A1. Prove Theorem 5.9 in the textbook.

Proof:

(i) Let \( A = \sum_{j=1}^{\nu} \sigma_j u_j v_j^* \Rightarrow A - A_\nu = \sum_{j=\nu+1}^{r} \sigma_j u_j v_j^* \).

\[
\|A - A_\nu\|_F^2 = \text{Tr}((A - A_\nu)^*(A - A_\nu)) = \sum_{j=1}^{n} v_j^*(A - A_\nu)^*(A - A_\nu)v_j = \sum_{j=\nu+1}^{r} \sigma_j^2.
\]

(ii) By the proof of Theorem 5.8, we know for any subspace of \( \mathbb{C}^n \) of dimension at least \( n - s \), there exists a nonzero vector \( x \) in the subspace such that \( \|Ax\| \geq \sigma_{s+1}\|x\| \). In fact, such an \( x \) lies in the intersection of the subspace with the space spanned by \( v_1, v_2, ..., v_{s+1} \).

For any \( B \) of rank up to \( \nu \), let \( W_1 = \text{Nul}B \). Then \( \dim \text{Nul} W_1 \geq n - \nu \). By the claim above, there is a unit vector \( x_1 \in W_1 \) such that \( \|Ax_1\| \geq \sigma_{\nu+1} \). Let

\[ W_2 = \{ x \in W_1 | x_1^* x = 0 \} \]

Then \( \dim \text{Nul} W_2 \geq n - \nu - 1 \), so there is a unit vector \( x_2 \in W_2 \) such that \( \|Ax_2\| \geq \sigma_{\nu+2} \). Let

\[ W_3 = \{ x \in W_2 | x_2^* x = 0 \} \]

Continue the process until we find \( x_{r-\nu} \in W_{r-\nu} \) such that \( \|Ax_{r-\nu}\| \geq \sigma_r \). \( <x_1, x_2, ..., x_{r-\nu}> \) is an orthonormal set in \( \mathbb{C}^n \), and so it be expanded to an orthonormal basis \( <x_1, x_2, ..., x_n> \). Finally,

\[
\|A - B\|_F^2 = \text{Tr}((A - B)^*(A - B)) = \sum_{j=1}^{n} x_j^*(A - B)^*(A - B)x_j \\
\geq \sum_{j=1}^{r-\nu} x_j^*(A - B)^*(A - B)x_j = \sum_{j=1}^{r-\nu} x_j^* A^* A x_j = \sum_{j=1}^{r-\nu} \|Ax_j\|^2 \geq \sum_{j=\nu+1}^{r} \sigma_j^2.
\]
A2. Exercise 12.3.

Solution:

(a) The eigenvalues are approximately uniformly distributed in the unit disc in the complex plane.

$$\lim_{m \to \infty} \rho(A) \approx 1.$$ 

The standard deviation of $\rho(A)$ goes to 0 as $m \to \infty$.

(b) 

$$\lim_{m \to \infty} \|A\|_2 \approx 2.$$ 

The standard deviation of $\|A\|_2$ goes to 0 as $m \to \infty$.

(c) $log(m\sigma_{min})$ seems to have a limit distribution with expectation value $\sim log(0.4)$.

(d) For triangular matrices, the eigenvalues are real, and have distribution $N(0, 1/\sqrt{m})$.

$$\lim_{m \to \infty} \rho(A) = 0.$$ 

$$\lim_{m \to \infty} \|A\|_2 \approx 1.64.$$ 

The standard deviation of $\|A\|_2$ goes to 0 as $m \to \infty$.

$$\sigma_{min} \ll 1/m.$$
A3. Exercise 17.3.

Solution:

(a) 

\[ m_{ii} = 1 \]

\[ m_{ij} = \sum_{k=j}^{i-1} l_{ik} m_{kj}, \quad i > j. \]

\( m_{ij} \) depends on \( l_{kl} \) with \( j \leq l < k \leq i \).

(b) 

\[ \mu_1 = 1 \]

\[ \mu_{i+1} = \sum_{j=1}^{i} S^j_i \mu_j. \]

\( S^j_i \) with \( 1 \leq j \leq i \) are independent random numbers \( \pm 1 \) with equal probability.

(c) \( C \approx 1.3 \).

\[ \text{m=100;} \]

\[ \text{C=cond(tril(2*(rand(m)>0.5)-1,-1)+eye(m))}^{\text{(1/m)}} \]

(d) \( C \approx 1.32 \).

\[ \text{m=2000;} \]

\[ \text{x(1)}=1; \]

\[ \text{for i=1:m-1} \]

\[ \text{x(i+1)}=\text{x}(1:i)\ast(2\ast(\text{rand}(i,1)>0.5)-1); \]

\[ \text{end} \]

\[ \text{C=abs(x(m))}^{\text{(1/m)}} \]

The \( C \) obtained in (c) and (d) would be the same in the limit \( m \to \infty \) because

\[ \kappa_m = \kappa(L) = \|L\|_2 \cdot \|M\|_2. \]

\[ \sqrt{m} \leq \|L\|_2 \leq \|L\|_F = \sqrt{m(m+1)/2}. \]

\[ \max |M_{ij}| \leq \|M\|_2 \leq \|M\|_F \leq m \cdot \max |M_{ij}|. \]

If \( \lim_{m \to \infty} \kappa_m^{1/m} = C \), then \( \lim_{m \to \infty} \max |M_{ij}|^{1/m} = C \). Since for fixed \( k \), \( m_{i+k,i} \) has the same distribution for all \( i \), we can look for \( \max |M_{ij}| \) in the first column. Since \( \text{E} \mu_i^2 \) increases with \( i \), \( \max_i |\mu_i| \) is approximately \( |\mu_m| \).

In fact, numerical experiments show that for \( \alpha_m \equiv (\log_2 \mu_m^2) / m \),

\[ \lim_{m \to \infty} \text{E} \alpha_m = C_1 \approx 0.814, \quad \lim_{m \to \infty} m \text{Var} \alpha_m = C_2 \approx 0.56. \]