbf{B}^d$ is introduced, whose direct/cross covariance functions are invariant or isotropic with respect to the distance on $\mathbf{B}^d$. Investigations of scalar and vector random fields on the unit sphere $\mathbf{S}^d = \{\mathbf{x} \in \mathbf{R}^{d+1} \colon \|\mathbf{x}\| = 1\}$ can be found in [2], [5], [7], [13], [15], [16], [17], [18], [21], [24], [25], [26], [27], [36], [37], [38], among others. In the present paper, we are also interested in series expansions of a fractional Brownian motion on $\mathbf{S}^d$, and we establish the strong local nondeterminism property of isotropic Gaussian random fields on $\mathbf{S}^d$ and of fractional and bifractional Brownian motions on $\mathbf{S}^d$.

The distance function $\rho(\mathbf{x}_1, \mathbf{x}_2)$ on $\mathbf{B}^d$ adopted in the present paper is defined by (see [4], [8])

$$\rho(\mathbf{x}_1, \mathbf{x}_2) = \arccos\bigl(\mathbf{x}_1'\mathbf{x}_2 + \sqrt{1-\|\mathbf{x}_1\|^2}\,\sqrt{1-\|\mathbf{x}_2\|^2}\,\bigr), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d, \tag{1.1}$$

where $\mathbf{x}_1'\mathbf{x}_2$ is the inner product in $\mathbf{R}^d$. Evidently, $0 \leqslant \rho(\mathbf{x}_1, \mathbf{x}_2) \leqslant \pi$. This distance function is deduced from the geodesic distance on the hemisphere $\mathbf{S}^d_+ = \{\mathbf{x} \in \mathbf{R}^{d+1} \colon \|\mathbf{x}\| = 1,\, x_{d+1} \geqslant 0\}$ of $\mathbf{R}^{d+1}$ by the bijection

$$\mathbf{B}^d \ni \mathbf{x} \mapsto \widetilde{\mathbf{x}} = \bigl(\mathbf{x}', \sqrt{1-\|\mathbf{x}\|^2}\,\bigr)' \in \mathbf{S}^d_+,$$

and hence it is a true distance on $\mathbf{B}^d$. It takes into account the difference between the points inside the ball, as well as those near the boundary, and the Euclidean distance.

*Received by the editors March 20, 2020; revised July 16, 2021; published electronically DATE. Originally published in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya*, 68 (2023), pp. 106–132.
https://doi.org/10.1137/S0040585X97T991301

†Department of Mathematics, Statistics, and Physics, Wichita State University, Wichita, KS 67260 USA (lu@math.wichita.edu, chunsheng.ma@wichita.edu).
‡Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609 USA (fwang4@wpi.edu).

Moreover, $(\mathbf{B}^d, \rho(\mathbf{x}_1, \mathbf{x}_2))$ is a metric space. For two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\mathbf{S}^d$, their spherical (angular, or geodesic) distance, denoted by $\vartheta(\mathbf{x}_1, \mathbf{x}_2)$, is the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$ on the largest circle on $\mathbf{S}^d$ that passes through them; more precisely,

$$\vartheta(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1' \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{S}^d. \tag{1.2}$$

Clearly, $0 \leqslant \vartheta(\mathbf{x}_1, \mathbf{x}_2) \leqslant \pi$, and $(\mathbf{S}^d, \vartheta(\mathbf{x}_1, \mathbf{x}_2))$ is also a metric space.

The primary interest of this paper is second-order random fields with the index set $\mathbf{B}^d$, whose covariance functions are merely functions of $\rho(\mathbf{x}_1, \mathbf{x}_2)$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d$. One of our motivating questions is as follows: Does a Brownian motion $\{Z(\mathbf{x}), \, \mathbf{x} \in \mathbf{B}^d\}$ exist, whose covariance function is of the form

$$\operatorname{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \rho(\mathbf{x}_1, \mathbf{x}_0) + \rho(\mathbf{x}_2, \mathbf{x}_0) - \rho(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d, \tag{1.3}$$

where $\mathbf{x}_0 \in \mathbf{B}^d$ is a fixed point? In the late 1940s Paul Lévy first introduced the notion of a Brownian motion indexed by a metric space, but its existence is questionable, unless the distance function is conditionally negative definite on the metric space [3]. A positive answer to the above question is provided in section 3, where a random field on $\mathbf{B}^d$ is constructed via a series expansion in ultraspherical polynomials. Moreover, it leads us to propose the definitions of fractional, bifractional, trifractional, and quadrifractional Brownian motions on $\mathbf{B}^d$ [22], [23]. We call $\{Z(\mathbf{x}), \, \mathbf{x} \in \mathbf{B}^d\}$ a fractional Brownian motion if it is a centered Gaussian random field with covariance function

$$\operatorname{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \rho^\nu(\mathbf{x}_1, \mathbf{x}_0) + \rho^\nu(\mathbf{x}_2, \mathbf{x}_0) - \rho^\nu(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d, \tag{1.4}$$

where ν is a positive constant and $\mathbf{x}_0 \in \mathbf{B}^d$ is a given point. Such a fractional Brownian motion exists on $\mathbf{B}^d$ if and only if $\nu \in (0, 1]$, which is explained in section 3. Series expansions of the fractional Brownian motion on the real line can been found in [6], [9], [10], [11], among others, and central and noncentral limit theorems are derived in [17] for the first Minkowski functional of the fractional Brownian motion on $\mathbf{S}^2$.

A centered Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{B}^d\}$ is called a bifractional Brownian motion if its covariance function is of the form

$$\begin{aligned} &\operatorname{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) \\ &\quad = \left(\rho^{\nu\kappa}(\mathbf{x}_1, \mathbf{x}_0) + \rho^{\nu\kappa}(\mathbf{x}_2, \mathbf{x}_0)\right)^{1/\kappa} - \rho^\nu(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d, \end{aligned} \tag{1.5}$$

where ν and κ are certain positive constants and $\mathbf{x}_0 \in \mathbf{B}^d$ is a given point. Clearly, (1.5) contains (1.4) as a special case. The existence of a bifractional Brownian motion on the Euclidean space and the unit sphere are addressed in, for instance, [19] and [23].

Section 3 also derives series expansions of a fractional Brownian motion on $\mathbf{B}^d$. Similar series expansions are established in section 4 for the fractional Brownian motion on $\mathbf{S}^d$, while a special case $d = 2$ is dealt with in [13].

In the Euclidean setting, the notion of strong local nondeterminism plays an important role in the derivation of a number of characterizations for sample trajectories [29], [31], [33], [34]. Section 5 establishes the strong local nondeterminism property for a large class of isotropic Gaussian random fields on $\mathbf{S}^d$ and for fractional and bifractional Brownian motions on $\mathbf{S}^d$ and $\mathbf{B}^d$, while the particular case of $d = 2$ is treated in [14] and [15].

Some preliminary results are presented in section 2, and all proofs are given in section 6.

2. Preliminary results. This section contains some basic properties of ultraspherical or Gegenbauer's polynomials [30] and spherical harmonics [1]. In addition, we also present four lemmas, of which the first three are essential for building series expansions of fractional Brownian motions on $\mathbf{B}^d$ and $\mathbf{S}^d$, and the last one is helpful in the proof of Theorem 6.

For $\lambda > 0$, the ultraspherical polynomials, $P_n^{(\lambda)}(x)$, $x \in \mathbf{R}$, $n \in \mathbf{N}_0$, can be defined via the recurrence formula

$$P_0^{(\lambda)}(x) \equiv 1, \quad P_1^{(\lambda)}(x) = 2\lambda x,$$
$$P_n^{(\lambda)}(x) = \frac{2(\lambda+n-1)xP_{n-1}^{(\lambda)}(x) - (2\lambda+n-2)P_{n-2}^{(\lambda)}(x)}{n}, \qquad n \geqslant 2,$$

where $\mathbf{N}_0$ stands for the set of nonnegative integers. These polynomials are the coefficients of u^n in the power series expansion of the function $(1-2ux+u^2)^{-\lambda}$, i.e.,

$$(1-2ux+u^2)^{-\lambda} = \sum_{n=0}^{\infty} u^n P_n^{(\lambda)}(x), \qquad x \in \mathbf{R}, \quad |u| < 1, \tag{2.1}$$

and satisfy the differential equation

$$(1-x^2)\,\frac{d^2y}{dx^2} - (2\lambda+1)x\,\frac{dy}{dx} + n(2\lambda+n)y = 0;$$

moreover, they are orthogonal with respect to the weight function $(1-x^2)^{\lambda-1/2}$ in the sense that

$$\int_{-1}^{1} P_i^{(\lambda)}(x)P_j^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}\,dx = \begin{cases} \dfrac{\pi\, 2^{1-2\lambda}\Gamma(i+2\lambda)}{i!\,(\lambda+i)(\Gamma(\lambda))^2} = \dfrac{\pi\, 2^{1-2\lambda}\Gamma(2\lambda)}{(\lambda+i)(\Gamma(\lambda))^2} P_i^{(\lambda)}(1), & i = j, \\ 0, & i \neq j. \end{cases}$$

In the particular case $\lambda = 1/2$, $P_n^{(1/2)}(x) = P_n(x)$ ($n \in \mathbf{N}_0$) are the Legendre polynomials; if $\lambda = 1$, then $P_n^{(1)}(\cos\vartheta) = (\sin\vartheta)^{-1}\sin((n+1)\vartheta)$, $n \in \mathbf{N}_0$. The functions $P_{2n}^{(\lambda)}(x)$ and $P_{2n+1}^{(\lambda)}(x)$ are even and odd, respectively. Moreover, $P_n^{(\lambda)}(x)$ is bounded in magnitude on $[-1,1]$ by $P_n^{(\lambda)}(1) = (\Gamma(n+1)\Gamma(2\lambda))^{-1}\Gamma(n+2\lambda)$, and[1] $P_n^{(\lambda)}(1) \sim (\Gamma(2\lambda))^{-1}n^{2\lambda-1}$.

With the help of the Gram–Schmidt orthogonalization, it is possible to choose $c_{n,d} = (n!\,(d-1)!)^{-1}(2n+d-1)(n+d-2)!$ spherical harmonics of degree n in $d+1$ variables that are orthonormal with respect to the invariant measure on $\mathbf{S}^d$ [1]. To be precise, denote the members of this orthonormal basis by $S_{nj}(\mathbf{x})$, for $\mathbf{x} \in \mathbf{S}^d$, $j = 1, \dots, c_{n,d}$. They satisfy $\int_{\mathbf{S}^d} S_{nj}(\mathbf{x})S_{kl}(\mathbf{x})\,d\sigma(\mathbf{x}) = \delta_{nk}\delta_{jl}$, where σ represents the invariant measure on $\mathbf{S}^d$ and δ_{nk} is the Kronecker delta function. The spherical harmonics are related to the ultraspherical polynomials via the identity

$$\sum_{j=1}^{c_{n,d}} S_{n,j}(\mathbf{x}_1)S_{n,j}(\mathbf{x}_2) = \frac{c_{n,d}}{\omega_d}\,\frac{P_n^{((d-1)/2)}(\mathbf{x}_1'\mathbf{x}_2)}{P_n^{((d-1)/2)}(1)}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{S}^d, \tag{2.2}$$

where $\omega_d = (\Gamma((d+1)/2))^{-1}\,2\pi^{(d+1)/2}$ is the surface area of $\mathbf{S}^d$; see, for instance, Theorem 9.6.3 of [1].

[1] We write $a_n \sim b_n$ if $\lim_{n\to\infty}(a_n/b_n) = 1$, and $a_n = O(b_n)$ if $\lim_{n\to\infty}(a_n/b_n)$ is nonzero.

As shown in [30], any continuous function $g(\vartheta)$ on $[0,\pi]$ can be expanded in ultraspherical polynomials,

$$g(\vartheta) = \sum_{n=0}^{\infty} b_n^{((d-1)/2)}(g)\, \frac{P_n^{((d-1)/2)}(\cos\vartheta)}{P_n^{((d-1)/2)}(1)}, \qquad \vartheta \in [0,\pi],$$

where

$$b_n^{((d-1)/2)}(g) = \frac{(2n+d-1)\, 2^{d-3}\Gamma^2((d-1)/2)}{\pi\Gamma(d-1)} \times \int_0^{\pi} g(\vartheta) P_n^{((d-1)/2)}(\cos\vartheta) \sin^{d-1}\vartheta\, d\vartheta, \qquad n \in \mathbf{N}_0.$$

In Lemmas 2 and 3 that follow, we derive such expansions for an exponential function and a power function, respectively. This enables us to build series expansions of fractional Brownian motions on $\mathbf{B}^d$ and $\mathbf{S}^d$ in later sections. The proof of Lemma 2 is based on the following useful lemma, which is interesting in itself as well.

LEMMA 1. *Let $g(\vartheta)$ be a continuous function on $[0,\pi]$.*

(i) *The following equality holds*:

$$d \cdot b_n^{((d+1)/2)}(g) = \frac{(n+d-1)(n+d)}{2n+d-1} b_n^{((d-1)/2)}(g) - \frac{(n+1)(n+2)}{2n+d+3} b_{n+2}^{((d-1)/2)}(g), \qquad n \in \mathbf{N}_0. \tag{2.3}$$

(ii) *If $g(\vartheta)$ has a continuous second-order derivative on $[0,\pi]$, then*

$$d \cdot b_n^{((d+1)/2)}(-g'') = \frac{(n+d-1)(n+d)}{2n+d-1} n^2 b_n^{((d-1)/2)}(g) - \frac{(n+1)(n+2)}{2n+d+3} (n+d+1)^2 b_{n+2}^{((d-1)/2)}(g), \qquad n \in \mathbf{N}_0. \tag{2.4}$$

LEMMA 2. *For a positive constant λ,*

$$\exp(-\lambda\vartheta) = \sum_{n=0}^{\infty} b_n \frac{P_n^{((d-1)/2)}(\cos\vartheta)}{P_n^{((d-1)/2)}(1)}, \tag{2.5}$$

where, for odd d,

$$b_n = \frac{(2n+d-1)\Gamma(n+d-1)\Gamma(d/2+1)}{n!\,\sqrt{\pi}\,\Gamma((d+1)/2)} \lambda\big(1-(-1)^n e^{-\lambda\pi}\big) \times \prod_{k=0}^{(d-1)/2} \{(n+2k)^2 + \lambda^2\}^{-1}, \qquad n \in \mathbf{N}_0, \tag{2.6}$$

and, for even d,

$$b_n = \frac{(2n+d-1)\Gamma(n+d-1)}{n!\,\sqrt{\pi}} \, \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \big(1+(-1)^n e^{-\lambda\pi}\big) h_n, \qquad n \in \mathbf{N}_0, \tag{2.7}$$

with

$$h_n = \begin{cases} \prod_{k=0}^{d/2} \big((2k+1)^2+\lambda^2\big)^{-1}, & n=0, \\ \prod_{k=0}^{d/2} \big((2k+2)^2+\lambda^2\big)^{-1}, & n=1, \\ \dfrac{\prod_{k=0}^{j-1}((2k)^2+\lambda^2)}{\prod_{k=0}^{j-1+d/2}((2k+1)^2+\lambda^2)}, & n=2j,\ j\geqslant 1, \\ \dfrac{\prod_{k=0}^{j-1}((2k+1)^2+\lambda^2)}{\prod_{k=0}^{j-1+d/2}((2k+2)^2+\lambda^2)}, & n=2j+1,\ j\geqslant 1. \end{cases}$$

It is known that the following Legendre polynomial expansion holds for $\vartheta \in [0,\pi]$:

$$\vartheta = \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{4n-1}{(2n-1)^2}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \big(1-P_{2n-1}(\cos\vartheta)\big); \tag{2.8}$$

see, for instance, formula 8.925 of [12]. Moreover, it also possesses an ultraspherical polynomial expansion given in the following lemma for $d \geqslant 2$. Let $\mathbf{N}$ be the set of positive integers.

LEMMA 3. (i) *For* $d \geqslant 2$,

$$\vartheta = \sum_{n=1}^{\infty} b_{2n-1}\left(1-\frac{P_{2n-1}^{((d-1)/2)}(\cos\vartheta)}{P_{2n-1}^{((d-1)/2)}(1)}\right), \qquad \vartheta\in[0,\pi], \tag{2.9}$$

where

$$\begin{aligned} b_{2n-1} &= \frac{(d-1)(\Gamma((d-1)/2))^2}{2\pi\Gamma(d-1)}\,\frac{4n+d-3}{(2n-1)^2} \\ &\quad\times \frac{\Gamma(2n+d-2)}{\Gamma(2n)}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+d/2)}\right)^2, \qquad n\in\mathbf{N}, \end{aligned} \tag{2.10}$$

and

$$b_{2n-1} \sim \frac{(d-1)(\Gamma((d-1)/2))^2}{2\pi\Gamma(d-1)} n^{-2}, \qquad n\to\infty. \tag{2.11}$$

(ii) *If* $\nu \in (0,1)$, *then*

$$\vartheta^{\nu} = \sum_{n=1}^{\infty} b_n\left(1-\frac{P_n^{((d-1)/2)}(\cos\vartheta)}{P_n^{((d-1)/2)}(1)}\right), \qquad \vartheta\in[0,\pi], \tag{2.12}$$

where the series on the right-hand side converges uniformly on $[0,\pi]$,

$$\begin{aligned} b_n &= -\frac{(2n+d-1)(\Gamma((d-1)/2))^2}{\pi\, 2^{3-d}\Gamma(d-1)} \\ &\quad\times \int_0^{\pi} \vartheta^{\nu} P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\, d\vartheta, \qquad n\in\mathbf{N}, \end{aligned} \tag{2.13}$$

with $b_n > 0$ $(n\in\mathbf{N})$, *and*

$$b_n \sim \frac{2^{\nu}\nu\Gamma((\nu+d)/2)}{\Gamma(1-\nu/2)\Gamma(d/2)} n^{-1-\nu}, \qquad n\to\infty. \tag{2.14}$$

In a particular case of $d = 2$, an incorrect rate of b_n was proposed in [13], as pointed out in [14]. It differs from (2.14).

LEMMA 4. *For a constant* $\varepsilon \in (0, \pi]$,

$$\left\{\max\left(1 - \frac{\vartheta}{\varepsilon}, 0\right)\right\}^d = \sum_{n=0}^{\infty} b_n^{((d-1)/2)} \frac{P_n^{((d-1)/2)}(\cos\vartheta)}{P_n^{((d-1)/2)}(1)}, \qquad \vartheta \in [0, \pi],$$

where $\{b_n^{((d-1)/2)}, n \in \mathbf{N}_0\}$ *is a sequence of nonnegative constants satisfying*

$$b_n^{((d-1)/2)} \leqslant \frac{\beta_0 \varepsilon}{(1 + n\varepsilon)^2}, \qquad n \in \mathbf{N}_0, \tag{2.15}$$

for some positive constant β_0 *that is invariant of* n *and* ε.

3. Series expansions of fractional Brownian motions on the ball. Based on Lemma 3, this section formulates a random field on $\mathbf{B}^d$ via an infinite series expansion, whose covariance function is the same as (1.4). It implies that the distance function (1.1) is indeed a conditionally negative definite function on $\mathbf{B}^d$, which confirms the existence of a (fractional) Brownian motion on $\mathbf{B}^d$ and enables us to introduce scalar or vector multifractional Brownian motions and related random fields such as those in [23]. We also establish series expansions of a fractional Brownian motion on $\mathbf{B}^d$ via Lemma 3 and identity (2.2).

In what follows, d is assumed to be at least 2. Let

$$\alpha_n = \left(\frac{2n + d - 1}{d - 1}\right)^{1/2}, \qquad n \in \mathbf{N}_0, \tag{3.1}$$

which relates $c_{n,d}$ and $P_n^{((d-1)/2)}(1)$ as follows: $c_{n,d} = \alpha_n^2 P_n^{((d-1)/2)}(1)$. For a point $\mathbf{x} \in \mathbf{B}^d$, we use $\widetilde{\mathbf{x}} = \left(\mathbf{x}', \sqrt{1 - \|\mathbf{x}\|^2}\,\right)'$ to denote the associated point on $\mathbf{S}_+^d$ or $\mathbf{S}^d$.

THEOREM 1. *Suppose that* $\{V_n, n \in \mathbf{N}\}$ *is a sequence of independent random variables (r.v.'s) with mean* 0 *and variance* $\mathrm{var}(V_n) = \alpha_n^2$, *and which is independent of a* $(d+1)$*-variate random vector* $\mathbf{U}$ *uniformly distributed on* $\mathbf{S}^d$. *Let* $\mathbf{x}_0 \in \mathbf{B}^d$ *be a fixed point.*

(i) *For* b_{2n-1}, *as given in* (2.10),

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_{2n-1}}{P_{2n-1}^{((d-1)/2)}(1)}\right)^{1/2} \times V_{2n-1}\left(P_{2n-1}^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U}) - P_{2n-1}^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\right), \qquad \mathbf{x} \in \mathbf{B}^d, \tag{3.2}$$

is a second-order random field on $\mathbf{B}^d$, *its mean function is identical to* 0, *its covariance function is the same as* (1.3), *and its variogram is* $\rho(\mathbf{x}_1, \mathbf{x}_2)$.

(ii) *For a positive constant* $\nu \in (0, 1)$ *and* b_n, *as given in* (2.13),

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_n}{P_n^{((d-1)/2)}(1)}\right)^{1/2} \times V_n\left(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U}) - P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\right), \qquad \mathbf{x} \in \mathbf{B}^d, \tag{3.3}$$

is a second-order random field on $\mathbf{B}^d$, *its mean function is identical to* 0, *its covariance function is the same as* (1.4), *and its variogram is* $\rho^\nu(\mathbf{x}_1, \mathbf{x}_2)$.

The random terms in the series (3.2) and (3.3) are uncorrelated, i.e.,

$$\begin{aligned}&\operatorname{cov} V_n\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_1'\mathbf{U}) - P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big),\\ &\qquad V_l\big(P_l^{((d-1)/2)}(\widetilde{\mathbf{x}}_2'\mathbf{U}) - P_l^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\big) = 0, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d, \quad n \neq l.\end{aligned}$$

It is interesting to observe that (3.2) just contains odd terms, which implies that its covariance function is not strictly positive definite.

Recall that a function $g(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{D}$, is said to be conditionally negative definite if $g(\mathbf{x}, \mathbf{x}) \equiv 0$, $\mathbf{x} \in \mathbf{D}$, and if the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j g(\mathbf{x}_i, \mathbf{x}_j) \leqslant 0 \tag{3.4}$$

holds for every $n \in \mathbf{N}$, any $\mathbf{x}_k \in \mathbf{D}$, and any $a_k \in \mathbf{R}$ $(k = 1, \dots, n)$ subject to $\sum_{k=1}^{n} a_k = 0$. It is well known that the Euclidean distance and ℓ_1-distance are conditionally negative definite functions in $\mathbf{R}^d$, as so is the spherical distance on $\mathbf{S}^d$. Theorem 1 implies the conditional negative definiteness of the distance function (1.1) and ensures the existence of a fractional Brownian motion on the ball. Moreover, it enables us, using the general results of [23], to generate vector multifractional Brownian motion and many other elliptically contoured random fields on $\mathbf{B}^d$, whose direct and cross covariance functions depend on $\rho(\mathbf{x}_1, \mathbf{x}_2)$.

THEOREM 2. *Let ν and κ be positive constants and $\mathbf{x}_0 \in \mathbf{B}^d$ be a fixed point.*

(i) *For the distance function $\rho(\mathbf{x}_1, \mathbf{x}_2)$ defined by* (1.1), *$\rho^\nu(\mathbf{x}_1, \mathbf{x}_2)$ is conditionally negative definite on $\mathbf{B}^d$ if and only if $0 < \nu \leqslant 1$.*

(ii) *There exists a fractional Brownian motion $\{B(\mathbf{x}), \mathbf{x} \in \mathbf{B}^d\}$ with covariance function given by* (1.4) *if and only if $0 < \nu \leqslant 1$.*

(iii) *For a given $\nu \in (0, 1]$, if $\kappa \geqslant 1$ and $\nu\kappa \leqslant 1$, then there exists a bifractional Brownian motion with covariance function given by* (1.5). *In this case,* (1.4) *is equal to the sum of* (1.5) *and the covariance function of a trifractional Brownian motion,*

$$\rho^\nu(\mathbf{x}_1, \mathbf{x}_0) + \rho^\nu(\mathbf{x}_2, \mathbf{x}_0) - \big(\rho^{\nu\kappa}(\mathbf{x}_1, \mathbf{x}_0) + \rho^{\nu\kappa}(\mathbf{x}_2, \mathbf{x}_0)\big)^{1/\kappa}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d. \tag{3.5}$$

(iv) *For a given $\nu \in (0, 1]$, if $1/2 \leqslant \kappa \leqslant 1$, then there exists a bifractional Brownian motion with covariance function given by* (1.5), *which is the same as the sum of* (1.4) *and the covariance function of a quadrifractional Brownian motion,*

$$\big(\rho^{\nu\kappa}(\mathbf{x}_1, \mathbf{x}_0) + \rho^{\nu\kappa}(\mathbf{x}_2, \mathbf{x}_0)\big)^{1/\kappa} - \rho^\nu(\mathbf{x}_1, \mathbf{x}_0) - \rho^\nu(\mathbf{x}_2, \mathbf{x}_0), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d. \tag{3.6}$$

Note that if $\nu \in (0, 1)$ and $\kappa \in (0, 1/2)$, then (1.5) is no longer positive definite, and no bifractional Brownian motion exists on $\mathbf{B}^d$. To see this, denote the right-hand side of (1.5) by $C(\mathbf{x}_1, \mathbf{x}_2)$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on $\mathbf{B}^d$ such that $\rho(\mathbf{x}_1, \mathbf{x}_2) = \rho(\mathbf{x}_1, \mathbf{x}_0) + \rho(\mathbf{x}_2, \mathbf{x}_0)$. Let $\delta = \rho(\mathbf{x}_2, \mathbf{x}_0)/\rho(\mathbf{x}_1, \mathbf{x}_0)$. Since $\nu\kappa < 1$ and $\nu > 2\nu\kappa$, we obtain

$$\begin{aligned}&\lim_{\delta \to 0} \frac{C(\mathbf{x}_1, \mathbf{x}_1)C(\mathbf{x}_2, \mathbf{x}_2) - C^2(\mathbf{x}_1, \mathbf{x}_2)}{\rho^{2\nu}(\mathbf{x}_1, \mathbf{x}_0)\delta^{2\nu\kappa}}\\ &\qquad = \lim_{\delta \to 0} \frac{\delta^\nu(2^{1/\kappa} - 1)^2 - ((1 + \delta^{\nu\kappa})^{1/\kappa} - (1 + \delta)^\nu)^2}{\delta^{2\nu\kappa}}\\ &\qquad = -\lim_{\delta \to 0} \frac{((1 + \delta^{\nu\kappa})^{1/\kappa} - (1 + \delta)^\nu)^2}{\delta^{2\nu\kappa}} = -\frac{1}{\kappa^2} < 0,\end{aligned}$$

which violates the Cauchy–Schwarz inequality for a covariance function.

Since $b_n = O(n^{-1-\nu})$ as (2.14) releases, the series in (3.3) converges in mean square. The advantage of (3.3) is its simple form for simulation, although it is not a Gaussian random field. Nevertheless, a fractional Brownian motion or a Gaussian random field with covariance function (1.4) on $\mathbf{B}^d$ may be simulated via spherical harmonics as described in the following theorem.

THEOREM 3. *Suppose that* $\{V_{nj},\, n \in \mathbf{N},\, j = 1, \ldots, c_{n,d}\}$ *is a sequence of independent normal r.v.'s with mean* 0 *and variance* $\operatorname{var}(V_{nj}) = \omega_d \alpha_n^{-2}$, *and that* $\mathbf{x}_0 \in \mathbf{B}^d$ *is a fixed point.*

(i) *A Brownian motion* $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{B}^d\}$ *possesses a series expansion*

(3.7)
$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_{2n-1}}{P_{2n-1}^{((d-1)/2)}(1)} \right)^{1/2} \sum_{j=1}^{c_{2n-1,d}} V_{2n-1,j} \{ S_{2n-1,j}(\widetilde{\mathbf{x}}) - S_{2n-1,j}(\widetilde{\mathbf{x}}_0) \}, \quad \mathbf{x} \in \mathbf{B}^d,$$

with covariance function (1.3), *where* $\{b_{2n-1},\, n \in \mathbf{N}\}$ *is given as in* (2.10).

(ii) *For* $\nu \in (0,1)$, *a fractional Brownian motion* $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{B}^d\}$ *with covariance function* (1.4) *possesses a series expansion*

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_n}{P_n^{((d-1)/2)}(1)} \right)^{1/2} \sum_{j=1}^{c_{n,d}} V_{nj} \{ S_{nj}(\widetilde{\mathbf{x}}) - S_{nj}(\widetilde{\mathbf{x}}_0) \}, \qquad \mathbf{x} \in \mathbf{B}^d, \tag{3.8}$$

where $\{b_n,\, n \in \mathbf{N}\}$ *is given as in* (2.13).

The random terms in (3.7) and (3.8) are uncorrelated. In contrast to (3.3), the expansion (3.8) contains $c_{n,d}$-terms at the level n, so that its use for simulation would be relatively inefficient.

4. Series expansions of spherical fractional Brownian motion. In this section, we derive two types of series expansions of a fractional Brownian motion on $\mathbf{S}^d$ ($d \geqslant 2$), one of which is based on ultraspherical polynomials and the other on spherical harmonics, analogously to those on $\mathbf{B}^d$ in section 3.

THEOREM 4. *Assume that* $\{V_n,\, n \in \mathbf{N}\}$ *is a sequence of independent r.v.'s with mean* 0 *and variance* $\operatorname{var}(V_n) = \alpha_n^2$ *and is independent of a* $(d+1)$*-variate random vector* $\mathbf{U}$ *uniformly distributed on* $\mathbf{S}^d$, *that* $\mathbf{x}_0 \in \mathbf{S}^d$ *is a fixed point, and that* α_n *is given as in* (3.1).

(i) *With* b_{2n-1}, *as given in* (2.10),

(4.1)
$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_{2n-1}}{P_{2n-1}^{((d-1)/2)}(1)} \right)^{1/2} V_{2n-1} \big(P_{2n-1}^{((d-1)/2)}(\mathbf{x}'\mathbf{U}) - P_{2n-1}^{((d-1)/2)}(\mathbf{x}_0'\mathbf{U}) \big), \quad \mathbf{x} \in \mathbf{S}^d,$$

is a second-order random field on $\mathbf{S}^d$, *its mean function is identical to* 0, *its covariance function is* $\vartheta(\mathbf{x}_1, \mathbf{x}_0) + \vartheta(\mathbf{x}_2, \mathbf{x}_0) - \vartheta(\mathbf{x}_1, \mathbf{x}_2)$, *and its variogram is* $\vartheta(\mathbf{x}_1, \mathbf{x}_2)$.

(ii) *For* $\nu \in (0,1)$ *and* b_n, *as given in* (2.13),

(4.2)
$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_n}{P_n^{((d-1)/2)}(1)} \right)^{1/2} V_n \big(P_n^{((d-1)/2)}(\mathbf{x}'\mathbf{U}) - P_n^{((d-1)/2)}(\mathbf{x}_0'\mathbf{U}) \big), \quad \mathbf{x} \in \mathbf{S}^d,$$

is a second-order random field on $\mathbf{S}^d$, *its mean function is identical to* 0, *its covariance function is* $\vartheta^\nu(\mathbf{x}_1, \mathbf{x}_0) + \vartheta^\nu(\mathbf{x}_2, \mathbf{x}_0) - \vartheta^\nu(\mathbf{x}_1, \mathbf{x}_2)$, *and its variogram is* $\vartheta^\nu(\mathbf{x}_1, \mathbf{x}_2)$.

THEOREM 5. *Suppose that* $\{V_{nj},\, n \in \mathbf{N},\, j = 1, \ldots, c_{n,d}\}$ *is a sequence of independent normal r.v.'s with mean* 0 *and variance* $\mathrm{var}(V_{nj}) = \omega_d \alpha_n^{-2}$, *and that* $\mathbf{x}_0 \in \mathbf{S}^d$ *is a fixed point.*

(i) *A spherical Brownian motion* $\{Z(\mathbf{x}),\, x \in \mathbf{S}^d\}$ *possesses a series expansion*

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_{2n-1}}{P_{2n-1}^{((d-1)/2)}(1)} \right)^{1/2} \times \sum_{j=1}^{c_{2n-1,d}} V_{2n-1,j} \big(S_{2n-1,j}(\mathbf{x}) - S_{2n-1,j}(\mathbf{x}_0) \big), \qquad \mathbf{x} \in \mathbf{S}^d, \tag{4.3}$$

with covariance function $\vartheta(\mathbf{x}_1, \mathbf{x}_0) + \vartheta(\mathbf{x}_2, \mathbf{x}_0) - \vartheta(\mathbf{x}_1, \mathbf{x}_2)$, *where* b_{2n-1} *is given as in* (2.10).

(ii) *For* $\nu \in (0, 1)$, *a spherical fractional Brownian motion* $\{Z(\mathbf{x}),\, x \in \mathbf{S}^d\}$, *whose covariance function is* $\vartheta^\nu(\mathbf{x}_1, \mathbf{x}_0) + \vartheta^\nu(\mathbf{x}_2, \mathbf{x}_0) - \vartheta^\nu(\mathbf{x}_1, \mathbf{x}_2)$, *possesses a series expansion*

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_n}{P_n^{((d-1)/2)}(1)} \right)^{1/2} \sum_{j=1}^{c_{n,d}} V_{nj} \big(S_{nj}(\mathbf{x}) - S_{nj}(\mathbf{x}_0) \big), \qquad \mathbf{x} \in \mathbf{S}^d, \tag{4.4}$$

where b_n *is as given in* (2.13).

Expressions (4.3) and (4.4) are similar in spirit to the expansions of the spherical fractional Brownian motions introduced in [13], as the former can be equivalently written as

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_{2n-1}\omega_d}{c_{2n-1,d}} \right)^{1/2} \sum_{j=1}^{c_{2n-1,d}} \epsilon_{2n-1,j} \big(S_{2n-1,j}(\mathbf{x}) - S_{2n-1,j}(\mathbf{x}_0) \big) \tag{4.5}$$

and

$$Z(\mathbf{x}) = \sum_{n=1}^{\infty} \left(\frac{b_n \omega_d}{c_{n,d}} \right)^{1/2} \sum_{j=1}^{c_{n,d}} \epsilon_{nj} \big(S_{nj}(\mathbf{x}) - S_{nj}(\mathbf{x}_0) \big), \tag{4.6}$$

respectively, where $\{\epsilon_{nj},\, n \in \mathbf{N},\, j = 1, \ldots, c_{n,d}\}$ is a sequence of independent standard normal r.v.'s. Strictly speaking, however, expansions (4.3) and (4.4) (or (4.5) and (4.6)) are not usual Karhunen–Loève expansions in the sense that $\{S_{nj}(\mathbf{x}) - S_{nj}(\mathbf{x}_0),\, n \in \mathbf{N},\, j = 1, \ldots, c_{n,d}\}$ is not a subset of an orthonormal basis, i.e.,

$$\int_{\mathbf{S}^d} \big(S_{ki}(\mathbf{x}) - S_{ki}(\mathbf{x}_0) \big) \big(S_{nj}(\mathbf{x}) - S_{nj}(\mathbf{x}_0) \big)\, d\sigma(\mathbf{x}) = \begin{cases} S_{ki}(\mathbf{x}_0) S_{nj}(\mathbf{x}_0) \omega_d & \text{if } k \neq n, \text{ or } k = n,\, i \neq j; \\ 1 + (S_{nj}(\mathbf{x}_0))^2 \omega_d & \text{if } k = n,\, i = j. \end{cases}$$

Nevertheless, V_{nj} can be recovered from (4.4) as follows:

$$\int_{\mathbf{S}^d} Z(\mathbf{x}) S_{nj}(\mathbf{x})\, d\sigma(\mathbf{x}) = \left(\frac{b_n}{P_n^{((d-1)/2)}(1)} \right)^{1/2} V_{nj}.$$

It is worth noting that in the particular case of $d = 2$, expansion (4.4) was given in [13], where the convergence rate of d_l in Theorem 1 is corrected in [15]; our results, i.e., Theorem 5 and Lemma 3, provide more precise convergence rates than those in Theorem 2 of [13] for a higher dimension.

5. Strong local nondeterminism. In this section, we establish the property of strong local nondeterminism (SLND) of an isotropic Gaussian random field on $\mathbf{S}^d$ under certain regularity conditions, which facilitates deriving the SLND property of fractional and bifractional Brownian motions on $\mathbf{B}^d$ and $\mathbf{S}^d$ for $\nu \in (0,1)$. The existence of bifractional, trifractional, and quadrifractional Brownian motions on $\mathbf{S}^d$ can be confirmed in a way similar to that on $\mathbf{B}^d$ in Theorem 2.

Denote by $\mathrm{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n))$ the conditional variance of $Z(\mathbf{x})$ given $Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)$. For a Gaussian random field $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{D}\}$, it is known that

$$\mathrm{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)) = \inf \mathbf{E}\left(Z(\mathbf{x}) - \sum_{k=1}^{n} a_k Z(\mathbf{x}_k) \right)^2, \tag{5.1}$$

where the infimum is taken over all $(a_1, \dots, a_n)' \in \mathbf{R}^n$.

For an isotropic and mean square continuous Gaussian random field on $\mathbf{S}^d$, the SLND property is described in the following theorem, under the condition that the coefficients in the ultraspherical expansion of its covariance function fulfill inequality (5.2) below. A particular case of $d = 2$ is derived in [14].

Theorem 6. *Suppose that $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ is an isotropic and mean square continuous Gaussian random field with mean 0 and covariance function*

$$\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \sum_{l=0}^{\infty} b_l \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}_1, \mathbf{x}_2))}{P_l^{((d-1)/2)}(1)}, \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{S}^d,$$

where $\{b_l,\, l \in \mathbf{N}_0\}$ is a summable sequence with nonnegative terms. If

$$b_l (1+l)^{1+\nu} \geqslant \beta_1, \qquad l \in \mathbf{N}_0, \tag{5.2}$$

for some constants $\beta_1 > 0$ and $\nu \in (0,2)$, then there is a positive constant β such that the inequality

$$\mathrm{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)) \geqslant \beta \varepsilon^{\nu} \tag{5.3}$$

holds for every $n \in \mathbf{N}$, any $\mathbf{x} \in \mathbf{S}^d$, any $\mathbf{x}_k \in \mathbf{S}^d$ $(k = 1, \dots, n)$, and any $\varepsilon \in (0, \min_{1 \leqslant k \leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k)]$, provided that $\min_{1 \leqslant k \leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k) > 0$.

The requirement that $\min_{1 \leqslant k \leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k) > 0$ is necessary in Theorem 6. Otherwise, if $\mathbf{x}_1 = \mathbf{x}$, then it follows from (5.1) that $\mathrm{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)) = 0$ by taking $a_1 = 1$ and $a_2 = \dots = a_n = 0$.

Part (i) of the next theorem contains Theorem 3.2 of [15] as a special case, where $d = 2$ is considered. Part (ii) works well for a bifractional Brownian motion on $\mathbf{S}^d$ whose parameter κ lies in the interval $[1/2, 1]$, but it is not clear whether a similar result is available for other cases.

Theorem 7. *Let ν and κ be positive constants, $0 < \nu < 1$, $1/2 \leqslant \kappa \leqslant 1$, and let $\mathbf{x}_0 \in \mathbf{S}^d$ be a fixed point.*

(i) *If $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{S}^d\}$ is a fractional Brownian motion with covariance function $\vartheta^{\nu}(\mathbf{x}_1, \mathbf{x}_0) + \vartheta^{\nu}(\mathbf{x}_2, \mathbf{x}_0) - \vartheta^{\nu}(\mathbf{x}_1, \mathbf{x}_2)$, then there exists a positive constant β depending only on ν such that*

$$\mathrm{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)) \geqslant \beta \varepsilon^{\nu} \tag{5.4}$$

for every $n \in \mathbf{N}$, any $\mathbf{x} \in \mathbf{S}^d$, any $\mathbf{x}_k \in \mathbf{S}^d$ $(k = 1, \dots, n)$ with $\min_{0 \leqslant k \leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k) > 0$, and any $\varepsilon \in (0, \min_{0 \leqslant k \leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k)]$.

(ii) *If* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{S}^d\}$ *is a bifractional Brownian motion with covariance function*

$$\big(\vartheta^{\nu\kappa}(\mathbf{x}_1, \mathbf{x}_0) + \vartheta^{\nu\kappa}(\mathbf{x}_2, \mathbf{x}_0)\big)^{1/\kappa} - \vartheta^{\nu}(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{S}^d,$$

then inequality (5.4) *holds for every* $n \in \mathbf{N}$, *any* $\mathbf{x} \in \mathbf{S}^d$, *any* $\mathbf{x}_k \in \mathbf{S}^d$ $(k = 1, \dots, n)$ *with* $\min_{0\leqslant k\leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k) > 0$, *and any* $\varepsilon \in (0, \min_{0\leqslant k\leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k)]$.

The SLND property similar to Theorem 6 can be derived for an isotropic and mean square Gaussian random field on $\mathbf{B}^d$ whose covariance function is given by equation (8) of [20], despite the fact that the general form of all isotropic covariance functions on $\mathbf{B}^d$ is not known. With the help of Theorem 7, we are able to establish the SLND property in the following theorem for a fractional Brownian motion and a bifractional Brownian motion on $\mathbf{B}^d$.

THEOREM 8. *Let* ν *and* κ *be positive constants*, $0 < \nu < 1$, $1/2 \leqslant \kappa \leqslant 1$, *and let* $\mathbf{x}_0 \in \mathbf{B}^d$ *be a fixed point.*

(i) *For a fractional Brownian motion* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{B}^d\}$ *with covariance function* (1.4), *there exists a positive constant* β *depending only on* ν *such that*

$$\operatorname{var}(Z(\mathbf{x}) \,|\, Z(\mathbf{x}_1), \dots, Z(\mathbf{x}_n)) \geqslant \beta\varepsilon^{\nu} \tag{5.5}$$

for every $n \in \mathbf{N}$, *any* $\mathbf{x} \in \mathbf{B}^d$, *any* $\mathbf{x}_k \in \mathbf{B}^d$ $(k = 1, \dots, n)$ *with* $\min_{0\leqslant k\leqslant n} \rho(\mathbf{x}, \mathbf{x}_k) > 0$, *and any* $\varepsilon \in (0, \min_{0\leqslant k\leqslant n} \rho(\mathbf{x}, \mathbf{x}_k)]$.

(ii) *For a bifractional Brownian motion* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{B}^d\}$ *with covariance function* (1.5), *inequality* (5.5) *holds for every* $n \in \mathbf{N}$, *any* $\mathbf{x} \in \mathbf{B}^d$, *any* $\mathbf{x}_k \in \mathbf{B}^d$ $(k = 1, \dots, n)$ *with* $\min_{0\leqslant k\leqslant n} \rho(\mathbf{x}, \mathbf{x}_k) > 0$, *and any* $\varepsilon \in (0, \min_{0\leqslant k\leqslant n} \rho(\mathbf{x}, \mathbf{x}_k)]$.

6. Proofs.

6.1. Proof of Lemma 1. (i) Identity (2.3) follows directly from formula (18.9.8) of [28].

(ii) Formula (18.9.20) of [28] implies that

$$\frac{d}{d\vartheta}\big(P_{n-1}^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\big) = \frac{n(n+d-1)}{d-1} P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta, \qquad \vartheta \in [0, \pi].$$

Using the last equation and integrating by parts, we obtain

$$\begin{aligned} &n(n+d-1)\int_0^\pi g(\vartheta) P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\, d\vartheta \\ &\quad= -(d-1)\int_0^\pi g'(\vartheta) P_{n-1}^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\, d\vartheta \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &(n+2)(n+d+1)\int_0^\pi g(\vartheta) P_{n+2}^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\, d\vartheta \\ &\quad= -(d-1)\int_0^\pi g'(\vartheta) P_{n+1}^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\, d\vartheta. \end{aligned}$$

Taking the difference between the last two equations, and using formula (4.7.29) of [30], we obtain

$$
\begin{aligned}
(n+2)(n+d+1)&\int_0^\pi g(\vartheta)P_{n+2}^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta \\
&\quad - n(n+d-1)\int_0^\pi g(\vartheta)P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta \\
&= -(d-1)\int_0^\pi g'(\vartheta)\big(P_{n+1}^{((d+1)/2)}(\cos\vartheta)-P_{n-1}^{((d+1)/2)}(\cos\vartheta)\big)\sin^d\vartheta\,d\vartheta \\
&= -(2n+d+1)\int_0^\pi g'(\vartheta)P_{n+1}^{((d-1)/2)}(\cos\vartheta)\sin^d\vartheta\,d\vartheta \\
&= -\frac{(2n+d+1)(d-1)}{(n+1)(n+d)}\int_0^\pi g'(\vartheta)\sin\vartheta\,d\big(P_n^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\big) \\
&= \frac{(2n+d+1)(d-1)}{(n+1)(n+d)}\int_0^\pi g''(\vartheta)P_n^{((d+1)/2)}(\cos\vartheta)\sin^{d+1}\vartheta\,d\vartheta \\
&\quad + \frac{(2n+d+1)(d-1)}{(n+1)(n+d)}\int_0^\pi g'(\vartheta)\cos\vartheta P_n^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\,d\vartheta \\
&= \frac{(2n+d+1)(d-1)}{(n+1)(n+d)}\int_0^\pi g''(\vartheta)P_n^{((d+1)/2)}(\cos\vartheta)\sin^{d+1}\vartheta\,d\vartheta \\
&\quad + \frac{(d-1)}{(n+1)(n+d)}\bigg((n+1)\int_0^\pi g'(\vartheta)P_{n+1}^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\,d\vartheta \\
&\qquad + (n+d)\int_0^\pi g'(\vartheta)P_{n-1}^{((d+1)/2)}(\cos\vartheta)\sin^d\vartheta\,d\vartheta\bigg) \\
&= \frac{(2n+d+1)(d-1)}{(n+1)(n+d)}\int_0^\pi g''(\vartheta)P_n^{((d+1)/2)}(\cos\vartheta)\sin^{d+1}\vartheta\,d\vartheta \\
&\quad - \frac{(n+2)(n+d+1)}{n+d}\int_0^\pi g(\vartheta)P_{n+2}^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta \\
&\quad - \frac{n(n+d-1)}{n+1}\int_0^\pi g(\vartheta)P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta,
\end{aligned}
$$

where the last equality is secured by (6.1), and the fifth equation is derived from formula (4.7.17) of [30]. This proves (2.4).

6.2. Proof of Lemma 2. For $g(\vartheta) = e^{-\lambda\vartheta}$, comparing the left-hand sides of (2.3) and (2.4), we get

$$
\frac{b_{n+2}^{((d-1)/2)}(e^{-\lambda\vartheta})}{b_n^{((d-1)/2)}(e^{-\lambda\vartheta})} = \frac{2n+d+3}{2n+d-1}\,\frac{(n+d-1)(n+d)}{(n+1)(n+2)}\,\frac{n^2+\lambda^2}{(n+d+1)^2+\lambda^2}.
$$

For odd d,

$$
\begin{aligned}
b_0^{((d-1)/2)}(e^{-\lambda\vartheta}) &= \frac{\Gamma(d)}{\sqrt{\pi}}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}\lambda(1-e^{-\lambda\pi})\prod_{k=0}^{(d-1)/2}\big((2k)^2+\lambda^2\big)^{-1}, \\
b_1^{((d-1)/2)}(e^{-\lambda\vartheta}) &= \frac{(d+1)\Gamma(d)}{\sqrt{\pi}}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}\lambda(1+e^{-\lambda\pi})\prod_{k=0}^{(d-1)/2}\big((2k+1)^2+\lambda^2\big)^{-1},
\end{aligned}
$$

and, by induction on n, we obtain (2.6). For even d,

$$b_0^{((d-1)/2)}(e^{-\lambda\vartheta}) = \frac{\Gamma(d)}{\sqrt{\pi}}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}(1+e^{-\lambda\pi})\prod_{k=0}^{d/2-1}\bigl((2k+1)^2+\lambda^2\bigr)^{-1},$$

$$b_1^{((d-1)/2)}(e^{-\lambda\vartheta}) = \frac{(d+1)\Gamma(d)}{\sqrt{\pi}}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}(1-e^{-\lambda\pi})\prod_{k=0}^{d/2-1}\bigl((2k+2)^2+\lambda^2\bigr)^{-1}.$$

Now (2.7) follows by induction on n.

6.3. Proof of Lemma 3. (i) Double factorials of even and odd numbers are related to the Gamma function by the identities

$$(2n)!! = 2^n\Gamma(n+1) \quad \text{and} \quad (2n-1)!! = \frac{2^n\Gamma(n+1/2)}{\sqrt{\pi}},$$

respectively, and so Lemma 4 of [24] reads

$$\int_0^\pi \vartheta P_{2n-1}^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta = -\frac{(d-1)\,2^{2-d}(2n+d-3)!}{(2n-1)!\,(2n-1)^2}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+d/2)}\right)^2$$

and

$$\int_0^\pi \vartheta P_{2n}^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta\,d\vartheta = 0, \qquad n\in\mathbf{N}.$$

As a result, $b_{2n}=0$ and

$$\begin{aligned} b_{2n-1} &= \frac{(d-1)(\Gamma((d-1)/2))^2}{2\pi\Gamma(d-1)}\,\frac{4n+d-3}{(2n-1)^2}\,\frac{\Gamma(2n+d-2)}{\Gamma(2n)}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+d/2)}\right)^2 \\ &\sim \frac{(d-1)(\Gamma((d-1)/2))^2}{2\pi\Gamma(d-1)}\,n^{-2}, \qquad n\to\infty, \end{aligned}$$

where the last asymptotic result follows from the fact that

$$\lim_{x\to\infty}\frac{\Gamma(x+\lambda)}{x^\lambda\Gamma(x)} = 1$$

for $\lambda>0$; see, for instance, formula 8.328.1 of [12].

(ii) In this proof we denote the term b_n in (2.13) by $b_n^{((d-1)/2)}$ to indicate the dependence on d. We employ the following identity for $\nu\in(0,1)$:

$$\vartheta^\nu = \frac{\nu}{\Gamma(1-\nu)}\int_0^\infty (1-e^{-u\vartheta})u^{-\nu-1}\,du, \qquad \vartheta\geqslant 0. \tag{6.2}$$

Since $e^{-u\vartheta}$ is an isotropic covariance function on $\mathbf{S}^d$, we have $b_n^{((d-1)/2)}>0$ $(n\in\mathbf{N})$. Next we prove (2.14). If d is odd, then, by Lemma 2,

$$b_n^{((d-1)/2)}(e^{-\lambda\vartheta}) \sim \frac{2}{\sqrt{\pi}}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}\lambda\bigl(1-(-1)^n e^{-\lambda\pi}\bigr)\frac{n^{d-1}}{(n^2+\lambda^2)^{(d+1)/2}}, \qquad n\to\infty,$$

and so, by identity (6.2),

$$b_n^{((d-1)/2)} \sim A_n - (-1)^n B_n,$$

where

$$\begin{aligned} A_n &= \frac{2\nu}{\Gamma(1-\nu)\sqrt{\pi}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \int_0^\infty u^{-\nu} \frac{n^{d-1}}{(n^2+u^2)^{(d+1)/2}}\,du \\ &= \frac{2\nu}{\Gamma(1-\nu)\sqrt{\pi}\,n^{1+\nu}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \int_0^\infty \frac{dx}{(1+x^2)^{(d+1)/2}x^\nu} \\ &= \frac{2^\nu \nu \Gamma((\nu+d)/2)}{n^{1+\nu}\Gamma(1-\nu/2)\Gamma(d/2)}, \\ B_n &= \frac{2\nu}{\Gamma(1-\nu)\sqrt{\pi}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \int_0^\infty u^{-\nu} e^{-u\pi} \frac{n^{d-1}}{(n^2+u^2)^{(d+1)/2}}\,du \\ &\sim \frac{2\nu}{n^2\Gamma(1-\nu)\sqrt{\pi}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \int_0^\infty u^{-\nu} e^{-u\pi}\,du \\ &= \frac{2\nu}{n^2\sqrt{\pi}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \pi^{\nu-1} \quad \text{for large } n. \end{aligned}$$

Since A_n is dominant over B_n for large n, Lemma 3(ii) is proved for an odd d.

If d is even, then $d+1$ is odd, and we use the following version of formula (4.10.29) of [30]:

$$P_n^{((d-1)/2)}(\cos\vartheta)\sin^{d-1}\vartheta = \sum_{k=0}^{\infty} \phi_{k,n} P_{n+2k}^{(d/2)}(\cos\vartheta)\sin^{2d}\vartheta,$$

where

$$\phi_{k,n} = \frac{4(d-1)\Gamma(n+2d-2)(n+2k+d)(n+2k)!}{n!\,\Gamma(n+2k+2d)}, \qquad k \in \mathbf{N}_0, \quad n \in \mathbf{N}_0.$$

It follows that

$$b_n^{((d-1)/2)} = \sum_{k=0}^{\infty} \frac{\phi_{k,n} P_{n+2k}^{(d/2)}(1)}{P_n^{((d-2)/2)}(1)}\, b_{n+2k}^{(d/2)}.$$

Notice that, for even d, the number $d+1$ is odd, and in this case (2.14) is proved. So, we have

$$\begin{aligned} &\lim_{n\to\infty} b_n^{((d-1)/2)} n^{\nu+1} = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\phi_{k,n} P_{n+2k}^{(d/2)}(1)}{P_n^{((d-2)/2)}(1)} b_{n+2k}^{(d/2)} n^{\nu+1} \\ &= \frac{2^\nu \nu \Gamma((\nu+d+1)/2)}{\Gamma(1-\nu/2)\Gamma((d+1)/2)} \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\phi_{k,n} P_{n+2k}^{(d/2)}(1)}{P_n^{((d-2)/2)}(1)} \frac{n^{\nu+1}}{(n+2k)^{\nu+1}} \\ &= \frac{2^\nu \nu \Gamma((\nu+d+1)/2)}{\Gamma(1-\nu/2)\Gamma((d+1)/2)} \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{2n^{d+\nu}}{(n+2k)^{d+\nu}(n+k)^{1/2}} \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \frac{\Gamma(k+1/2)}{\Gamma(1/2)k!} \\ &= \frac{2^\nu \nu \Gamma((\nu+d+1)/2)}{\Gamma(1-\nu/2)\Gamma(d/2)\Gamma(1/2)} \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{2n^{d+\nu}}{(n+2k)^{d+\nu}((n+k)k)^{1/2}} \\ &= \frac{2^\nu \nu \Gamma((\nu+d+1)/2)}{\Gamma(1-\nu/2)\Gamma(d/2)\Gamma(1/2)} \int_0^\infty \frac{2\,dx}{(1+2x)^{d+\nu}(x+x^2)^{1/2}} \\ &= \frac{2^\nu \nu \Gamma((\nu+d+1)/2)}{\Gamma(1-\nu/2)\Gamma(d/2)\Gamma(1/2)} \int_0^\infty \frac{dy}{(1+y)^{(d+\nu+1)/2}y^{1/2}} \\ &= \frac{2^\nu \nu \Gamma((\nu+d)/2)}{\Gamma(1-\nu/2)\Gamma(d/2)}, \end{aligned}$$

where in the derivation of the fourth equality we approximate $\Gamma(k+1/2)/k!$ by $k^{-1/2}$ because for $k > \sqrt{n}$ the relative error is $O(n^{-1/2})$, while the contribution of the terms with $k \leqslant \sqrt{n}$ in the summation or, equivalently, in the integral, is $O(n^{-1/4})$.

6.4. Proof of Lemma 4. For odd d ($d = 2k+1$, say) by Theorem 5 of [21], there exists $\xi \in [n, n+d-1]$ such that

$$b_n^{((d-1)/2)} = \frac{(2n+d-1)\Gamma(n+d-1)}{n!\,\Gamma(d/2)\sqrt{\pi}}\, g(\xi),$$

where

$$\begin{aligned} g(x) &= \left(-\frac{d}{2x\,dx}\right)^k \int_0^\pi \left(1-\frac{u}{\varepsilon}\right)_+^d \cos(xu)\,du \\ &= \varepsilon^d \left(-\frac{d}{2t\,dt}\right)^k \int_0^1 (1-y)^d \cos(ty)\,dy, \end{aligned}$$

and the last equality is obtained by making a transform in which $t = \varepsilon x$. A direct evaluation shows that

$$\int_0^1 (1-y)^d \cos(ty)\,dy = \sum_{i=0}^k \frac{(-1)^i d!}{(d-2i-1)!\,t^{2i+2}} - (-1)^k \frac{d!}{t^{d+1}} \cos t,$$

and, subsequently,

$$\left(-\frac{d}{2t\,dt}\right)^k \int_0^1 (1-y)^d \cos(ty)\,dy = \frac{k!\,d}{t^{d+1}} + O\!\left(\frac{1}{t^{d+3}}\right) + O\!\left(\frac{d!}{t^{d+1}(2t)^k}\right) = O\!\left(\frac{1}{t^{d+1}}\right).$$

Noticing that $\xi \in [n, n+d-1]$, we have

$$b_n^{((d-1)/2)} = O(n^{d-1})\,\varepsilon^d O\big((n\varepsilon)^{-(d+1)}\big) = \varepsilon O\big((n\varepsilon)^{-2}\big).$$

For even d, we can show similarly that $b_n^{(d/2)} = \varepsilon O((n\varepsilon)^{-2})$. Notice that, for any continuous function $g(x)$ on $[0,\pi]$ and $\nu > 0$, the associated $b_n^{((d-1)/2)}(g)$ satisfies

$$\limsup_{n\to\infty} |b_n^{((d-1)/2)}(g)| n^{\nu+1} \leqslant \frac{\Gamma((\nu+d)/2)\Gamma((d+1)/2)}{\Gamma((\nu+d+1)/2)\Gamma(d/2)} \limsup_{n\to\infty} |b_n^{(d/2)}(g)| n^{\nu+1}.$$

Substituting $\nu = 1$ we get $b_n^{((d-1)/2)} = \varepsilon O((n\varepsilon)^{-2})$ for even d. As a result, inequality (2.15) is obtained for a $\beta_0 > 0$.

6.5. Proof of Theorem 1. We give a proof of part (ii) only, while part (i) can be derived similarly.

Since the positive series $\sum_{n=1}^\infty b_n$ is convergent by Lemma 3(ii), the series on the right-hand side of (3.3) converges in mean square. Indeed, for $\mathbf{x} \in \mathbf{B}^d$ and $n_k \in \mathbf{N}$

$(k=1,2)$, we have

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{n=n_1}^{n_1+n_2}\Big(\frac{b_n}{P_n^{((d-1)/2)}(1)}\Big)^{1/2}V_n\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\Big]^2\\
&\quad=\sum_{n=n_1}^{n_1+n_2}\sum_{l=n_1}^{n_1+n_2}\Big(\frac{b_nb_l}{P_n^{((d-1)/2)}(1)P_l^{((d-1)/2)}(1)}\Big)^{1/2}\mathbf{E}(V_nV_l)\\
&\qquad\times\mathbf{E}\big[\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\\
&\qquad\quad\times\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\big]\\
&\quad=\sum_{n=n_1}^{n_1+n_2}\frac{b_n}{P_n^{((d-1)/2)}(1)}\,\alpha_n^2\operatorname{var}\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\\
&\quad=\sum_{n=n_1}^{n_1+n_2}b_n\Big(1-\frac{P_n^{((d-1)/2)}(\cos\rho(\mathbf{x},\mathbf{x}_0))}{P_n^{((d-1)/2)}(1)}\Big)\to 0,\qquad n_1\to\infty,\quad n_2\to\infty,
\end{aligned}$$

where the second equality follows from the independence assumption between $\mathbf{U}$ and $\{V_n,\, n\in\mathbf{N}\}$, the third equality is secured by Lemma 2 of [25], and the last one is a consequence of Lemma 3(ii).

It follows from Lemma 2 of [25] that $\mathbf{E}Z(\mathbf{x})=0$, and, from (1.1) and (2.12), we obtain

$$\begin{aligned}
&\operatorname{cov}(Z(\mathbf{x}_1),Z(\mathbf{x}_2))\\
&\quad=\sum_{n=1}^{\infty}\frac{b_n}{P_n^{((d-1)/2)}(1)}\mathbf{E}(V_n^2)\,\mathbf{E}\big\{\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_1'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\\
&\qquad\times\big(P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_2'\mathbf{U})-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\mathbf{U})\big)\big\}\\
&\quad=\sum_{n=1}^{\infty}\frac{b_n}{P_n^{((d-1)/2)}(1)}\big\{P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_1'\widetilde{\mathbf{x}}_2)-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\widetilde{\mathbf{x}}_1)\\
&\qquad-P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_0'\widetilde{\mathbf{x}}_2)+P_n^{((d-1)/2)}(1)\big\}\\
&\quad=\rho^\nu(\mathbf{x}_1,\mathbf{x}_0)+\rho^\nu(\mathbf{x}_2,\mathbf{x}_0)-\rho^\nu(\mathbf{x}_1,\mathbf{x}_2),\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbf{B}^d.
\end{aligned}$$

Moreover, the variogram of $\{Z(\mathbf{x}),\mathbf{x}\in\mathbf{B}^d\}$ is

$$\begin{aligned}
\frac{1}{2}\operatorname{var}(Z(\mathbf{x}_1)-Z(\mathbf{x}_2))&=\frac{1}{2}\{\operatorname{var}(Z(\mathbf{x}_1))+\operatorname{var}(Z(\mathbf{x}_2))-2\operatorname{cov}(Z(\mathbf{x}_1),Z(\mathbf{x}_2))\}\\
&=\rho^\nu(\mathbf{x}_1,\mathbf{x}_2),\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbf{B}^d.
\end{aligned}$$

6.6. Proof of Theorem 2. (i) Since $\rho^\nu(\mathbf{x}_1,\mathbf{x}_2)$ is the variogram of the random field (3.2) or (3.3) on $\mathbf{B}^d$, it has to be conditionally negative definite for a constant $\nu\in(0,1]$.

To show that $\rho^\nu(\mathbf{x}_1,\mathbf{x}_2)$ is conditionally negative definite on $\mathbf{B}^d$ if and only if $\nu\in(0,1]$, it suffices to provide a counterexample that violates inequality (3.4) if $\nu>1$.

Let $n=4$. For $d\geqslant 2$, we take the following four points on $\mathbf{B}^d$:

$$\begin{aligned}
\mathbf{x}_1&=(1,0,0,\dots,0)', &\qquad \mathbf{x}_2&=(0,1,0,\dots,0)',\\
\mathbf{x}_3&=(-1,0,0,\dots,0)', &\qquad \mathbf{x}_4&=(0,-1,0,\dots,0)'.
\end{aligned}$$

We have

$$\arccos(\widetilde{\mathbf{x}}_1'\widetilde{\mathbf{x}}_2) = \arccos(\widetilde{\mathbf{x}}_1'\widetilde{\mathbf{x}}_4) = \arccos(\widetilde{\mathbf{x}}_2'\widetilde{\mathbf{x}}_3) = \arccos(\widetilde{\mathbf{x}}_3'\widetilde{\mathbf{x}}_4) = \arccos 0 = s\frac{\pi}{2},$$
$$\arccos(\widetilde{\mathbf{x}}_1'\widetilde{\mathbf{x}}_3) = \arccos(\widetilde{\mathbf{x}}_2'\widetilde{\mathbf{x}}_4) = \arccos(-1) = \pi.$$

For $a_4 = -a_1 - a_2 - a_3$,

$$\begin{aligned}
-\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} a_i a_j \rho^\nu(\mathbf{x}_i,\mathbf{x}_j) &= a_1^2\rho^\nu(\mathbf{x}_1,\mathbf{x}_4) + a_2^2\rho^\nu(\mathbf{x}_2,\mathbf{x}_4) + a_3^2\rho^\nu(\mathbf{x}_3,\mathbf{x}_4) \\
&\quad + a_1a_2\big(\rho^\nu(\mathbf{x}_1,\mathbf{x}_4) + \rho^\nu(\mathbf{x}_2,\mathbf{x}_4) - \rho^\nu(\mathbf{x}_1,\mathbf{x}_2)\big) \\
&\quad + a_1a_3\big(\rho^\nu(\mathbf{x}_1,\mathbf{x}_4) + \rho^\nu(\mathbf{x}_3,\mathbf{x}_4) - \rho^\nu(\mathbf{x}_1,\mathbf{x}_3)\big) \\
&\quad + a_2a_3\big(\rho^\nu(\mathbf{x}_2,\mathbf{x}_4) + \rho^\nu(\mathbf{x}_3,\mathbf{x}_4) - \rho^\nu(\mathbf{x}_2,\mathbf{x}_3)\big) \\
&= \left(\frac{\pi}{2}\right)^\nu \big(a_1^2 + 2^\nu a_2^2 + a_3^2 + 2^\nu a_1a_2 + (2-2^\nu)\, a_1a_3 + 2^\nu a_2a_3\big) \\
&= \left(\frac{\pi}{2}\right)^\nu (a_1,a_2,a_3)\begin{pmatrix} 1 & 2^{\nu-1} & 1-2^{\nu-1} \\ 2^{\nu-1} & 2^\nu & 2^{\nu-1} \\ 1-2^{\nu-1} & 2^{\nu-1} & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.
\end{aligned}$$

Note that if $\nu > 1$, then the determinant is

$$\det(\mathbf{A}) := \begin{vmatrix} 1 & 2^{\nu-1} & 1-2^{\nu-1} \\ 2^{\nu-1} & 2^\nu & 2^{\nu-1} \\ 1-2^{\nu-1} & 2^{\nu-1} & 1 \end{vmatrix} = 2^{2\nu}(1-2^{\nu-1}) < 0.$$

Let $\mathbf{a} = (a_1,a_2,a_3)'$ be the eigenvector that is associated with the negative eigenvalue of $\mathbf{A}$. Clearly, $\mathbf{a}'\mathbf{A}\mathbf{a} < 0$ and thus $\sum_{i=1}^{4}\sum_{j=1}^{4} a_ia_j\rho^\nu(\mathbf{x}_i,\mathbf{x}_j) > 0$, which violates inequality (3.4).

(ii) This is a consequence of part (i).

(iii) If $\nu\kappa \leqslant 1$, then $\rho^{\nu\kappa}(\mathbf{x}_1,\mathbf{x}_2)$ is a variogram on $\mathbf{B}^d$, by part (i). Taking $m=1$ in Corollary 2.1 of [23] we obtain the covariance function (1.5), and from Corollary 3.4 of [23] we reach the covariance function (3.5) of a trifractional Brownian motion on $\mathbf{B}^d$.

(iv) This follows from part (ii) and Theorem 2.2 of [22], and the latter ensures the existence of a quadrifractional Brownian motion with the covariance function (3.6).

6.7. Proof of Theorem 3. We give a proof of part (ii) only, while part (i) can be derived similarly.

According to Lemma 3(ii), the positive series $\sum_{n=1}^\infty b_n$ converges, and it implies the mean square convergence of the series on the right-hand side of (3.7). In fact, for $\mathbf{x} \in \mathbf{B}^d$ and $n_k \in \mathbf{N}$ $(k=1,2)$, applying (2.2) and noting that $c_{n,d} = \alpha_n^2 P_n^{((d-1)/2)}(1)$, we obtain

$$\begin{aligned}
&\mathbf{E}\left|\sum_{n=n_1}^{n_1+n_2}\left(\frac{b_n}{P_n^{((d-1)/2)}(1)}\right)^{1/2}\sum_{j=1}^{c_{n,d}} V_{nj}\{S_{nj}(\widetilde{\mathbf{x}}) - S_{nj}(\widetilde{\mathbf{x}}_0)\}\right|^2 \\
&\quad= \sum_{n=n_1}^{n_1+n_2}\frac{b_n}{P_n^{((d-1)/2)}(1)}\sum_{j=1}^{c_{n,d}}\operatorname{var}(V_{nj})\{S_{nj}(\widetilde{\mathbf{x}}) - S_{nj}(\widetilde{\mathbf{x}}_0)\}^2 \\
&\quad= \sum_{n=n_1}^{n_1+n_2}\frac{b_n\omega_d\alpha_n^{-2}}{P_n^{((d-1)/2)}(1)}\sum_{j=1}^{c_{n,d}}\{S_{nj}(\widetilde{\mathbf{x}}) - S_{nj}(\widetilde{\mathbf{x}}_0)\}^2
\end{aligned}$$

$$= \sum_{n=n_1}^{n_1+n_2} b_n \frac{\omega_d \alpha_n^{-2}}{P_n^{((d-1)/2)}(1)} \frac{2c_{n,d}}{\omega_d} \left(1 - \frac{P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\widetilde{\mathbf{x}}_0)}{P_n^{((d-1)/2)}(1)}\right)$$

$$= 2 \sum_{n=n_1}^{n_1+n_2} b_n \left(1 - \frac{P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}'\widetilde{\mathbf{x}}_0)}{P_n^{((d-1)/2)}(1)}\right) \to 0, \qquad n_1 \to \infty, \quad n_2 \to \infty.$$

Let $Z(\mathbf{x})$ be the right-hand side of (3.8). Clearly, the mean function of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{B}^d\}$ is identically 0, and by (2.2) its variogram is given by

$$\frac{1}{2} \operatorname{var}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2)) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{b_n}{P_n^{((d-1)/2)}(1)} \sum_{j=1}^{c_{n,d}} \operatorname{var}(V_{nj}) \{S_{nj}(\widetilde{\mathbf{x}}_1) - S_{nj}(\widetilde{\mathbf{x}}_2)\}^2$$
$$= \sum_{n=1}^{\infty} b_n \left(1 - \frac{P_n^{((d-1)/2)}(\widetilde{\mathbf{x}}_1'\widetilde{\mathbf{x}}_2)}{P_n^{((d-1)/2)}(1)}\right) = \rho^\nu(\mathbf{x}_1, \mathbf{x}_2);$$

its covariance function is given by

$$\operatorname{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \rho^\nu(\mathbf{x}_1, \mathbf{x}_0) + \rho^\nu(\mathbf{x}_2, \mathbf{x}_0) - \rho^\nu(\mathbf{x}_1, \mathbf{x}_2), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}^d.$$

6.8. Proof of Theorem 4. The proof is similar to that of Theorem 1.

6.9. Proof of Theorem 5. It can be derived analogously to that of Theorem 3.

6.10. Proof of Theorem 6. Since $\beta\varepsilon^\nu$ increases in ε, we need only prove (5.3) for sufficiently small ε.

For the Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbf{S}^d\}$, in order to verify inequality (5.3), it suffices to show that there is a positive constant β such that the inequality

$$\mathbf{E}\left(Z(\mathbf{x}) - \sum_{k=1}^{n} a_k Z(\mathbf{x}_k)\right)^2 \geqslant \beta\varepsilon^\nu \tag{6.3}$$

holds for every $a_k \in \mathbf{R}$ $(k = 1, \ldots, n)$. We have

$$\mathbf{E}\left(Z(\mathbf{x}) - \sum_{k=1}^{n} a_k Z(\mathbf{x}_k)\right)^2$$
$$= C(\mathbf{x}, \mathbf{x}) - 2 \sum_{i=1}^{n} a_i C(\mathbf{x}, \mathbf{x}_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j C(\mathbf{x}_i, \mathbf{x}_j)$$
$$= \sum_{l=0}^{\infty} b_l \left(1 - \sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}, \mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\right)^2$$
$$+ \sum_{l=0}^{\infty} b_l \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \left(\frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}_i, \mathbf{x}_j))}{P_l^{((d-1)/2)}(1)} - \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}, \mathbf{x}_i))}{P_l^{((d-1)/2)}(1)} \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}, \mathbf{x}_j))}{P_l^{((d-1)/2)}(1)} \right)$$
$$\geqslant \sum_{l=0}^{\infty} b_l \left(1 - \sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}, \mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\right)^2,$$

where the last inequality holds, since, for every $l \in \mathbf{N}$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x}_i,\mathbf{x}_j))}{P_l^{((d-1)/2)}(1)} \geqslant \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_i))}{P_l^{((d-1)/2)}(1)} \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_j))}{P_l^{((d-1)/2)}(1)}. \tag{6.4}$$

To verify inequality (6.4), consider an isotropic random field

$$Y_l(\mathbf{y}) = \alpha_l \frac{P_l^{((d-1)/2)}(\mathbf{y}'\mathbf{U})}{\sqrt{P_l^{((d-1)/2)}(1)}}, \qquad \mathbf{y} \in \mathbf{S}^d,$$

where $\mathbf{U}$ is a $(d+1)$-variate random vector uniformly distributed on $\mathbf{S}^d$. Using Lemma 2 of [25], we see that

$$\operatorname{cov}(Y_l(\mathbf{y}_1), Y_l(\mathbf{y}_2)) = \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{y}_1,\mathbf{y}_2))}{P_l^{((d-1)/2)}(1)}, \qquad \mathbf{y}_1,\mathbf{y}_2 \in \mathbf{S}^d,$$

and inequality (6.4) follows from the Cauchy–Schwarz inequality

$$\mathbf{E}Y_l^2(\mathbf{x})\,\mathbf{E}\left(\sum_{i=1}^{n} a_i Y_l(\mathbf{x}_i)\right)^2 \geqslant \left\{\mathbf{E}\left(Y_l(\mathbf{x})\sum_{i=1}^{n} a_i Y_l(\mathbf{x}_i)\right)\right\}^2.$$

By Theorem 2.1 of [35] or Theorem 4 of [21], there exists an isotropic Gaussian random field $\{Y(\mathbf{y}), \mathbf{y} \in \mathbf{S}^d\}$ with a compactly supported covariance function

$$\operatorname{cov}(Y(\mathbf{y}_1), Y(\mathbf{y}_2)) = \left\{\max\left(1 - \frac{\vartheta(\mathbf{y}_1,\mathbf{y}_2)}{\varepsilon}, 0\right)\right\}^d, \qquad \mathbf{y}_1,\mathbf{y}_2 \in \mathbf{S}^d,$$

whose ultraspherical expansion is

$$\left\{\max\left(1 - \frac{\vartheta(\mathbf{y}_1,\mathbf{y}_2)}{\varepsilon}, 0\right)\right\}^d = \sum_{l=0}^{\infty} b_l(\varepsilon)\frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{y}_1,\mathbf{y}_2))}{P_l^{((d-1)/2)}(1)}, \qquad \mathbf{y}_1,\mathbf{y}_2 \in \mathbf{S}^d,$$

where the positive series $\sum_{l=0}^{\infty} b_l(\varepsilon)$ converges to 1. According to Lemma 4, there is a positive constant β_0 such that

$$b_l(\varepsilon) \leqslant \frac{\beta_0\varepsilon}{(1+l\varepsilon)^2}, \qquad l \in \mathbf{N}_0.$$

Consequently, it follows from (5.2) that, for $\varepsilon \leqslant 1$,

$$\frac{b_l^2(\varepsilon)}{b_l} \leqslant \frac{\beta_0^2\varepsilon^2}{\beta_1}\frac{(1+l)^{1+\nu}}{(1+l\varepsilon)^4} = \frac{\beta_0^2\varepsilon^{-\nu+1}}{\beta_1}\frac{(\varepsilon+l\varepsilon)^{1+\nu}}{(1+l\varepsilon)^4} \leqslant \frac{\beta_0^2\varepsilon^{-\nu+1}}{\beta_1}(1+l\varepsilon)^{-(3-\nu)}$$

and

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{b_l^2(\varepsilon)}{b_l} &\leqslant \frac{\beta_0^2\varepsilon^{-\nu}}{\beta_1}\sum_{l=0}^{\infty}\frac{\varepsilon}{(1+l\varepsilon)^{3-\nu}} \leqslant \frac{\beta_0^2\varepsilon^{-\nu}}{\beta_1}\left(\varepsilon + \int_0^{\infty}\frac{dx}{(1+x)^{3-\nu}}\right)\\ &\leqslant \frac{\beta_0^2\varepsilon^{-\nu}}{\beta_1}\left(1 + \frac{1}{2-\nu}\right) = \frac{1}{\beta\varepsilon^{\nu}},\end{aligned}$$

where $\beta = (2-\nu)\beta_1/((3-\nu)\beta_0^2)$.

Now we consider

$$I=\sum_{l=0}^{\infty} b_l(\varepsilon)\Bigg(1-\sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\Bigg).$$

On the one hand, it follows from $\vartheta(\mathbf{x},\mathbf{x}_k)\geqslant \varepsilon$ $(k=1,\ldots,n)$ that

$$\begin{aligned}I&=\sum_{l=0}^{\infty} b_l(\varepsilon)-\sum_{k=1}^{n} a_k\sum_{l=0}^{\infty} b_l(\varepsilon)\frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\\&=1-\sum_{k=1}^{n} a_k\Bigg\{\max\Bigg(1-\frac{\vartheta(\mathbf{x},\mathbf{x}_k)}{\varepsilon},0\Bigg)\Bigg\}^d=1.\end{aligned}$$

On the other hand, an application of the Cauchy–Schwarz inequality yields that

$$\begin{aligned}I^2&=\Bigg\{\sum_{l=0}^{\infty}\frac{b_l(\varepsilon)}{\sqrt{b_l}}\sqrt{b_l}\Bigg(1-\sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\Bigg)\Bigg\}^2\\&\leqslant\sum_{l=0}^{\infty}\frac{b_l^2(\varepsilon)}{b_l}\sum_{l=0}^{\infty} b_l\Bigg(1-\sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\Bigg)^2\\&\leqslant\frac{1}{\beta\varepsilon^2}\sum_{l=0}^{\infty} b_l\Bigg(1-\sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\Bigg)^2,\end{aligned}$$

and, thus,

$$\begin{aligned}&\mathbf{E}\Bigg(Z(\mathbf{x})-\sum_{k=1}^{n} a_k Z(\mathbf{x}_k)\Bigg)^2\\&\qquad\geqslant\sum_{l=0}^{\infty} b_l\Bigg(1-\sum_{k=1}^{n} a_k \frac{P_l^{((d-1)/2)}(\cos\vartheta(\mathbf{x},\mathbf{x}_k))}{P_l^{((d-1)/2)}(1)}\Bigg)^2\geqslant\beta\varepsilon^2.\end{aligned}$$

6.11. Proof of Theorem 7. (i) Note that (4.4) implies $Z(\mathbf{x})=Y(\mathbf{x})-Y(\mathbf{x}_0)$, $\mathbf{x}\in\mathbf{S}^d$, where

$$Y(\mathbf{x})=\sum_{n=0}^{\infty}\Bigg(\frac{b_n}{P_n^{((d-1)/2)}(1)}\Bigg)^{1/2}\sum_{j=1}^{c_{n,d}} V_{nj}S_{nj}(\mathbf{x}),\qquad \mathbf{x}\in\mathbf{S}^d,$$

is an isotropic and mean square continuous Gaussian random field on $\mathbf{S}^d$ and possesses the covariance function

$$\operatorname{cov}(Y(\mathbf{x}_1),Y(\mathbf{x}_2))=\sum_{n=0}^{\infty} b_n\frac{P_n^{((d-1)/2)}(\cos(\mathbf{x}_1,\mathbf{x}_2))}{P_n^{((d-1)/2)}(1)},\qquad \mathbf{x}_1,\mathbf{x}_2\in\mathbf{S}^d,$$

where $b_0>0$, and the sequence $\{b_n,\, n\in\mathbf{N}_0\}$ and its rate given in Lemma 3(ii) satisfy inequality (5.2). It follows from Theorem 6 that there is a positive constant β

such that

$$\begin{aligned}\operatorname{var}(Z(\mathbf{x})\,|\,Z(\mathbf{x}_1),\ldots,Z(\mathbf{x}_n)) &= \inf \mathbf{E}\Big(Z(\mathbf{x})-\sum_{k=1}^n a_k Z(\mathbf{x}_k)\Big)^2\\ &= \inf \mathbf{E}\Big(Y(\mathbf{x})-Y(\mathbf{x}_0)-\sum_{k=1}^n a_k (Y(\mathbf{x}_k)-Y(\mathbf{x}_0))\Big)^2\\ &= \inf \mathbf{E}\Big\{Y(\mathbf{x})-\Big(1-\sum_{k=1}^n a_k\Big)Y(\mathbf{x}_0)-\sum_{k=1}^n a_k Y(\mathbf{x}_k)\Big\}^2 \geqslant \beta\varepsilon^2\end{aligned}$$

holds for every $n \in \mathbf{N}$, any $a_k \in \mathbf{R}$, any $\mathbf{x} \in \mathbf{S}^d$, any $\mathbf{x}_k \in \mathbf{S}^d$, $k = 1, \ldots, n$, and any $\varepsilon \in (0, \min_{0\leqslant k\leqslant n} \vartheta(\mathbf{x}, \mathbf{x}_k)]$.

(ii) For $\kappa \in [1/2, 1]$, it follows from Theorem 2.2 of [22] that there is a quadri-fractional Brownian motion $\{Z_1(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ with covariance function

$$\begin{aligned}\operatorname{cov}(Z_1(\mathbf{x}_1), Z_2(\mathbf{x}_2)) &= \big(\vartheta^{\nu\kappa}(\mathbf{x}_1,\mathbf{x}_0)+\vartheta^{\nu\kappa}(\mathbf{x}_2,\mathbf{x}_0)\big)^{1/\kappa}\\ &\quad -\vartheta^{\nu}(\mathbf{x}_1,\mathbf{x}_0)-\vartheta^{\nu}(\mathbf{x}_2,\mathbf{x}_0), \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathbf{S}^d.\end{aligned}$$

Clearly, the difference of the covariance function of the bifractional Brownian motion $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ and that of $\{Z_1(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ is $\vartheta^{\nu}(\mathbf{x}_1,\mathbf{x}_0)+\vartheta^{\nu}(\mathbf{x}_2,\mathbf{x}_0)-\vartheta^{\nu}(\mathbf{x}_1,\mathbf{x}_2)$, which is the covariance function of a fractional Brownian motion $\{Z_2(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$. By Theorem 2 of [32], $\{Z_2(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ is smaller than $\{Z(\mathbf{x}),\, \mathbf{x} \in \mathbf{S}^d\}$ in the convex order, so that

$$\mathbf{E}g\big(Z_2(\mathbf{x}), Z_2(\mathbf{x}_1),\ldots,Z_2(\mathbf{x}_n)\big) \leqslant \mathbf{E}g\big(Z(\mathbf{x}), Z(\mathbf{x}_1),\ldots,Z(\mathbf{x}_n)\big)$$

for every convex function $g(x, x_1, \ldots, x_n)$. In particular,

$$g(x, x_1, \ldots, x_n) = \Big(x-\sum_{k=1}^n a_k x_k\Big)^2$$

is a convex function. From part (i) we obtain

$$\begin{aligned}\operatorname{var}(Z(\mathbf{x})|Z(\mathbf{x}_1),\ldots,Z(\mathbf{x}_n)) &= \inf \mathbf{E}\Big(Z(\mathbf{x})-\sum_{k=1}^n a_k Z(\mathbf{x}_k)\Big)^2\\ &\geqslant \inf \mathbf{E}\Big(Z_2(\mathbf{x})-\sum_{k=1}^n a_k Z_2(\mathbf{x}_k)\Big)^2 \geqslant \beta\varepsilon^2.\end{aligned}$$

6.12. Proof of Theorem 8. We give a proof for part (i) only, while part (ii) can be derived analogously to the proof of Theorem 7(ii).

For $\mathbf{x}, \mathbf{x}_k \in \mathbf{B}^d$ $(k = 1, \ldots, n)$, it follows from (1.4) and (2.12) that

$$\begin{aligned}&\mathbf{E}\Big(Z(\mathbf{x})-\sum_{k=1}^n a_k Z(\mathbf{x}_k)\Big)^2\\ &\quad = C(\mathbf{x},\mathbf{x})-2\sum_{i=1}^n a_i C(\mathbf{x},\mathbf{x}_i)+\sum_{i=1}^n\sum_{j=1}^n a_i a_j C(\mathbf{x}_i,\mathbf{x}_j)\end{aligned}$$

$$
\begin{aligned}
&= 2\rho^{\nu}(\mathbf{x},\mathbf{x}_0) - 4\sum_{i=1}^{n} a_i\big(\rho^{\nu}(\mathbf{x},\mathbf{x}_0) + \rho^{\nu}(\mathbf{x}_i,\mathbf{x}_0) - \rho^{\nu}(\mathbf{x},\mathbf{x}_i)\big)\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\big(\rho^{\nu}(\mathbf{x}_i,\mathbf{x}_0) + \rho^{\nu}(\mathbf{x}_j,\mathbf{x}_0) - \rho^{\nu}(\mathbf{x}_i,\mathbf{x}_j)\big)\\
&= 2\vartheta^{\nu}(\widetilde{\mathbf{x}},\widetilde{\mathbf{x}}_0) - 4\sum_{i=1}^{n} a_i\big(\vartheta^{\nu}(\widetilde{\mathbf{x}},\widetilde{\mathbf{x}}_0) + \vartheta^{\nu}(\widetilde{\mathbf{x}}_i,\widetilde{\mathbf{x}}_0) - \vartheta^{\nu}(\widetilde{\mathbf{x}},\widetilde{\mathbf{x}}_i)\big)\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\big(\vartheta^{\nu}(\widetilde{\mathbf{x}}_i,\widetilde{\mathbf{x}}_0) + \vartheta^{\nu}(\widetilde{\mathbf{x}}_j,\widetilde{\mathbf{x}}_0) - \vartheta^{\nu}(\widetilde{\mathbf{x}}_i,\widetilde{\mathbf{x}}_j)\big) \geqslant \beta\varepsilon^2
\end{aligned}
$$

holds for every $n \in \mathbf{N}$, any $\widetilde{\mathbf{x}} \in \mathbf{S}^d$, any $\widetilde{\mathbf{x}}_k \in \mathbf{S}^d$ $(k = 0, 1, \ldots, n)$, and any $\varepsilon \in (0, \min_{0\leqslant k\leqslant n} \vartheta(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}_k)] = (0, \min_{0\leqslant k\leqslant n} \rho(\mathbf{x}, \mathbf{x}_k)]$, where the last inequality follows from Theorem 7(i).

Acknowledgment. The referee's helpful comments and useful suggestions are gratefully acknowledged.

REFERENCES

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia Math. Appl. 71, Cambridge Univ. Press, Cambridge, 1999, https://doi.org/10.1017/CBO9781107325937.

[2] N. H. Bingham, *Positive definite functions on spheres*, Proc. Cambridge Philos. Soc., 73 (1973), pp. 145–156, https://doi.org/10.1017/S0305004100047551.

[3] N. H. Bingham, A. Mijatović, and T. L. Symons, *Brownian manifolds, negative type and geo-temporal covariances*, Commun. Stoch. Anal., 10 (2016), 3, https://doi.org/10.31390/cosa.10.4.03.

[4] L. Bos, N. Levenberg, and S. Waldron, *Metrics associated to multivariate polynomial inequalities*, in Advances in Constructive Approximation: Vanderbilt 2003 (Nashville, TN, 2003), Mod. Methods Math., Nashboro Press, Brentwood, TN, 2004, pp. 133–147.

[5] D. Cheng and P. Liu, *Extremes of spherical fractional Brownian motion*, Extremes, 22 (2019), pp. 433–457, https://doi.org/10.1007/s10687-019-00344-4.

[6] P. Chigansky and M. Kleptsyna, *Exact asymptotics in eigenproblems for fractional Brownian covariance operators*, Stochastic Process. Appl., 128 (2018), pp. 2007–2059, https://doi.org/10.1016/j.spa.2017.08.019.

[7] S. Cohen and M. A. Lifshits, *Stationary Gaussian random fields on hyperbolic spaces and on Euclidean spheres*, ESAIM Probab. Stat., 16 (2012), pp. 165–221, https://doi.org/10.1051/ps/2011105.

[8] F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, Springer Monogr. Math., Springer, New York, 2013, https://doi.org/10.1007/978-1-4614-6660-4.

[9] K. Dzhaparidze and H. van Zanten, *A series expansion of fractional Brownian motion*, Probab. Theory Related Fields, 130 (2004), pp. 39–55, https://doi.org/10.1007/s00440-003-0310-2.

[10] K. Dzhaparidze and H. van Zanten, *Optimality of an explicit series expansion of the fractional Brownian sheet*, Statist. Probab. Lett., 71 (2005), pp. 295–301, https://doi.org/10.1016/j.spl.2004.11.004.

[11] K. Dzhaparidze, H. van Zanten, and P. Zareba, *Representations of fractional Brownian motion using vibrating strings*, Stochastic Process. Appl., 115 (2005), pp. 1928–1953, https://doi.org/10.1016/j.spa.2005.06.006.

[12] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier/Academic Press, Amsterdam, 2007.

[13] J. Istas, *Karhunen–Loève expansion of spherical fractional Brownian motions*, Statist. Probab. Lett., 76 (2006), pp. 1578–1583, https://doi.org/10.1016/j.spl.2006.03.019.

[14] X. Lan, D. Marinucci, and Y. Xiao, *Strong local nondeterminism and exact modulus of continuity for spherical Gaussian fields*, Stochastic Process. Appl., 128 (2018), pp. 1294–1315, https://doi.org/10.1016/j.spa.2017.07.008.

[15] X. Lan and Y. Xiao, *Strong local nondeterminism of spherical fractional Brownian motion*, Statist. Probab. Lett., 135 (2018), pp. 44–50, https://doi.org/10.1016/j.spl.2017.11.007.

[16] A. Lang and C. Schwab, *Isotropic Gaussian random fields on the sphere: Regularity, fast simulation and stochastic partial differential equations*, Ann. Appl. Probab., 25 (2015), pp. 3047–3094, https://doi.org/10.1214/14-AAP1067.

[17] N. N. Leonenko and M. D. Ruiz-Medina, *Increasing domain asymptotics for the first Minkowski functional of spherical random fields*, Theory Probab. Math. Statist., 97 (2018), pp. 127–149, https://doi.org/10.1090/tpms/1053.

[18] N. Leonenko and L. Sakhno, *On spectral representation of tensor random fields on the sphere*, Stoch. Anal. Appl., 30 (2012), pp. 44–66, https://doi.org/10.1080/07362994.2012.628912.

[19] M. Lifshits and K. Volkova, *Bifractional Brownian motion: Existence and border cases*, ESAIM Probab. Stat., 19 (2015), pp. 766–781, https://doi.org/10.1051/ps/2015015.

[20] T. Lu, N. Leonenko, and C. Ma, *Series representations of isotropic vector random fields on balls*, Statist. Probab. Lett., 156 (2020), 108583, https://doi.org/10.1016/j.spl.2019.108583.

[21] T. Lu and C. Ma, *Isotropic covariance matrix functions on compact two-point homogeneous spaces*, J. Theoret. Probab., 33 (2020), pp. 1630–1656, https://doi.org/10.1007/s10959-019-00920-1.

[22] C. Ma, *The Schoenberg–Lévy kernel and relationships among fractional Brownian motion, bifractional Brownian motion, and others*, Theory Probab. Appl., 57 (2013), pp. 619–632, https://doi.org/10.1137/S0040585X97986230.

[23] C. Ma, *Multifractional vector Brownian motions, their decompositions, and generalizations*, Stoch. Anal. Appl., 33 (2015), pp. 535–548, https://doi.org/10.1080/07362994.2015.1017108.

[24] C. Ma, *Isotropic covariance matrix polynomials on spheres*, Stoch. Anal. Appl., 34 (2016), pp. 679–706, https://doi.org/10.1080/07362994.2016.1170612.

[25] C. Ma, *Time varying isotropic vector random fields on spheres*, J. Theoret. Probab., 30 (2017), pp. 1763–1785, https://doi.org/10.1007/s10959-016-0689-1.

[26] C. Ma and A. Malyarenko, *Time-varying isotropic vector random fields on compact two-point homogeneous spaces*, J. Theoret. Probab., 33 (2020), pp. 319–339, https://doi.org/10.1007/s10959-018-0872-7.

[27] A. Malyarenko, *Invariant Random Fields on Spaces with a Group Action*, Probab. Appl. (N.Y.), Springer, Heidelberg, 2013, https://doi.org/10.1007/978-3-642-33406-1.

[28] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, eds., *NIST Handbook of Mathematical Functions*, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC, 2010.

[29] L. D. Pitt, *Local times for Gaussian vector fields*, Indiana Univ. Math. J., 27 (1978), pp. 309–330, https://doi.org/10.1512/iumj.1978.27.27024.

[30] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 1975.

[31] M. Talagrand, *Hausdorff measure of trajectories of multiparameter fractional Brownian motion*, Ann. Probab., 23 (1995), pp. 767–775, https://doi.org/10.1214/aop/1176988288.

[32] F. Wang and C. Ma, *Peakedness and convex ordering for elliptically contoured random fields*, J. Statist. Plann. Inference, 197 (2018), pp. 21–34, https://doi.org/10.1016/j.jspi.2017.12.001.

[33] Y. Xiao, *Strong local nondeterminism and sample path properties of Gaussian random fields*, in Asymptotic Theory in Probability and Statistics with Applications, Adv. Lect. Math. (ALM) 2, Int. Press, Somerville, MA, 2008, pp. 136–176.

[34] Y. Xiao, *Recent developments on fractal properties of Gaussian random fields*, in Further Developments in Fractals and Related Fields, Trends Math., Birkhäuser/Springer, New York, 2013, pp. 255–288, https://doi.org/10.1007/978-0-8176-8400-6_13.

[35] Y. Xu, *Positive definite functions on the unit sphere and integrals of Jacobi polynomials*, Proc. Amer. Math. Soc., 146 (2018), pp. 2039–2048, https://doi.org/10.1090/proc/13913.

[36] M. I. Yadrenko, *Spectral Theory of Random Fields*, Optimization Software, New York, 1983.

[37] A. M. Yaglom, *Second-order homogeneous random fields*, in Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, Vol. 2, Univ. California Press, Berkeley, CA, 1961, pp. 593–622.

[38] A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, Vol. 1, Springer Ser. Statist., Springer-Verlag, New York, 1987, https://doi.org/10.1007/978-1-4612-4628-2.