MATH555
Differential Equations
Practice Exam II

1. Solve the given initial value problem

$$y'' + 4y = 3\sin 2t, \quad y(0) = 2, \quad y'(0) = -1.$$

Solution:
First solve the homogeneous equation.

$$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y = c_1 \cos 2t + c_2 \sin 2t.$$

Next we find the particular solution to $y'' + 4y = 3\sin 2t$. Since $2i$ is a root of the characteristic equation, let $Y = at\cos 2t + bt\sin 2t$.

$$Y'' + 4Y = -4a\sin 2t + 4b\cos 2t = 3\sin 2t \Rightarrow a = -\frac{3}{4},\ b = 0 \Rightarrow Y = -\frac{3}{4}t\cos 2t.$$

So the general solution is

$$y(t) = c_1\cos 2t + c_2\sin 2t - \frac{3}{4}t\cos 2t.$$

$$y'(t) = -2c_1\sin 2t + 2c_2\cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t.$$

$$y(0) = 2,\ y'(0) = -1 \Rightarrow c_1 = 2,\ 2c_2 - \frac{3}{4} = -1 \Rightarrow c_1 = 2,\ c_2 = -\frac{1}{8}.$$

$$y(t) = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t.$$

2. Find the Wronskian determinant (up to a constant factor) of two independent solutions to the following equation.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$

Solution:

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0.$$

$$p(x) = \frac{1}{x} \Rightarrow W = \exp(-\int p(x)\mathrm{d}x) = \exp\int \frac{-\mathrm{d}x}{x} = e^{-\ln x + C} = \frac{A}{x}.$$

3. Find the general solution to the given equation

$$y'' + 2y' + y = 6te^{-t}.$$

Solution:

First solve the homogeneous equation.

$$y'' + 2y' + y = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1 \Rightarrow y = e^{-t}(c_1 t + c_2).$$

Next we find a particular solution to $y''+2y'+y = 6te^{-t}$. Since -1 is a double root of the characteristic equation, let $Y = t^2 e^{-t}(at+b)$.

$$Y'' + 2Y' + Y = e^{-t}(6at + 2b) = 6te^{-t} \Rightarrow a = 1,\ b = 0 \Rightarrow Y = t^3 e^{-t}.$$

So the general solution is

$$y(t) = e^{-t}(t^3 + c_1 t + c_2).$$

4. Find the general solution to the given equation

$$t^2y'' - 3ty' + 4y = 2t^2.$$

Solution:

First solve the homogeneous equation. Assume $y(t) = t^n$ is a solution to

$$t^2y'' - 3ty' + 4y = 0.$$

then

$$n(n-1)t^n - 3nt^n + 4t^n = t^n(n^2 - 4n + 4) = t^n(n-2)^2 = 0 \Rightarrow n_1 = n_2 = 2.$$

$$y(t) = c_1t^2 + c_2t^2 \ln t.$$

Next we find a particular solution to $t^2y'' - 3ty' + 4y = 2t^2$. Since 2 is a double root of the characteristic equation, let $Y = at^2 \ln^2 t$.

$$t^2Y'' - 3tY' + 4Y = 2at^2 = 2t^2 \Rightarrow a = 1 \Rightarrow Y = t^2 \ln^2 t.$$

So the general solution is

$$y(t) = t^2(c_1 + c_2 \ln t + \ln^2 t).$$

5.

$$ty'' - (1+t)y' + y = t^2e^{2t}.$$

(a) (*10 points*) Verify that $y_1(t) = 1+t$ is a solution to the homogeneous equation. Find another linearly independent solution.
(b) (*10 points*) Find the general solution to the nonhomogeneous equation.

Solution:

(a) It's easy to verify that

$$ty_1'' - (1+t)y_1' + y_1 = -(1+t) + (1+t) = 0,$$

so $y_1(t) = 1+t$ is indeed a solution to the homogeneous equation. Rewrite the equation as

$$y'' - (1+\frac{1}{t})y' + \frac{1}{t}y = te^{2t}.$$

$$p(t) = -(1+\frac{1}{t}), \quad g(t) = te^{2t}.$$

The Wronskian determinant is

$$W(t) = \exp(-\int p(t)\mathrm{d}t) = te^t.$$

The other linearly independent solution is

$$y_2(t) = y_1\int \mathrm{d}t\frac{W(t)}{y_1^2(t)} = (1+t)\int \mathrm{d}t\frac{te^t}{(1+t)^2} = (1+t)\frac{e^t}{1+t} = e^t.$$

(b) A particular solution to the nonhomogeneous equation is

$$\begin{aligned} Y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t \\ &= -(t+1)\int \frac{e^t\cdot te^{2t}}{te^t}\mathrm{d}t + e^t\int \frac{(t+1)\cdot te^{2t}}{te^t}\mathrm{d}t \\ &= -(t+1)\frac{e^{2t}}{2} + e^t te^t = \frac{t-1}{2}e^{2t}. \end{aligned}$$

So the general solution is

$$y(t) = \frac{t-1}{2}e^{2t} + c_1(t+1) + c_2e^t.$$

6.

$$x'' + \gamma x' + 4x = \cos 2t + \sin 2t, \quad x(0) = 0, \quad x'(0) = 0.$$

(a) Assume $\gamma > 0$, find the steady state $x(t)$ and its amplitude.
(b) Assume $\gamma = 0$, find $x(t)$.

Solution:

(a) Let the steady state be $x(t) = a\cos 2t + b\sin 2t$,

$$x'' + x' + 4x = \gamma(2b\cos 2t - 2a\sin 2t) = \cos 2t + \sin 2t \Rightarrow a = -\frac{1}{2\gamma}, \ b = \frac{1}{2\gamma}.$$

$$x(t) = \frac{1}{2\gamma}(\sin 2t - \cos 2t) = \frac{1}{\sqrt{2}\gamma}\sin(2t - \frac{\pi}{4}).$$

So the amplitude of the steady state is $1/(\sqrt{2}\gamma)$.

(b)

$$x'' + 4x = 0 \Rightarrow x(t) = c_1\cos 2t + c_2\sin 2t.$$

For $x''+4x = \cos 2t+\sin 2t$, we look for a particular solution in the form of $X(t) = t(a\cos 2t+b\sin 2t)$.

$$X'' + 4X = 4b\cos 2t - 4a\sin 2t = \cos 2t + \sin 2t \Rightarrow b = \frac{1}{4}, \ a = -\frac{1}{4}.$$

$$X = \frac{t}{4}(\sin 2t - \cos 2t) \Rightarrow x(t) = \frac{t}{4}(\sin 2t - \cos 2t) + c_1\cos 2t + c_2\sin 2t.$$

$$x(0) = 0, \ x'(0) = 0 \Rightarrow c_1 = 0, \ c_2 = -\frac{1}{8}.$$

$$x(t) = \frac{t}{4}(\sin 2t - \cos 2t) - \frac{1}{8}\sin 2t.$$