MATH555
Differential Equations
Practice Exam III

1. Determine the Taylor series of the following function about the point x_0, and determine the radius of convergence of the series.

$$f(x) = \ln x, \qquad x_0 = 1.$$

Solution:

$$(\ln x)' = \frac{1}{x},\ (\ln x)'' = -\frac{1}{x^2},\ (\ln x)''' = \frac{2}{x^3},\ (\ln x)'''' = -\frac{3!}{x^4}, \ldots,\ (\ln x)^{(n)} = (-1)^{n-1}\frac{(n-1)!}{x^n}.$$

$$\ln 1 = 0,\ (\ln x)'(1) = 1,\ (\ln x)''(1) = -1, \ldots,\ (\ln x)^{(n)}(1) = (-1)^{n-1}(n-1)!.$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(\ln x)^{(n)}(1)}{n!}(x-1)^n = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}.$$

Apply the ratio test:

$$\lim_{n\to\infty} \frac{|n(-1)^n(x-1)^{n+1}|}{|(n+1)(-1)^{n-1}(x-1)^n|} = |x-1| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = |x-1|.$$

Hence the series converges absolutely for $|x-1| < 1$. The radius of convergence is $\rho = 1$.

2. Determine all the singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x^2)y'' + \frac{2}{x}y' + 4y = 0.$$

Solution:

$$y'' + \frac{2}{x^3(1-x)(1+x)}y' + \frac{4}{x^2(1-x)(1+x)}y = 0.$$

The three singular points are 0, 1 and -1. Near $x = 0$,

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to 0}\frac{2}{x^2} \quad \text{does not exist.}$$

So $x = 0$ is an irregular singular point. Near $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to 1}\frac{-2}{x^3(1+x)} = -1.$$

$$\lim_{x\to 1}(x-1)^2q(x) = \lim_{x\to 1}(x-1)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to 1}\frac{4(1-x)}{x^2(1+x)} = 0.$$

So $x = 1$ is a regular singular point. Near $x = -1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to -1}\frac{2}{x^3(1-x)} = -1.$$

$$\lim_{x\to -1}(x+1)^2q(x) = \lim_{x\to -1}(x+1)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to -1}\frac{4(1+x)}{x^2(1-x)} = 0.$$

So $x = -1$ is a regular singular point.

3. Determine two linearly independent solutions about $x = 0$ to the following equation.

$$(1-x^2)y'' - 2xy' + 2y = 0.$$

Solution:

$x = 0$ is an ordinary point of the equation. Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$(1-x^2)y'' = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n, \quad xy' = \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n.$$

$$y'' - 2xy' + 2y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (n(n-1) + 2n - 2)a_n]x^n = 0$$

$$a_{n+2} = \frac{(n-1)(n+2)}{(n+1)(n+2)}a_n = \frac{n-1}{n+1}a_n, \ n \geq 0,$$

which is equivalent to

$$a_n = \frac{n-3}{n-1}a_{n-2}, \ n \geq 2.$$

a_0 and a_1 are arbitrary, so the general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

For $y_1(x)$,

$$a_{2n} = \frac{(-1)\cdot 1 \cdot 3 \cdots (2n-3)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}a_0 = -\frac{a_0}{2n-1}.$$

$$y_1(x) = 1 - \sum_{n=1}^{\infty} \frac{x^{2n}}{2n-1}.$$

For $y_2(x)$,

$$a_3 = 0 \Rightarrow a_{2n+1} = 0, \ n \geq 1.$$

$$y_2(x) = x.$$

4. Determine two linearly independent solutions about $x = 0$ to the following equation.

$$x^2y'' + 3xy' + (1+x)y = 0.$$

Solution:

It's easy to verify that $x = 0$ is a regular singular point. The corresponding Euler equation is

$$x^2y'' + 3xy' + y = 0.$$

The indicial equation is

$$F(r) = r(r-1) + 3r + 1 = (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1.$$

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n,$$

where $r = r_1 = -1$.

$$x^2y'' = x^r \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^n, \quad xy' = x^r \sum_{n=0}^{\infty}(n+r)a_nx^n, \quad (1+x)y = x^r[\sum_{n=1}^{\infty}(a_n+a_{n-1})x^n + a_0].$$

$$x^2y'' + 3xy' + (1+x)y = x^r\{\sum_{n=1}^{\infty}[F(n+r)a_n + a_{n-1}]x^n + F(r)a_0\} = 0.$$

Since $F(r_1) = 0$, a_0 is arbitrary, we set $a_0 = 1$. For $n \geq 1$,

$$a_n = \frac{-1}{F(r+n)}a_{n-1} = \frac{-1}{(r+n+1)^2}a_{n-1} \Rightarrow a_n(r) = \frac{(-1)^n}{(r+2)^2 \dots (r+n+1)^2}.$$

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n = x^{-1}[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n!)^2}].$$

Since $r_1 = r_2$,

$$y_2(x) = y_1(x)\ln x + x^{r_1} \sum_{n=1}^{\infty} b_nx^n.$$

$$a_n'(r) = (\frac{(-1)^n}{(r+2)^2 \dots (r+n+1)^2})'(r) = \frac{(-1)^n}{(r+2)^2 \dots (r+n+1)^2}(\frac{-2}{r+2} + \dots + \frac{-2}{r+n+1}).$$

$$b_n = a_n'(r_1) = -(-1)^n 2\frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{(n!)^2}.$$

Therefore,

$$y_2(x) = y_1(x)\ln x - 2x^{-1}\sum_{n=1}^{\infty}(-1)^n \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{(n!)^2}x^n.$$

5. Determine two linearly independent solutions about $x = 0$ to the following equation.

$$x^2y'' + xy' + (x^2 - \frac{9}{4})y = 0.$$

Solution:

It's easy to verify that $x = 0$ is a regular singular point. The corresponding Euler equation is

$$x^2y'' + xy' - \frac{9}{4}y = 0.$$

The indicial equation is

$$F(r) = r(r-1) + r - \frac{9}{4} = (r - \frac{3}{2})(r + \frac{3}{2}) = 0 \Rightarrow r_1 = \frac{3}{2},\ r_2 = -\frac{3}{2}.$$

$$y_1(x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n,$$

where $r = r_1 = 3/2$.

$$x^2y'' = x^r \sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^n, \quad xy' = x^r \sum_{n=0}^{\infty}(n+r)a_nx^n.$$

$$(x^2 - \frac{9}{4})y = x^r[\sum_{n=2}^{\infty}(a_{n-2} - \frac{9}{4}a_n)x^n - \frac{9}{4}(a_0 + a_1x)].$$

$$x^2y'' + xy' + (x^2 - \frac{9}{4})y = x^r\{\sum_{n=2}^{\infty}[F(n+r)a_n + a_{n-2}]x^n + F(r)a_0 + F(r+1)a_1x\} = 0.$$

Since $F(r_1) = 0$, a_0 is arbitrary, we set $a_0 = 1$. $F(r_1 + 1) \neq 0 \Rightarrow a_1 = 0 \Rightarrow a_{2n+1} = 0$.

$$a_n = \frac{-a_{n-2}}{(r+n-\frac{3}{2})(r+n+\frac{3}{2})} \Rightarrow a_{2n}(r) = \frac{(-1)^n}{(r+2-\frac{3}{2})(r+2+\frac{3}{2})\dots(r+2n-\frac{3}{2})(r+2n+\frac{3}{2})}.$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_{2n}(r_1)x^{2n} = x^{3/2}[1 + \sum_{n=1}^{\infty}\frac{(-1)^n 3x^{2n}}{(2n+3)(2n+1)!}] = 3x^{3/2}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)(2n+1)!}.$$

Since $N = r_1 - r_2 = 3$,

$$y_2(x) = ay_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} c_nx^n.$$

$$a = \lim_{r\to r_2}(r - r_2)a_N(r) = \lim_{r\to r_2}(r - r_2)a_3(r) = \lim_{r\to r_2}(r - r_2)0 = 0.$$

Since y_2 does not contain the logarithm term, $c_n = a_n(r_2)$. $c_0 = 1$, and

$$c_{2n} = \frac{(-1)^n}{(r_2+2-\frac{3}{2})(r_2+2+\frac{3}{2})\dots(r_2+2n-\frac{3}{2})(r_2+2n+\frac{3}{2})} = \frac{(-1)^{n+1}}{2n(2n-2)!}.$$

$$y_2(x) = x^{-3/2}[1 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2n(2n-2)!}].$$