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Strong Local Nondeterminism and Exact Modulus of Continuity for Isotropic Gaussian Random Fields on Compact Two-Point Homogeneous Spaces

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Abstract
This paper is concerned with sample path properties of real-valued isotropic Gaussian fields on compact two-point homogeneous spaces. In particular, we establish the property of strong local nondeterminism of an isotropic Gaussian field and then exploit this result to establish an exact uniform modulus of continuity for its sample paths.

Keywords Isotropy · Jacobi polynomial · Strong local nondeterminism · Uniform modulus of continuity

Mathematics Subject Classification (2020) 60G60 · 60G17 · 60G15 · 42C40

1 Introduction

The regularity and fractal properties of sample paths of Gaussian random fields have been studied extensively by many authors, for example [2, 13, 18, 35, 37–39], [28, 42, 48, 49, 51–53], but the index set of the random fields is typically restricted to be the Euclidean space $\mathbb{R}^d$. In many of the aforementioned references, the properties of strong local nondeterminism (SLND) have played important roles in studies of the

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exact regularity properties, fractal measure functions, multifractal analysis (e.g., the set of the fast points), local times of Gaussian random fields. Moreover, the properties of strong local nondeterminism are also closely related to the prediction theory of Gaussian random fields in spatial statistics (cf. [44–46]). Recently, the investigation of sample path properties of random fields over the unit sphere $\mathbb{S}^2$ has been conducted by [19], [20], [22]. The main result of [19] provides a sufficient condition in terms of the angular power spectrum for an isotropic spherical Gaussian random field on $\mathbb{S}^2$ to have the property of strong local nondeterminism. As an application of SLND, [19, Theorem 2] establishes the exact uniform modulus of continuity of the isotropic spherical Gaussian random field. Their result is much more precise than the Hölder continuity proved in [22, Theorem 4.5].

This paper is concerned with isotropic Gaussian random fields on a d-dimensional compact two-point homogeneous space $\mathbb{M}^d$. Our main objective is to extend the method in [19] from the setting of $\mathbb{S}^2$ to $\mathbb{M}^d$ and provide a condition in terms of the angular power spectrum for an isotropic Gaussian random field to be strongly local nondeterministic. This latter property is not only useful for studying precise sample path properties (we show this by proving the exact uniform modulus of continuity) but also important for prediction theory of isotropic Gaussian random fields on $\mathbb{M}^d$, which will be studied separately.

It is well known that $\mathbb{M}^d$ is a compact Riemannian symmetric space of rank one and belongs to one of the following five families ([17], [50]): the unit spheres $\mathbb{S}^d$ $(d = 1, 2, \ldots)$, the real projective spaces $\mathbb{P}^d(\mathbb{R})$ $(d = 2, 4, \ldots)$, the complex projective spaces $\mathbb{P}^d(\mathbb{C})$ $(d = 4, 6, \ldots)$, the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$ $(d = 8, 12, \ldots)$, and the Cayley elliptic plane $\mathbb{P}^{16}(Cay)$ or $\mathbb{P}^{16}(\mathbb{O})$. There are at least two different approaches to the subject of compact two-point homogeneous spaces [31], including an approach based on Lie algebras and a geometric approach, which are used in probability and statistics literature [4, 15, 33, 41]. All compact two-point homogeneous spaces share the same property that all geodesics in a given one of these spaces are closed and have the same length [15]. In particular, when the unit sphere $\mathbb{S}^d$ is embedded into the space $\mathbb{R}^{d+1}$, the length of any geodesic line is equal to that of the unit circle, that is, 2π. In what follows, the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$ between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on $\mathbb{M}^d$ is defined in such a way that the length of any geodesic line on all $\mathbb{M}^d$ is equal to 2π, or the distance between any two points is bounded between 0 and π, *i.e.*, $0 \leq \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \pi$. Over $\mathbb{S}^d$, for instance, $\rho(\mathbf{x}_1, \mathbf{x}_2)$ is defined by $\rho(\mathbf{x}_1, \mathbf{x}_2) = \arccos(\mathbf{x}_1'\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{S}^d$, where $\mathbf{x}_1'\mathbf{x}_2$ is the inner product between $\mathbf{x}_1$ and $\mathbf{x}_2$. Expressions of $\rho(\mathbf{x}_1, \mathbf{x}_2)$ on other spaces may be found in [6] and [24].

Gaussian random fields on $\mathbb{M}^d$ have been studied in [4, 9, 15, 26, 31, 33], among others, while theoretical investigations and practical applications of scalar and vector random fields on spheres may be found in [4, 7, 11, 15, 23, 29, 30, 33, 34], [55–57]. A series representation for a real-valued isotropic Gaussian random field on $\mathbb{M}^d$ is presented in [33, Chapter 2]. More generally, a series representation is provided in [31] for a vector random field that is isotropic and mean square continuous on $\mathbb{M}^d$ and stationary on a temporal domain, and a general form of the covariance matrix function is derived for such a vector random field, which involve Jacobi polynomials and the distance defined on $\mathbb{M}^d$. Parametric and semiparametric covariance matrix structures on $\mathbb{M}^d$ are constructed in [26].

Table 1 Parameters α and β associated with Jacobi polynomials over $\mathbb{M}^d$

$\mathbb{M}^d$	α	β
$\mathbb{S}^d$, $d = 1, 2, \ldots$	$\frac{d-2}{2}$	$\frac{d-2}{2}$
$\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$	$\frac{d-2}{2}$	$-\frac{1}{2}$
$\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$	$\frac{d-2}{2}$	0
$\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$	$\frac{d-2}{2}$	1
$\mathbb{P}^{16}(Cay)$	7	3

A second-order random field $Z = \{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is called stationary (homogeneous) and isotropic, if its mean function $\mathrm{E}Z(\mathbf{x})$ does not depend on $\mathbf{x}$, and its covariance function,

$$\mathrm{cov}(Z(\mathbf{x}_1), Z(\mathbf{x}_2)) = \mathrm{E}[(Z(\mathbf{x}_1) - \mathrm{E}Z(\mathbf{x}_1))(Z(\mathbf{x}_2) - \mathrm{E}Z(\mathbf{x}_2))], \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d,$$

depends only on the distance $\rho(\mathbf{x}_1, \mathbf{x}_2)$ between $\mathbf{x}_1$ and $\mathbf{x}_2$. We denote such a covariance function by $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d$, and call it an isotropic covariance function on $\mathbb{M}^d$. An isotropic random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is said to be mean square continuous if

$$\mathrm{E}|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|^2 \to 0, \text{ as } \rho(\mathbf{x}_1, \mathbf{x}_2) \to 0, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d.$$

It implies the continuity of the associated covariance function in terms of $\rho(\mathbf{x}_1, \mathbf{x}_2)$.

For a real-valued isotropic and mean square continuous random field on $\mathbb{M}^d$, its covariance function is of the form ([26], [31])

$$C(\rho(\mathbf{x}_1, \mathbf{x}_2)) = \sum_{n=0}^{\infty} b_n p_n^{(\alpha,\beta)} \left(\cos \rho(\mathbf{x}_1, \mathbf{x}_2)\right), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d, \tag{1}$$

where $\{b_n, n \in \mathbb{N}_0\}$ is a summable sequence of nonnegative constants ($\mathbb{N}_0$ denotes the set of nonnegative integers), $p_0^{(\alpha,\beta)}(x) \equiv 1$ and

$$p_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}, \quad x \in \mathbb{R}, \ n \in \mathbb{N}.$$

Here and in what follows,

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+\beta+n+k+1)}{\Gamma(\alpha+k+1)} \left(\frac{x-1}{2}\right)^k, \quad x \in \mathbb{R}, \ n \in \mathbb{N}_0, \tag{2}$$

are the Jacobi polynomials [47] with specific pair of parameters α and β given in Table 1, and $\Gamma(x)$ represents the Gamma function. It is known that $|p_n^{(\alpha,\beta)}(x)| \leq 1$, and

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}. \tag{3}$$

On the other hand, if $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$ is a function of the form (1), then there exists a real-valued isotropic Gaussian random field on $\mathbb{M}^d$ with $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$ as its covariance function [26], [31].

Many of the probabilistic and regularity properties of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ are determined by the property of the coefficient sequence $\{b_n, n \in \mathbb{N}_0\}$ in (1). For instance, a necessary and sufficient condition obtained in [10] for the covariance function $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$ to be strictly positive definite on $\mathbb{M}^d = \mathbb{S}^d$ $(d \geq 2)$ is that there are infinitely many positive b_{2n}'s and infinitely many positive b_{2n+1}'s in (1), and a necessary and sufficient condition on $\mathbb{M}^d \neq \mathbb{S}^d$ is that [5] there are infinitely many positive b_n's in (1). Otherwise, there exist an integer $m \geq 1$, $\mathbf{x}_i \in \mathbb{M}^d$ and nonzero $a_i \in \mathbb{R}$ $(i = 0, \ldots, m)$ such that $\operatorname{var}\left(\sum_{i=0}^m a_i Z(\mathbf{x}_i)\right) = 0$, or, equivalently, $\sum_{i=0}^m a_i Z(\mathbf{x}_i)$ equals a constant with probability 1 so that a linear (deterministic) relationship exists among $Z(\mathbf{x}_0), \ldots, Z(\mathbf{x}_m)$. It implies that the mean square prediction error vanishes when $Z(\mathbf{x}_0)$ is predicted via the linear combination of others. Thus, a necessary condition for $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ to have the property of local nondeterminism is to keep infinitely many b_n's in (1) away from zero, while the only known sufficient conditions in the literature are those on $\mathbb{S}^2$ in [19]. We refer the readers to Ref. [19] and [51] for a detailed description of the property of strong local nondeterminism over $\mathbb{R}^d$ as well as $\mathbb{S}^2$.

The property of strong local nondeterminism is established in this paper under an asymptotic condition on $\{b_n, n \in \mathbb{N}_0\}$ for an isotropic, mean square continuous, and centered Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$. The Gaussian setting is highlighted here, because of the following identity for the conditional variance of $Z(\mathbf{x})$ given $Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m)$,

$$\operatorname{var}(Z(\mathbf{x})|Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m)) = \inf \mathrm{E}\left(Z(\mathbf{x}) - \sum_{i=1}^m a_i Z(\mathbf{x}_i)\right)^2, \tag{4}$$

where the infimum is taken over all $(a_1, \ldots, a_m)' \in \mathbb{R}^m$. Notice that the right-hand side of (4) may be interpreted as the mean square prediction error, when one tries to predict $Z(\mathbf{x})$ through the linear combination of $Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m)$. A positive lower bound in terms of the locations $\mathbf{x}, \mathbf{x}_1, \ldots, \mathbf{x}_m$ for $\operatorname{var}(Z(\mathbf{x})|Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m))$ obtained in Theorem 1 of [19] reflects a property of strong local nondeterminism, in the particular case of $\mathbb{M}^d = \mathbb{S}^d$ and $d = 2$. In this paper, we pursue this line of investigation further from $\mathbb{S}^2$ to $\mathbb{M}^d$. In particular, we establish a property of strong local nondeterminism in Sect. 2 for a large class of isotropic Gaussian fields on $\mathbb{M}^d$. As an application of the SLND property, we determine the exact uniform modulus of continuity of the sample function $Z(\mathbf{x})$ in Sect. 3. This result extends that of [19] from $\mathbb{S}^2$ to $\mathbb{M}^d$ and significantly improves the Hölder continuity for $Z(\mathbf{x})$ in [9, Corollary 5.3]; see Remark 2 for a comparison of these two results. Finally, the proofs of propositions and theorems are given in Sect. 4.

2 Strong Local Nondeterminism

In what follows let $\alpha = \frac{d-2}{2}$, and let β be given in the last column of Table 1 associated with α. Our focus is on a Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ that is isotropic and mean square continuous on $\mathbb{M}^d$, whose covariance function is known ([26], [31]) to be of the form (1). The main result of this section is Theorem 1, which establishes the property of strong local nondeterminism for $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ under certain asymptotic condition on the coefficient sequence $\{b_n, n \in \mathbb{N}_0\}$ in (1). In the particular case of $\mathbb{M}^d = \mathbb{S}^d$ and $d = 2$, the SLND property was derived in [19].

Theorem 1 *Suppose that $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is a real-valued isotropic and mean square continuous Gaussian random field with mean 0 and covariance function (1), where $\{b_n, n \in \mathbb{N}_0\}$ is a summable sequence with nonnegative terms. If there are $n_0 \in \mathbb{N}$ and positive constants ν and γ_1, such that*

$$b_n(n+1)^{\nu+1} \geq \gamma_1, \qquad \forall\, n \geq n_0, \tag{5}$$

then there is a positive constant γ such that the inequality

$$\operatorname{var}(Z(\mathbf{x})|Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m)) \geq \gamma \left(\min_{1 \leq i \leq m} \rho(\mathbf{x}, \mathbf{x}_i) \right)^{\nu} \tag{6}$$

holds for all $m \in \mathbb{N}$ and all $\mathbf{x},\ \mathbf{x}_i \in \mathbb{M}^d$ ($i = 1, \ldots, m$).

Remark 1 The following are some remarks about Theorem 1.

(i) Inequality (6) was obtained in [19] in the particular case of $\mathbb{M}^d = \mathbb{S}^2$ and $\nu \in (0, 2]$, where the condition (A) requires both lower bound and upper bound on b_n's for large n. In contrast, (5) is just a lower bound on b_n's, while the series of $\sum_{n=0}^{\infty} b_n$ is assumed to be convergent.

(ii) Inequality (5) implies that $\{b_n, n \in \mathbb{N}_0\}$ is away from zero for all large $n \in \mathbb{N}$. In this case, the covariance function (1) of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is strictly positive definite [5].

(iii) It follows from (6) and Proposition 3 that the lower bound in (6) is optimal under (12) with $\nu \in (0, 2)$. In this case, the SLND is a useful tool for studying various regularity and fractal properties of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$. See Sect. 3 for more information.

(iv) When $\nu \geq 2$, the lower bound in (6) is not optimal as shown by Lemma 4 in [19] for the special case of $\mathbb{M}^d = \mathbb{S}^d$ and $d = 2$. As far as we know, an optimal lower bound for $\operatorname{var}(Z(\mathbf{x})|Z(\mathbf{x}_1), \ldots, Z(\mathbf{x}_m))$ is not known when $\nu \geq 2$.

To prove Theorem 1, we will make use of Propositions 1 and 2. Proposition 1 is quite interesting in its own right, since $\mathbb{D}$ is an arbitrary index set.

Proposition 1 *If $C(\mathbf{x}_1, \mathbf{x}_2)$ is a covariance function on $\mathbb{D}$, then so is the function $C(\mathbf{x}_0, \mathbf{x}_0)C(\mathbf{x}_1, \mathbf{x}_2) - C(\mathbf{x}_1, \mathbf{x}_0)C(\mathbf{x}_2, \mathbf{x}_0)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{D}$, for any fixed point $\mathbf{x}_0 \in \mathbb{D}$.*

For a continuous function $g(\theta)$ on $[0, \pi]$, we expand it in terms of the Jacobi polynomials [47] as follows

$$g(\theta) = \sum_{n=0}^{\infty} b_n^{(\alpha,\beta)}(g) p_n^{(\alpha,\beta)}(\cos\theta), \qquad \theta \in [0, \pi], \tag{7}$$

where the coefficients $b_n^{(\alpha,\beta)}(g)$ are given by

$$\begin{aligned} b_n^{(\alpha,\beta)}(g) = &\frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)} \\ &\times \int_0^{\pi} g(x) P_n^{(\alpha,\beta)}(\cos x) \sin^{2\alpha+1}\left(\frac{x}{2}\right) \cos^{2\beta+1}\left(\frac{x}{2}\right) dx, \quad n \in \mathbb{N}_0. \end{aligned} \tag{8}$$

In the proof of Proposition 2, we actually construct a specific function $g_\epsilon(\theta)$ that satisfies the following properties (i) - (iii). Such a function is termed as a spherical bump function [19] in the case of $\mathbb{M}^d = \mathbb{S}^d$ and $d = 2$.

Proposition 2 *For any constants $r > 1$, $n_0 \in \mathbb{N}$, and parameters α and β associated with $\mathbb{M}^d$ as listed in Table 1*, there exists a positive constant M_r such that, for any $\epsilon \in (0, \pi]$, there is a continuous function $g_\epsilon(\theta)$ on $[0, \pi]$ that satisfies

(i) $g_\epsilon(0) = 1$, and $g_\epsilon(\theta) = 0$ for $\epsilon \le \theta \le \pi$,
(ii) $b_n^{(\alpha,\beta)}(g_\epsilon) = 0$ for $0 \le n < n_0$,
(iii) $|b_n^{(\alpha,\beta)}(g_\epsilon)| \le M_r \epsilon (1 + n\epsilon)^{-r}$.

The proofs of Propositions 1 and 2 and Theorem 1 are presented in Sect. 4.

3 Modulus of Continuity

As we have mentioned in Introduction, the property of strong local nondeterminism plays important roles for studying precise regularity and fractal properties, multifractal analysis, local times, and prediction theory of Gaussian random fields. It would be interesting to apply Theorem 1 to study these problems for isotropic Gaussian random fields on compact two-point homogeneous spaces.

In this section, as an application of Theorem 1, we determine the exact uniform modulus of continuity of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ on $\mathbb{M}^d$. Theorem 2 extends Theorem 2 of [19] from $\mathbb{S}^2$ to $\mathbb{M}^d$ and improves Corollary 5.3 in [9] significantly (see Remark 2 for details).

Theorem 2 *Assume that* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ *is a real-valued isotropic and mean square continuous Gaussian random field with mean 0 and covariance function (1). If there are* $n_0 \in \mathbb{N}$ *and positive constants* γ_1, γ_2 *and* $\nu \in (0, 2)$*, such that*

$$\gamma_1 \leq b_n(n+1)^{\nu+1} \leq \gamma_2, \qquad \forall\, n \geq n_0, \tag{9}$$

then there is a positive and finite constant κ *such that, with probability 1,*

$$\lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d \\ \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \varepsilon}} \frac{|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2} \sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}} = \kappa. \tag{10}$$

Remark 2 We compare Theorem 2 with the regularity property of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ proved in [9]. Proposition 5.1 of [9] characterizes the Hölder regularity of the covariance function (1). Combining this with the Kolmogorov–Chentsov theorem for Riemannian Manifolds in [21], Cleanthous *et al*. [9, Corollary 5.3] proved the Hölder continuity of the sample function $Z(\mathbf{x})$. Under our condition (9), condition (5.1) in [9] is satisfied for every constant $N < \frac{\nu}{2}$. Hence, it follows from Corollary 5.3 in [9] (Notice that N in (5.1) is written as η in Corollary 5.3) that for any $0 < \gamma < \frac{\nu}{2}$,

$$\sup_{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d} \frac{|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\gamma}} < \infty, \quad \text{a.s.}$$

This only provides an upper bound for the uniform modulus of continuity of $Z(\mathbf{x})$, which is larger than $\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2}\sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}$, at least when $\rho(\mathbf{x}_1, \mathbf{x}_2)$ is sufficiently small. Meanwhile, (10) provides an exact uniform modulus of continuity of $Z(\mathbf{x})$ on $\mathbb{M}^d$. We remark that (10) cannot be proved by applying the Kolmogorov–Chentsov theorem as in [9]. Our method of proof, as well as that in [19], is based on the Gaussian techniques (cf. [2]), the property of strong local nondeterminism, and a conditioning argument from Meerschaert *et al*. [37].

For a centered isotropic Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ with covariance function $C(\rho(\mathbf{x}_1, \mathbf{x}_2))$ given by (1), its variogram is

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}\mathrm{E}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2))^2 = \sum_{n=0}^{\infty} b_n (1 - p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}_1, \mathbf{x}_2))), \ \ \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d. \tag{11}$$

Under the assumption (9) on the coefficients $\{b_n,\ n \in \mathbb{N}_0\}$ in (1), we obtain upper and lower bounds for the variogram of the Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ in terms of the distance function over $\mathbb{M}^d$ in Proposition 3, which will be employed to prove Theorem 2. Some related asymptotic relationships between $\{b_n, n \in \mathbb{N}_0\}$ and the variogram may be found in [32]. Under a stronger condition than (9), the upper bound in (12) is also obtained in [9, Proposition 5.2].

Proposition 3 *For a real-valued isotropic and mean square continuous Gaussian random field* $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ *with mean 0 and covariance function* (1), if (9) holds for a fixed $n_0 \in \mathbb{N}$ and positive constants γ_1, γ_2, and $\nu \in (0, 2)$, then there are positive constants δ_0, K_1 and K_2 such that

$$K_1 \rho(\mathbf{x}_1, \mathbf{x}_2)^\nu \le \mathrm{E}\,(Z(\mathbf{x}_1) - Z(\mathbf{x}_2))^2 \le K_2 \rho(\mathbf{x}_1, \mathbf{x}_2)^\nu \tag{12}$$

holds for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d$ with $\rho(\mathbf{x}_1, \mathbf{x}_2) \le \delta_0$.

Proposition 3 implies that many regularity and fractal properties of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ are determined by the index ν. For example, the upper bound in (12) and the Kolmogorov–Chentsov theorem together imply that $Z(\mathbf{x})$ is Hölder continuous of any order less than $\frac{\nu}{2}$, as having proved in [9, Corollary 5.3]. It can also be shown in a standard way that the Hausdorff dimension of the trajectory (the graph set) of Z, $\mathrm{Gr}Z(\mathbb{M}^d) = \{(\mathbf{x}, Z(\mathbf{x})), \mathbf{x} \in \mathbb{M}^d\} \subseteq \mathbb{M}^d \times \mathbb{R}$, is given by

$$\dim_{\mathrm{H}} \mathrm{Gr}Z(\mathbb{M}^d) = d + 1 - \frac{\nu}{2}, \quad \text{a.s.},$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension [12]. Because of these results, we call $\frac{\nu}{2}$ the fractal index of the random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$. When one uses such a random field as a statistical model to fit data sampled from values defined on $\mathbb{M}^d$, it will be important to estimate the fractal index $\frac{\nu}{2}$. We refer to [36] for more information on statistical inference of random fields on the sphere and to [6] for general nonparametric theory of statistics on manifolds.

Remark 3 If (9) holds for some $\nu > 2$, we believe that the sample function of $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ is continuously differentiable and it would be interesting to study its topological properties and excursion probabilities (see [2] for more information). We will not pursue this open problems in the present paper.

Next we give two examples of isotropic covariance functions on $\mathbb{M}^d$ and derive their expansions of the form (1). The covariance function in Example 1 satisfies (9), one of two functions in Example 2 satisfies (9), but the other does not. In what follows $b_n \sim a_n$ means that $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = 1$.

Example 1 For $\nu \in (0, 2]$, it is shown in Example 4 of [26] that

$$C(\rho(\mathbf{x}_1, \mathbf{x}_2)) = 1 - \left(\sin \frac{\rho(\mathbf{x}_1, \mathbf{x}_2)}{2}\right)^\nu, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d,$$

is the covariance function of an isotropic Gaussian random field on $\mathbb{M}^d$. For $\nu \in (0, 2)$, it can be shown that

$$b_n^{(\alpha,\beta)} \sim \frac{-2\Gamma\left(\alpha + \frac{\nu}{2} + 1\right)}{\Gamma(\alpha + 1)\Gamma\left(-\frac{\nu}{2}\right) n^{\nu+1}},$$

after the coefficients in (1) are found. In fact, applying the formula (8) to $C(x) = 1 - \left(\sin\frac{x}{2}\right)^{\nu}$, the coefficients in (1) are

$$
\begin{aligned}
b_n^{(\alpha,\beta)} &= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)} \\
&\quad \int_0^{\pi} C(x) P_n^{(\alpha,\beta)}(\cos x)\sin^{2\alpha+1}\left(\frac{x}{2}\right)\cos^{2\beta+1}\left(\frac{x}{2}\right)dx \\
&= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)} \\
&\quad \left\{\int_0^{\pi} P_n^{(\alpha,\beta)}(\cos x)\sin^{2\alpha+1}\left(\frac{x}{2}\right)\cos^{2\beta+1}\left(\frac{x}{2}\right)dx\right. \\
&\quad \left. - \int_0^{\pi} P_n^{(\alpha,\beta)}(\cos x)\sin^{2\alpha+\nu+1}\left(\frac{x}{2}\right)\cos^{2\beta+1}\left(\frac{x}{2}\right)dx\right\} \\
&= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2\Gamma(n+\beta+1)\Gamma(\alpha+1)}\left\{\int_{-1}^{1} P_n^{(\alpha,\beta)}(y)\left(\frac{1-y}{2}\right)^{\alpha}\left(\frac{1+y}{2}\right)^{\beta}dy\right. \\
&\quad \left. - \int_{-1}^{1} P_n^{(\alpha,\beta)}(y)\left(\frac{1-y}{2}\right)^{\alpha+\frac{\nu}{2}}\left(\frac{1+y}{2}\right)^{\beta}dy\right\} \\
&= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)}\left\{\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\delta_{n0}\right. \\
&\quad \left. - \frac{\Gamma\left(\alpha+\frac{\nu}{2}+1\right)\Gamma(\beta+n+1)\Gamma\left(n-\frac{\nu}{2}\right)}{\Gamma(n+1)\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\alpha+\beta+\frac{\nu}{2}+n+2\right)}\right\}, \quad n \in \mathbb{N}_0,
\end{aligned}
$$

where the third equality is obtained by making the transform $y = \cos x$, the last one from formulas (7.391.1) and (7.391.4) of [16], and $\delta_{n0} = \begin{cases} 1, & n = 0, \\ 0, & n \in \mathbb{N}. \end{cases}$

Example 2 Let

$$
g_1(x) = (\pi - x)^2, \quad g_2(x) = 2\pi^2(\pi - x)^2 - (\pi - x)^4, \quad x \in [0, \pi].
$$

In the following, we show that the functions $g_1(\rho(\mathbf{x}_1, \mathbf{x}_2))$ and $g_2(\rho(\mathbf{x}_1, \mathbf{x}_2))$ are covariance functions of isotropic Gaussian random fields on $\mathbb{M}^d$. For $\beta = -\frac{1}{2}$, applying formula (8) and integrating by parts we get

$$
b_n^{\left(\alpha,-\frac{1}{2}\right)}(g_1) = \frac{2\sqrt{\pi}\Gamma(n)}{n\Gamma(n+\frac{1}{2})}\frac{2n+\alpha+1}{n+\alpha+\frac{1}{2}}\frac{\Gamma(\alpha+\frac{3}{2})\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha+\frac{3}{2})},
$$

and

$$
b_n^{\left(\alpha,-\frac{1}{2}\right)}(g_2) = 12 b_n^{\left(\alpha,-\frac{1}{2}\right)}(g_1)\left(\sum_{k=0}^{\infty}\frac{1}{(n+k)^2} - \sum_{k=0}^{\infty}\frac{1}{(n+k+\alpha+\frac{3}{2})^2}\right), \quad n \in \mathbb{N}_0.
$$

Therefore,

$$\lim_{n\to\infty} n^2 b_n^{\left(\alpha,-\frac{1}{2}\right)}(g_1) = \frac{4\sqrt{\pi}\,\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+1)}, \quad \lim_{n\to\infty} n^4 b_n^{\left(\alpha,-\frac{1}{2}\right)}(g_2) = \frac{48\sqrt{\pi}\,\Gamma\left(\alpha+\frac{5}{2}\right)}{\Gamma(\alpha+1)}.$$

Following Example 1 of [26], it can be shown that $g_1(\rho(\mathbf{x}_1, \mathbf{x}_2))$ is a positive definite function on all $\mathbb{M}^d$, which implies that their asymptotic spectral coefficients are the same on all $\mathbb{M}^d$ with the same dimension d, *i.e.*,

$$\lim_{n\to\infty} n^2 b_n^{(\alpha,\beta)}(g_1) = \frac{4\sqrt{\pi}\,\Gamma\left(\alpha+\frac{3}{2}\right)}{\Gamma(\alpha+1)}.$$

Similar conclusion applies to $g_2(x)$, and

$$\lim_{n\to\infty} n^4 b_n^{(\alpha,\beta)}(g_2) = \frac{48\sqrt{\pi}\,\Gamma\left(\alpha+\frac{5}{2}\right)}{\Gamma(\alpha+1)}.$$

Since the spectral coefficients for $g_1(\rho(\mathbf{x}_1, \mathbf{x}_2))$ and $g_2(\rho(\mathbf{x}_1, \mathbf{x}_2))$ decay at the rates n^{-2} and n^{-4}, respectively, the corresponding Gaussian random fields on $\mathbb{M}^d$ have the SLND properties with $\nu = 1$ and $\nu = 3$, respectively, in Theorem 1.

4 Proofs

In this section, we provide proofs for our main results, in the order of Propositions 1-3 and Theorems 1-2.

4.1 Proof of Proposition 1

Suppose that $C(\mathbf{x}_1, \mathbf{x}_2)$ is the covariance function of a Gaussian random field $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{D}\}$. For every $n \in \mathbb{N}$, any $\mathbf{x}_k \in \mathbb{D}$ and any $a_k \in \mathbb{R}$ $(k = 1, \ldots, n)$, by applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}&\left\{\mathrm{E}\left((Z(\mathbf{x}_0) - \mathrm{E}Z(\mathbf{x}_0))\sum_{k=1}^n a_k(Z(\mathbf{x}_k) - \mathrm{E}Z(\mathbf{x}_k))\right)\right\}^2\\ &\le \mathrm{var}(Z(\mathbf{x}_0))\,\mathrm{var}\left(\sum_{k=1}^n a_k(Z(\mathbf{x}_k) - \mathrm{E}Z(\mathbf{x}_k))\right),\end{aligned}$$

or

$$\sum_{i=1}^n\sum_{j=1}^n a_i a_j C(\mathbf{x}_i, \mathbf{x}_0)C(\mathbf{x}_j, \mathbf{x}_0) \le C(\mathbf{x}_0, \mathbf{x}_0)\sum_{i=1}^n\sum_{j=1}^n a_i a_j C(\mathbf{x}_i, \mathbf{x}_j).$$

This implies that $C(\mathbf{x}_0, \mathbf{x}_0)C(\mathbf{x}_1, \mathbf{x}_2) - C(\mathbf{x}_1, \mathbf{x}_0)C(\mathbf{x}_2, \mathbf{x}_0)$, $\mathbf{x}_i, \mathbf{x}_2 \in \mathbb{D}$, is a positive definite function and thus is a covariance function on $\mathbb{D}$.

4.2 Proof of Proposition 2 (Construction of a Bump Function)

Given constants $r > 1$, $n_0 \in \mathbb{N}$, and a manifold $\mathbb{M}^d$ with parameters α and β as given in Table 1, we construct the following function $g_\epsilon(\theta)$ on $[0, \pi]$,

$$g_\epsilon(\theta) = \begin{cases} \phi\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\epsilon}{2}}\right), & \text{if } \beta \text{ is an integer,} \\ \cos\left(\frac{\theta}{2}\right)\phi\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\epsilon}{2}}\right), & \text{if } \beta - \frac{1}{2} \text{ is an integer,} \end{cases} \tag{13}$$

where

$$\phi(x) = \begin{cases} (1-x^2)_+^R \frac{P_{2K}(x)}{P_{2K}(0)}, & \text{if } d \text{ is odd,} \\ (1-x^2)_+^R P_K(1-2x^2), & \text{if } d \text{ is even,} \end{cases}$$

in which $y_+ = (y + |y|)/2$, R and K are integers such that

$$R \geq r + \alpha - \frac{1}{2}, \quad K = n_0 + R + \lceil\alpha\rceil + \lceil\beta\rceil,$$

and $P_K(x)$ is the Legendre polynomial of degree K.

Notice that $g_\epsilon(\theta)$ also depends on α, β, n_0, and r. We will show that this function satisfies the conditions (i), (ii), and (iii) in Proposition 2. It is clear that $g_\epsilon(\theta)$ is continuous on $[0, \pi]$, $g_\epsilon(0) = 1$, and $g_\epsilon(\theta) = 0$ for $\theta \geq \epsilon$, so the condition (i) is satisfied.

For an odd d, $\beta + \frac{1}{2}$ is an integer, as is seen from Table 1. Making the transform $y = \sin\frac{\theta}{2}/\sin\frac{\epsilon}{2}$, we obtain

$$\begin{aligned}
&\int_0^\pi g_\epsilon(\theta) P_n^{(\alpha,\beta)}(\cos\theta)\sin^{2\alpha+1}\left(\frac{\theta}{2}\right)\cos^{2\beta+1}\left(\frac{\theta}{2}\right)d\theta \\
&= \int_0^\epsilon \cos\left(\frac{\theta}{2}\right)\left(1 - \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\epsilon}{2}}\right)^R \frac{P_{2K}\left(\frac{\sin\frac{\theta}{2}}{\sin\frac{\epsilon}{2}}\right)}{P_{2K}(0)} P_n^{(\alpha,\beta)}(\cos\theta)\sin^{2\alpha+1} \\
&\left(\frac{\theta}{2}\right)\cos^{2\beta+1}\left(\frac{\theta}{2}\right)d\theta \\
&= \frac{2\sin^{2\alpha+2}\frac{\epsilon}{2}}{P_{2K}(0)}\int_0^1 (1-y^2)^R y^{2\alpha+1} \\
&\left(1 - y^2\sin^2\frac{\epsilon}{2}\right)^{\beta+\frac{1}{2}} P_{2K}(y) P_n^{(\alpha,\beta)}\left(1 - 2y^2\sin^2\frac{\epsilon}{2}\right)dy \\
&= \int_{-1}^1 P_{2K}(y) h_1(y^2)dy,
\end{aligned}$$

where

$$h_1(y^2) = \frac{\sin^{2\alpha+2}\frac{\epsilon}{2}}{P_{2K}(0)}(1-y^2)^R y^{2\alpha+1}\left(1-y^2\sin^2\frac{\epsilon}{2}\right)^{\beta+\frac{1}{2}} P_n^{(\alpha,\beta)}\left(1-2y^2\sin^2\frac{\epsilon}{2}\right)$$

is a polynomial of y of degree $2n+2R+2\alpha+2\beta+1$. For $0 \le n < n_0$, we have

$$2K = 2n_0 + 2R + 2\alpha + 2\beta + 1 > 2n + 2R + 2\alpha + 2\beta + 1,$$

so that $\int_{-1}^{1} P_{2K}(y)h_1(y^2)dy = 0$ by the orthogonality of the Legendre polynomial. It implies that $b_n^{(\alpha,\beta)}(g_\epsilon) = 0$ for $0 \le n < n_0$.

For an even d and $\beta \ne -\frac{1}{2}$, α and β are integers, as is seen from Table 1. By making the transform $y = \sin\frac{\theta}{2}/\sin\frac{\epsilon}{2}$ and followed by a change of variable $w = 1-2y^2$, we obtain

$$\begin{aligned}
&\int_0^\pi g_\epsilon(\theta)P_n^{(\alpha,\beta)}(\cos\theta)\sin^{2\alpha+1}\frac{\theta}{2}\cos^{2\beta+1}\frac{\theta}{2}d\theta \\
&= \int_0^\epsilon \left(1-\frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\epsilon}{2}}\right)^R P_K\left(1-\frac{2\sin^2\frac{\theta}{2}}{\sin^2\frac{\epsilon}{2}}\right)P_n^{(\alpha,\beta)}(\cos\theta)\sin^{2\alpha+1}\frac{\theta}{2}\cos^{2\beta+1}\frac{\theta}{2}d\theta \\
&= 2\sin^{2\alpha+2}\frac{\epsilon}{2}\int_0^1 (1-y^2)^R y^{2\alpha+1}\left(1-y^2\sin^2\frac{\epsilon}{2}\right)^\beta P_K \\
&\left(1-2y^2\right)P_n^{(\alpha,\beta)}\left(1-2y^2\sin^2\frac{\epsilon}{2}\right)dy \\
&= \frac{1}{2}\sin^{2\alpha+2}\frac{\epsilon}{2}\int_{-1}^1\left(\frac{1+w}{2}\right)^R\left(\frac{1-w}{2}\right)^\alpha\left(1-\frac{1-w}{2}\sin^2\frac{\epsilon}{2}\right)^\beta \\
&\times P_K(w)P_n^{(\alpha,\beta)}\left(1-(1-w)\sin^2\frac{\epsilon}{2}\right)dw \\
&= \int_{-1}^1 P_K(w)h_2(w)dw,
\end{aligned}$$

where $h_2(w)$ is the polynomial of degree of $n+R+\alpha+\beta$ defined by

$$\begin{aligned}
h_2(w) = \frac{1}{2}\sin^{2\alpha+2}\frac{\epsilon}{2}\left(\frac{1+w}{2}\right)^R\left(\frac{1-w}{2}\right)^\alpha\left(1-\frac{1-w}{2}\sin^2\frac{\epsilon}{2}\right)^\beta \\
\times P_n^{(\alpha,\beta)}\left(1-(1-w)\sin^2\frac{\epsilon}{2}\right).
\end{aligned}$$

For $0 \le n < n_0$, it follows from the orthogonality of the Legendre polynomials and $K = n_0 + R + \alpha + \beta > n + R + \alpha + \beta$ that $\int_{-1}^1 P_K(w)h_2(w)dw = 0$ and thus $b_n^{(\alpha,\beta)}(g_\epsilon) = 0$. Therefore the condition (ii) is satisfied for all cases.

Now we prove $|b_n^{(\alpha,\beta)}(g_\epsilon)| \le M_r\epsilon(1+n\epsilon)^{-r}$ for some $M_r > 0$. By Theorem 8.1.1 of [47],

$$\lim_{n\to\infty}\left(\frac{x}{2n}\right)^\alpha P_n^{(\alpha,\beta)}\left(\cos\frac{x}{n}\right) = J_\alpha(x),$$

where $J_\alpha(x)$ is the Bessel function of the first kind, we have

$$\lim_{\epsilon \to 0,\ n\epsilon = k} \frac{b_n^{(\alpha,\beta)}(g_\epsilon)}{\epsilon} = \frac{2k^{2\alpha+1}}{\Gamma(\alpha+1)} I_k^\alpha(\phi), \tag{14}$$

where

$$I_k^\alpha(\phi) = \int_0^1 \phi(y) \frac{J_\alpha(ky)}{k^\alpha} \left(\frac{y}{2}\right)^{\alpha+1} dy.$$

In particular, for $\alpha = -\frac{1}{2}$,

$$I_k^{-\frac{1}{2}}(\phi) = \frac{1}{\sqrt{\pi}} \int_0^1 \phi(y) \cos(ky) dy.$$

Since $\phi(y)$ is an even polynomial with derivatives up to order $R-1$ vanishing at 1, integration by parts gives

$$I_k^{-\frac{1}{2}}(\phi) = \frac{1}{\sqrt{\pi} k^{R+1}} \left(\phi^{(R)}(1) g_R(k) - \int_0^1 \phi^{(R+1)}(y) g_R(ky) dy \right),$$

where

$$g_R(y) = \begin{cases} (-1)^{\lfloor R/2 \rfloor} \sin(y), & \text{if } R \text{ is even,} \\ (-1)^{\lfloor R/2 \rfloor} \cos(y), & \text{if } R \text{ is odd.} \end{cases}$$

For $\alpha = 0$, integrating by parts and using the derivative formulas,

$$(J_1(x)x)' = J_0(x)x, \quad J_0(x)' = -J_1(x),$$

we get

$$\begin{aligned} I_k^0(\phi) &= \frac{1}{2} \int_0^1 \phi(y) J_0(ky) y dy \\ &= \frac{1}{2k^{R+1}} \left(\phi_R(1) h_R(k) - \int_0^1 \phi_{R+1}(y) h_R(ky) y dy \right), \end{aligned}$$

where

$$h_R(y) = \begin{cases} (-1)^{\lfloor R/2 \rfloor} J_1(y), & \text{if } R \text{ is even,} \\ (-1)^{\lfloor R/2 \rfloor} J_0(y), & \text{if } R \text{ is odd,} \end{cases}$$

and $\phi_m(y)$ are defined recursively by $\phi_0(y) = \phi(y)$, and for $m \geq 0$,

$$\phi_{2m+1}(y) = \phi_{2m}'(y), \quad \phi_{2m+2}(y) = (y\phi_{2m+1}(y))'/y,$$

which are well defined since $\phi(y)$ is an even polynomial. For higher dimensions, using the derivative formula $(J_\alpha(x)/x^\alpha)' = -J_{\alpha+1}(x)/x^\alpha$, we get

$$I_k^{\alpha+1}(\phi) = -\frac{d}{2kdk} I_k^\alpha(\phi).$$

For large k,

$$I_k^\alpha(\phi) = \begin{cases} \frac{\phi^{(R)}(1)}{\sqrt{\pi} k^{R+1}(-2k)^{\alpha+\frac{1}{2}}} \left(g_{R+\alpha+\frac{1}{2}}(k) + O\left(\frac{1}{k}\right)\right), & \text{odd-dimensional } \mathbb{M}^d, \\ \frac{\phi_R(1)}{2k^{R+1}(-2k)^\alpha} \left(h_{R+\alpha}(k) + O\left(\frac{1}{k}\right)\right), & \text{even-dimensional } \mathbb{M}^d. \end{cases}$$

Notice that for even-dimensional $\mathbb{M}^d$, the derivatives of ϕ up to order $R-1$ vanish at 1, so $\phi_R(1) = \phi^{(R)}(1)$. Using the asymptotic form of Bessel functions,

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}} \left(\cos(z - \frac{\alpha}{2}\pi - \frac{\pi}{4}) + O\left(\frac{1}{z}\right)\right),$$

we get the unified asymptotic form of $I_k^\alpha(\phi)$,

$$I_k^\alpha(\phi) = \frac{\phi^{(R)}(1)}{\sqrt{2\pi} 2^\alpha k^{R+\alpha+\frac{3}{2}}} \left(\sin(k + \frac{\pi}{2}(R - \alpha - \frac{1}{2})) + O\left(\frac{1}{k}\right)\right).$$

Substituting it in Eq. (14), we see that there is a constant $C_1 > 0$ such that for large $k = n\epsilon$,

$$\begin{aligned} |b_n^{(\alpha,\beta)}(g_\epsilon)| &\le C_1 \epsilon \frac{2k^{2\alpha+1}}{\Gamma(\alpha+1)} \frac{|\phi^{(R)}(1)|}{\sqrt{2\pi} 2^\alpha k^{R+\alpha+\frac{3}{2}}} = \frac{C_1 2^{1-\alpha} |\phi^{(R)}(1)|}{\Gamma(\alpha+1)\sqrt{2\pi}} \epsilon k^{-(R+\frac{1}{2}-\alpha)} \\ &\le \frac{C_1 2^{1-\alpha} |\phi^{(R)}(1)|}{\Gamma(\alpha+1)\sqrt{2\pi}} \epsilon (n\epsilon)^{-r}. \end{aligned}$$

On the other hand, by Eq. (8), there exists $C_2 > 0$, such that for any $n > 0$ and $\epsilon > 0$,

$$\begin{aligned} |b_n^{(\alpha,\beta)}(g_\epsilon)| &= \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)} P_n^{(\alpha,\beta)}(1) \\ &\quad \times \int_0^\epsilon g_\epsilon(\theta) p_n^{(\alpha,\beta)}(\cos\theta) \sin^{2\alpha+1}\left(\frac{\theta}{2}\right) \cos^{2\beta+1}\left(\frac{\theta}{2}\right) d\theta \\ &\le C_2 \epsilon (n\epsilon)^{2\alpha+1}. \end{aligned}$$

Therefore there exists $M_r > 0$ such that $|b_n^{(\alpha,\beta)}(g_\epsilon)| \le M_r \epsilon (1 + n\epsilon)^{-r}$.

4.3 Proof of Proposition 3 (Estimation of the Variogram)

To show (12), we apply the following recurrence relation of the Jacobi polynomials [47],

$$p_n^{(\alpha,\beta)}(\cos\theta) - p_{n+1}^{(\alpha,\beta)}(\cos\theta) = \frac{2n+\alpha+\beta+2}{\alpha+1} p_n^{(\alpha+1,\beta)}(\cos\theta)\sin^2\frac{\theta}{2}.$$

It is known that $|p_n^{(\alpha,\beta)}(x)| \le 1$ for $x \in [-1,1]$, which gives that for all $0 \le \theta \le \pi$,

$$\begin{aligned}
1 - p_n^{(\alpha,\beta)}(\cos\theta) &= \sum_{j=0}^{n-1}(p_j^{(\alpha,\beta)}(\cos\theta) - p_{j+1}^{(\alpha,\beta)}(\cos\theta)) \\
&= \sum_{j=0}^{n-1} \frac{2j+\alpha+\beta+2}{\alpha+1} p_j^{(\alpha+1,\beta)}(\cos\theta)\sin^2\frac{\theta}{2} \\
&\le \frac{n(n-1)+(\alpha+\beta+2)n}{\alpha+1}\sin^2\frac{\theta}{2} \\
&\le K_1 n^2\theta^2
\end{aligned}$$

for some constant $K_1 > 0$ that depends only on α and β. Hence, for $\theta = \rho(\mathbf{x}_1, \mathbf{x}_2)$, by (11) and the condition that $0 < \nu < 2$ and $b_n(n+1)^{\nu+1} \le \gamma_2$, we obtain

$$\begin{aligned}
\mathrm{E}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2))^2 &= 2\sum_{n=0}^{\infty} b_n\left(1 - p_n^{(\alpha,\beta)}(\cos\theta)\right) \\
&\le 2\sum_{0\le n<1/\theta} b_n K_1 n^2\theta^2 + 4\sum_{n\ge 1/\theta} b_n \\
&\le 2\gamma_2 \sum_{1\le n<1/\theta} K_1 n^{1-\nu}\theta^2 + 4\gamma_2\sum_{n\ge 1/\theta} n^{-\nu-1} \\
&\le K_2\theta^{\nu},
\end{aligned} \tag{15}$$

for some finite constant $K_2 > 0$. For obtaining the last inequality, we have used integrals to bound the two sums from above. This proves the upper bound in (12).

On the other hand, the lower bound in (12) follows from Theorem 1 because

$$\mathrm{E}(Z(\mathbf{x}_1) - Z(\mathbf{x}_2))^2 \ge \mathrm{var}(Z(\mathbf{x}_1)|Z(\mathbf{x}_2)) \ge \gamma\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu}.$$

Alternatively, it can be proved in the following elementary way. For $\theta \in [0, \pi]$,

$$p_n^{(\alpha,\beta)}(\cos\theta) = \sum_{m=0}^{n}(-1)^m\binom{n}{m}\frac{\Gamma(n+\alpha+\beta+m+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(\alpha+m+1)}\sin^2 m\frac{\theta}{2}.$$

Let

$$g(\theta)=\frac{1-p_n^{(\alpha,\beta)}(\cos\theta)}{\sin^2\frac{\theta}{2}}=\frac{n(n+\alpha+\beta+1)}{\alpha+1}$$
$$-\sum_{m=2}^{n}(-1)^m\binom{n}{m}\frac{\Gamma(n+\alpha+\beta+m+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(\alpha+m+1)}\sin^{2m-2}\frac{\theta}{2}.$$

We will bound $g(\theta)$ from below. Since

$$\binom{n}{m}\frac{\Gamma(n+\alpha+\beta+m+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(\alpha+m+1)}\sin^{2m-2}\frac{\theta}{2}$$
$$\leq\frac{(n+\alpha+\beta+m)^{2m}\Gamma(\alpha+1)}{m!\Gamma(\alpha+m+1)}\left(\frac{\theta}{2}\right)^{2m-2},$$

for any $\epsilon>0$, there exists $\delta>0$ such that for $n\geq\alpha+\beta+1$ and $0\leq n\theta\leq\delta$,

$$\sum_{m=2}^{n}\frac{(n+\alpha+\beta+m)^{2m}\Gamma(\alpha+1)}{m!\Gamma(\alpha+m+1)}\left(\frac{\theta}{2}\right)^{2m-2}\leq\sum_{m=2}^{n}\frac{4n^2(n\theta)^{2m-2}\Gamma(\alpha+1)}{m!\Gamma(\alpha+m+1)}\leq n^2\epsilon.$$

Therefore, there exists $K_3>0$ for which

$$1-p_n^{(\alpha,\beta)}(\cos\theta)=g(\theta)\sin^2\frac{\theta}{2}\geq K_3(n\theta)^2.$$

Given that $b_n\geq\gamma_1 n^{-\nu-1}$ for large n, there exists $K_4>0$ such that

$$\sum_{n=0}^{\infty}b_n\left(1-p_n^{(\alpha,\beta)}(\cos\theta)\right)\geq\sum_{n=\alpha+\beta+1}^{\delta/\theta}\frac{K_3(n\theta)^2}{n^{\nu+1}}\geq K_4\theta^{\nu}.$$

4.4 Proof of Theorem 1 (Property of SLND)

Write $\varepsilon=\min_{1\leq i\leq m}\rho(\mathbf{x},\mathbf{x}_i)$. For the Gaussian random field $\{Z(\mathbf{x}),\mathbf{x}\in\mathbb{M}^d\}$, in order to verify inequality (6), because of identity (4) it suffices to show that there is a positive constant γ such that

$$\mathrm{E}\left(Z(\mathbf{x})-\sum_{i=1}^{m}a_iZ(\mathbf{x}_i)\right)^2\geq\gamma\varepsilon^{\nu},\tag{16}$$

holds for all $n \in \mathbb{N}$, $\mathbf{x}$, $\mathbf{x}_k \in \mathbb{M}^d$ $(k = 1, \ldots, n)$ with $\min_{1 \le k \le n} \rho(\mathbf{x}, \mathbf{x}_k) > 0$ and all $a_k \in \mathbb{R}$ $(k = 1, \ldots, n)$. We have

$$
\begin{aligned}
&\mathrm{E}\left(Z(\mathbf{x}) - \sum_{i=1}^{m} a_i Z(\mathbf{x}_i) \right)^2 \\
&= C(\mathbf{x}, \mathbf{x}) - 2\sum_{i=1}^{m} a_i C(\mathbf{x}, \mathbf{x}_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j C(\mathbf{x}_i, \mathbf{x}_j) \\
&= \sum_{n=0}^{\infty} b_n \left[1 - 2\sum_{i=1}^{m} a_i p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_i)) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}_i, \mathbf{x}_j)) \right] \\
&= \sum_{n=0}^{\infty} b_n \left[\left(1 - \sum_{i=1}^{m} a_i p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_i)) \right)^2 \right. \\
&\left. + \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \left(p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}_i, \mathbf{x}_j)) - p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_i)) p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_j)) \right) \right] \\
&\ge \sum_{n=0}^{\infty} b_n \left(1 - \sum_{i=1}^{m} a_i p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_i)) \right)^2,
\end{aligned}
$$

where the last inequality holds since, for every $n \in \mathbb{N}$,

$$
\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j \left(p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}_i, \mathbf{x}_j)) - p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_i)) p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_j)) \right) \ge 0,
$$

which is due to Proposition 1, while $p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}_1, \mathbf{x}_2))$ is known to be a covariance function on $\mathbb{M}^d$ by Lemma 3 of [31].

For a continuous function $g_\epsilon(\theta)$ on $[0, \pi]$ satisfying conditions (i)-(iii) in Proposition 2 with $r > 1 + \frac{\nu}{2}$, we consider

$$
I = \sum_{n=0}^{\infty} b_n^{(\alpha,\beta)}(g_\epsilon) \left(1 - \sum_{k=1}^{m} a_k p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_k)) \right).
$$

On one hand, it follows from $\rho(\mathbf{x}, \mathbf{x}_k) \ge \epsilon$ and Proposition 2 (i) that $g_\epsilon(\rho(\mathbf{x}, \mathbf{x}_k)) = 0$ $(k = 1, \ldots, m)$, so that

$$
\begin{aligned}
I &= \sum_{n=0}^{\infty} b_n^{(\alpha,\beta)}(g_\epsilon) - \sum_{k=1}^{m} a_k \sum_{n=0}^{\infty} b_n^{(\alpha,\beta)}(g_\epsilon) p_n^{(\alpha,\beta)}(\cos \rho(\mathbf{x}, \mathbf{x}_k)) \\
&= g_\epsilon(0) - \sum_{k=1}^{m} a_k g_\epsilon(\rho(\mathbf{x}, \mathbf{x}_k)) \\
&= 1.
\end{aligned}
$$

On the other hand, an application of the Cauchy–Schwarz inequality yields that

$$
\begin{aligned}
I^2 &= \left\{ \sum_{n=n_0}^{\infty} \frac{b_n^{(\alpha,\beta)}(g_\epsilon)}{\sqrt{b_n}} \sqrt{b_n} \left(1 - \sum_{k=1}^{m} a_k p_n^{\left(\frac{d-1}{2}\right)} (\cos \rho(\mathbf{x}, \mathbf{x}_k)) \right) \right\}^2 \\
&\le \sum_{n=n_0}^{\infty} \frac{\left(b_n l^{(\alpha,\beta)}(g_\epsilon)\right)^2}{b_n} \sum_{n=n_0}^{\infty} b_n \left(1 - \sum_{k=1}^{m} a_k p_n^{(\alpha,\beta)} (\cos \rho(\mathbf{x}, \mathbf{x}_k)) \right)^2 \\
&\le \frac{1}{M_r \epsilon^{\nu}} \sum_{n=n_0}^{\infty} b_n \left(1 - \sum_{k=1}^{m} a_k p_n^{\left(\frac{d-1}{2}\right)} (\cos \theta(\mathbf{x}, \mathbf{x}_k)) \right)^2,
\end{aligned}
$$

where the last inequality follows from $|b_n^{(\alpha,\beta)}(g_\epsilon)| \le M_r \epsilon (n\epsilon + 1)^{-r}$ by Proposition 2 (iii), $b_n (n+1)^{\nu+1} \ge K$ for $n \ge n_0$ by (5), and

$$
\begin{aligned}
\sum_{n=n_0}^{\infty} \frac{\left(b_n^{(\alpha,\beta)}(g_\epsilon)\right)^2}{b_n} &\le \sum_{n=0}^{\infty} \frac{M_r^2 \epsilon^2 (n\epsilon + 1)^{-2r}}{K(n+1)^{-\nu-1}} \\
&\le \frac{M_r^2}{K} \epsilon^2 \int_0^{\infty} \frac{(1+x)^{\nu+1}}{(\epsilon x + 1)^{2r}}\, dx = \frac{M_r^2}{K} \frac{\epsilon^{-\nu}}{2r - \nu - 2}.
\end{aligned}
$$

Consequently, inequality (6) is obtained.

4.5 Proof of Theorem 2 (Exact Uniform Modulus of Continuity)

We start with the following zero-one law, which is proved by applying the Karhunen–Loève expansion for $Z(\mathbf{x})$ (cf. [33, Chapter 2]) and Kolmogorov's zero-one law.

Lemma 1 *Let $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathbb{M}^d\}$ be a real-valued, centered isotropic Gaussian random field on $\mathbb{M}^d$ that satisfies the conditions of Theorem 2. Then there is a constant $K \in [0, \infty]$ such that*

$$
\lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d \\ \rho(\mathbf{x}_1, \mathbf{x}_2) \le \varepsilon}} \frac{|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2} \sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}} = K, \quad a.s. \tag{17}
$$

Proof Recall from Malyarenko [33, Chapter 2] that $\{Z(\mathbf{x}), \mathbf{x} \in \mathbb{M}^d\}$ has the following Karhunen–Loève expansion

$$
Z(\mathbf{x}) = C_d \sum_{l=0}^{\infty} \sum_{m=1}^{h(\mathbb{M}^d, l)} \sqrt{\frac{b_l}{h(\mathbb{M}^d, l)}}\, X_{l,m} Y_{l,m}(\mathbf{x}), \tag{18}
$$

with convergence in $L^2(\Omega, L^2(\mathbb{M}^d))$. In the above, $C_d = \sqrt{\frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)}}$ if $\mathbb{M}^d = \mathbb{S}^d$ and $C_d = 1$ in all the other cases in Table 1; $h(\mathbb{M}^d, l)$ is given by

$$h(\mathbb{M}^d, l) = \frac{(2l + \alpha + \beta + 1)\Gamma(\beta + 1)\Gamma(l + \alpha + \beta + 1)\Gamma(l + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + 2)l!\Gamma(l + \beta + 1)},$$

where α and β are the parameters given in Table 1, $\{b_l, l \geq 0\}$ is the coefficient sequence in (1), and the sequences $\{X_{l,m}\}$, and $\{Y_{l,m}\}$ are specified as follows:

- $\{X_{l,m}\}$ is a sequence of i.i.d. standard normal random variables.
- $\{Y_{l,m}\}$ are the eigenfunctions of the Laplace-Beltrami operator $\Delta_{\mathbb{M}^d}$ on $\mathbb{M}^d$, i.e.,

$$-\Delta_{\mathbb{M}^d} Y_{l,m} = \lambda_l Y_{l,m},$$

where the eigenvalues $\lambda_l = \lambda_l(\alpha, \beta) = l(l + \alpha + \beta + 1)$, for all $l \in \mathbb{N}_0$.

For every $l \geq 0$, the eigenfunctions $\{Y_{l,m}, 1 \leq m \leq h(\mathbb{M}^d, l)\}$ corresponding to the same eigenvalue λ_l form a finite-dimensional vector space of dimension $h(\mathbb{M}^d, l)$. It is known that, for every (l, m), the eigenfunction $Y_{l,m}(\mathbf{x})$ is continuously differentiable and $\mathbb{M}^d$ is compact. Recall that $\nu \in (0, 2)$. Hence for every integer $L \geq 0$,

$$\lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d \\ \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \varepsilon}} \frac{|Z_L(\mathbf{x}_1) - Z_L(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2}\sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}} = 0, \quad \text{a.s.},$$

where

$$Z_L(\mathbf{x}) = C_d \sum_{l=0}^{L} \sum_{m=1}^{h(\mathbb{M}^d, l)} \sqrt{\frac{b_l}{h(\mathbb{M}^d, l)}}\, X_{l,m} Y_{l,m}(\mathbf{x}).$$

Hence for every constant $\kappa_1 \geq 0$, the event

$$E_{\kappa_1} = \left\{ \lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d \\ \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \varepsilon}} \frac{|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2}\sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}} \leq \kappa_1 \right\}$$

is a tail event with respect to $\{X_{l,m}\}$. By Kolmogorov's zero-one law, we have $\mathbb{P}(E_{\kappa_1}) = 0$ or 1. This implies (17) with $K = \sup\{\kappa_1 \geq 0 : \mathbb{P}(E_{\kappa_1}) = 0\}$. □

Now we prove Theorem 2.

Proof of Theorem 2 Because of the zero-one law in Lemma 1, it is sufficient to prove the existence of positive and finite constants K_5 and K_6 such that

$$\lim_{\varepsilon \to 0} \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{M}^d \\ \rho(\mathbf{x}_1, \mathbf{x}_2) \leq \varepsilon}} \frac{|Z(\mathbf{x}_1) - Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1, \mathbf{x}_2)^{\nu/2}\sqrt{|\ln \rho(\mathbf{x}_1, \mathbf{x}_2)|}} \leq K_5 \quad \text{a.s.} \tag{19}$$

and

$$\lim_{\varepsilon\to 0} \sup_{\substack{\mathbf{x}_1,\mathbf{x}_2\in\mathbb{M}^d\\ \rho(\mathbf{x}_1,\mathbf{x}_2)\le\varepsilon}} \frac{|Z(\mathbf{x}_1)-Z(\mathbf{x}_2)|}{\rho(\mathbf{x}_1,\mathbf{x}_2)^{\nu/2}\sqrt{|\ln\rho(\mathbf{x}_1,\mathbf{x}_2)|}} \ge K_6 \quad \text{a.s.} \tag{20}$$

The proof of (19) is quite standard. By Proposition 3, the canonical metric d_Z defined by $d_Z(\mathbf{x}_1,\mathbf{x}_2)=\sqrt{\mathrm{E}\big[(Z(\mathbf{x}_1)-Z(\mathbf{x}_2))^2\big]}$ satisfies $d_Z(\mathbf{x}_1,\mathbf{x}_2)\le K\,\rho(x_1,x_2)^{\nu/2}$ for all $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{M}^d$. This implies that for every $\varepsilon\in(0,\pi)$, we have

$$N(\mathbb{M}^d,d_Z,\varepsilon)\le K\varepsilon^{-\frac{2d}{\nu}}, \tag{21}$$

where $N(\mathbb{M}^d,d_Z,\varepsilon)$ denotes the minimum number of d_Z-balls of radius ε that are needed to cover $\mathbb{M}^d$. Hence, (19) follows from (21) and Theorem 1.3.5 in [2].

For any $n\ge n_0$, we choose a sequence of 2^n points $\{x_{n,i},1\le i\le 2^n\}\subseteq\mathbb{M}^d$ that are equally spaced along a geodesic of length s, where s is an arbitrary positive constant. Then for every $2\le k\le 2^n$, we have

$$\rho(\mathbf{x}_{n,k},\ \mathbf{x}_{n,k-1})=\frac{s}{2^n-1}. \tag{22}$$

With the choice of $\{\mathbf{x}_{n,i},1\le i\le 2^n\}$, we now prove (20) in a way that is similar to the proof in [19]. Notice that

$$\lim_{\varepsilon\to 0} \sup_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{M}^d,\\ \rho(\mathbf{x},\mathbf{y})\le\varepsilon}} \frac{|Z(\mathbf{x})-Z(\mathbf{y})|}{\rho(\mathbf{x},\mathbf{y})^{\nu/2}\sqrt{|\ln\rho(\mathbf{x},\mathbf{y})|}} \ge \liminf_{n\to\infty}\max_{2\le k\le 2^n}\frac{|Z(\mathbf{x}_{n,k})-Z(\mathbf{x}_{n,k-1})|}{2^{-n\nu/2}\sqrt{n}}. \tag{23}$$

It is sufficient to prove that, almost surely, the last limit in (23) is bounded below by a positive constant. This is done by applying the property of strong local nondeterminism in Theorem 1 and a standard Borel–Cantelli argument.

Let $\eta>0$ be a constant whose value will be chosen later. We consider the events

$$A_m=\left\{\max_{2\le k\le m}\big|Z(\mathbf{x}_{n,k})-Z(\mathbf{x}_{n,k-1})\big|\le\eta 2^{-n\nu/2}\sqrt{n}\right\}$$

for $m=2,3,\ldots,2^n$. By conditioning on A_{2^n-1} first, we can write

$$\begin{aligned}\mathbb{P}\big(A_{2^n}\big)&=\mathbb{P}\big(A_{2^n-1}\big)\\ &\quad\times\mathbb{P}\Big\{\big|Z(\mathbf{x}_{n,2^n})-Z(\mathbf{x}_{n,2^n-1})\big|\le\eta 2^{-n\nu/2}\sqrt{n}\,\big|A_{2^n-1}\Big\}.\end{aligned} \tag{24}$$

Recall that, given the random variables in A_{2^n-1}, the conditional distribution of the Gaussian random variable $Z(\mathbf{x}_{n,2^n})-Z(\mathbf{x}_{n,2^n-1})$ is still Gaussian, with the corresponding conditional mean and variance as its mean and variance. By Theorem 1 and (22), there exists $\gamma_3>0$ independent of n such that

$$\mathrm{var}\big(Z(\mathbf{x}_{n,2^n})-Z(\mathbf{x}_{n,2^n-1})\big|A_{2^n-1}\big)\ge\gamma_3\,2^{-n\nu}.$$

This and Anderson's inequality (see [3]) imply

$$\begin{aligned}
&\mathbb{P}\Big\{\big|Z(\mathbf{x}_{n,2^n}) - Z(\mathbf{x}_{n,2^n-1})\big| \le \eta 2^{-n\nu/2}\sqrt{n}\big|\, A_{2^n-1}\Big\} \\
&\le \mathbb{P}\Big\{N(0,1) \le \frac{\eta}{\sqrt{\gamma_3}}\sqrt{n}\Big\} \\
&\le 1 - \frac{\sqrt{\gamma_3}}{\eta\sqrt{n}} \exp\Big(-\frac{\eta^2 n}{2\gamma_3}\Big) \\
&\le \exp\Big(-\frac{\sqrt{\gamma_3}}{\eta\sqrt{n}} \exp\Big(-\frac{\eta^2 n}{2\gamma_3}\Big)\Big).
\end{aligned} \tag{25}$$

In deriving the last two inequalities, we have applied Mill's ratio and the elementary inequality $1 - x \le e^{-x}$ for $x > 0$. Iterating this procedure in (24) and (25) for $2^n - 1$ more times, we obtain

$$\mathbb{P}\big(A_{2^n}\big) \le \exp\Big(-\frac{\sqrt{\gamma_3}}{\eta\sqrt{n}} 2^n \exp\Big(-\frac{\eta^2 n}{2\gamma_3}\Big)\Big). \tag{26}$$

By taking $\eta > 0$ small enough such that $\eta^2 < 2\gamma_3 \ln 2$, we have $\sum_{n=1}^{\infty} \mathbb{P}(A_{2^n}) < \infty$. Hence the Borel–Cantelli lemma implies that

$$\mathbb{P}\left(\liminf_{n\to\infty} \max_{2\le k\le 2^n} \frac{|Z(\mathbf{x}_{n,k}) - Z(\mathbf{x}_{n,k-1})|}{2^{-n\nu/2}\sqrt{n}} \ge \eta\right) = 1.$$

This finishes the proof of Theorem 2. □

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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