1 Solution to Homework 2

6.2

\[ E_{ij} = \frac{\delta_{i,j} + \delta_{i,m+1-j}}{2} \]

\( E \) is a projector because \( E^2 = E \). It is an orthogonal projector because \( E = E^* \).

7.3

\( A = QR \). \(| \det A | = | \det R | = \prod_{i=1}^{m} |r_{ii}| \). Denote the \( i \)th column of \( R \) by \( r_{i} \), then

\[ \| a_{i} \|_{2} = \| Qa_{i} \|_{2} = \| r_{i} \|_{2} \geq |r_{ii}|. \]

Therefore \( | \det A | \leq \prod_{i=1}^{m} \| a_{i} \|_{2} \). The geometric interpretation is that the volume of an \( m \)-dimensional parallelepiped is bounded by the product of the length of the \( m \) spanning edges.

8.2

function [Q,R]=mgs(A)
[m,n]=size(A);
Q=A;
R=eye(n);
for i=1:n
R(i,i)=norm(Q(:,i));
Q(:,i)=Q(:,i)/R(i,i);
if i==n
break
end
for j=i+1:n
R(i,j)=Q(:,i)'*Q(:,j); % modified GS
Q(:,j)=Q(:,j)-R(i,j)*Q(:,i);
end
end

10.2

(a)
function [W,R]=house(A)
[m,n]=size(A);
W=zeros(m,n); % store Householder vectors
for k=1:n
x=A(k:m,k);
if x(1)==0
    sgn=1;
else
    sgn=sign(x(1));
end
A(k:m,k)=0;
\textbf{10.4}

(a) $F$ reflects a vector with respect to the line of azimuthal angle $(\pi - \theta)/2$. $J$ rotates the plane by $-\theta$.

(b) For each nonzero vector $<a, b> \in \mathbb{R}^2$, we can apply $J$ with $\theta = \arctan(b/a)$ to it and turn it into a vector with 2nd entry zero. For each nonzero vector in $\mathbb{C}^2$, we can modify $J$ to include the relative phase of the two entries, so that $J$ still maps it into a vector with 2nd entry zero. For any given matrix $A$, we can apply such $J$ for the 1st and 2nd rows of $A$ to to get $A' = JA$ with $A'_{21} = 0$; then apply such $J$ for the 1st and 3rd rows of $A'$, and so on, until we get a matrix whose 1st column is all zero except the 1st entry. The we can apply such $J$ for the 2nd and 3rd row, without computation involving the 1st column because the relevant entries are all 0. The process can be continued until we transform $A$ into an upper-triangular matrix by successive rotations, whose product is a unitary matrix.

(c) For the first $J$, which acts on the 1st and 2nd rows of length $n$, the multiplication by $J$ takes $6n$ flops. The introduce all the zeros in the first column, we need $6n(m - 1)$ flops, so the average per entry is 6, as compared to the average of 4 in the Householder triangularization.

\textbf{11.1}

Denote by $P$ the orthogonal projection onto $\text{Col}A$. For any $x \in \mathbb{C}^m$, there exists a $y \in \mathbb{C}^n$, such that

\[ x = Px + (1 - P)x = Ay + (1 - P)x. \]

Since $A^+A = I$, and $\text{Nul}A^+ = \text{Col}(1 - P)$,

\[ A^+x = A^+Ay + A^+(1 - P)x = y. \]
On the other hand,

\[ Ay = \begin{pmatrix} A_1 y \\ A_2 y \end{pmatrix}, \]

so

\[ \| x \|_2^2 = \| Ay \|_2^2 + \| (1 - P)x \|_2^2 \geq \| Ay \|_2^2 \geq \| A_1 y \|_2^2. \]

As a result,

\[ \begin{aligned} \frac{\| A^+ x \|}{\| x \|} &= \frac{\| y \|}{\| x \|} \leq \frac{\| y \|}{\| A_1 y \|} = \frac{\| A_1^{-1} A_1 y \|}{\| A_1 y \|}. \end{aligned} \]

Taking the sup on both ends, we get

\[ \| A^+ \| \leq \| A_1^{-1} \|. \]