# MATH555 Fall 2009
# Differential Equations
# Exam II

1. (*20 points*) Solve the given initial value problem

$$y'' + 2y' - 3y = 10\cos t, \quad y(0) = 0, \quad y'(0) = 9.$$

*Solution:*

First solve the homogeneous equation.

$$y'' + 2y' - 3y = 0 \Rightarrow (r-1)(r+3) = 0 \Rightarrow y = c_1 e^t + c_2 e^{-3t}.$$

Next we find the particular solution to $y'' + 2y' - 3y = 10\cos t$. Fince $i$ is not a root of the characteristic equation, let $Y = a\cos t + b\sin t$.

$$Y'' + 2Y' - 3Y = (8b - 4a)\cos t - (2a + 4b)\sin t = 10\cos t \Rightarrow a = -2,\ b = 1 \Rightarrow Y = \sin t - 2\cos t.$$

So the general solution is

$$y(t) = c_1 e^t + c_2 e^{-3t} + \sin t - 2\cos t \Rightarrow y'(t) = c_1 e^t - 3c_2 e^{-3t} + \cos t + 2\sin t.$$

$$y(0) = 0, y'(0) = 3 \Rightarrow c_1 = 2,\ c_2 = 0 \Rightarrow y(t) = 2e^t + \sin t - 2\cos t.$$

2. (*20 points*) Let $y_1(t)$ be the solution to the initial value problem

$$(1-x^2)y'' - 2xy' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 1;$$

$y_2(t)$ be the solution to the initial value problem

$$(1-x^2)y'' - 2xy' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 0;$$

Find their Wronskian determinant $W(t) = y_1(t)y_2'(t) - y_2(t)y_1'(t)$. (Hint: you don't need to solve the equation.)

*Solution:*

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = 0.$$

$$p(x) = -\frac{2x}{1-x^2} \Rightarrow W(x) = \exp(-\int p(x)\mathrm{d}x) = \exp\int \frac{2x\mathrm{d}x}{1-x^2} = e^{-\ln(1-x^2)+C} = \frac{A}{1-x^2}.$$

We can determine the constant $A$ by $W0$).

$$A = W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 0 - 1 = -1.$$

So

$$W(t) = -\frac{1}{1-x^2}.$$

3. (*30 points*) Find the general solution to the given equation

$$y'' - 2y' + y = 2e^t + t.$$

*Solution:*

First solve the homogeneous equation.

$$y'' - 2y' + y = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Rightarrow y = e^t(c_1 t + c_2).$$

Next we find the particular solution to $y'' - 2y' + y = 2e^t$ and $y'' - 2y' + y = t$ and add them up. For $y'' - 2y' + y = 2e^t$, since 1 is a double root of the characteristic equation, let $Y_1 = ae^t t^2$ .

$$Y_1'' - 2Y_1' + Y_1 = ae^t[t^2 + 4t + 2 - 2(t^2 + 2t) + t^2] = 2ae^t = 2e^t \Rightarrow a = 1 \Rightarrow Y_1 = e^t t^2.$$

For $y'' - 2y' + y = t$, since 0 is not a root of the characteristic equation, let $Y_2 = at + b$.

$$Y_2'' - 2Y_2' + Y_2 = -2a + at + b = at + (b - 2a) = t \Rightarrow a = 1,\ b = 2 \Rightarrow Y_2 = t + 2.$$

So the general solution is

$$y(t) = e^t(t^2 + c_1 t + c_2) + t + 2.$$

4.

$$(1-x^2)y'' - 2xy' + 2y = 4.$$

(a) (*15 points*) Verify that $y_1(x) = x$ is a solution to the homogeneous equation. Find another linearly independent solution.
(b) (*15 points*) Find the general solution to the nonhomogeneous equation.

*Solution:*

(a) It's easy to verify that

$$(1-x^2)y_1'' - 2xy_1' + 2y_1 = -2x + 2x = 0,$$

so $y_1(x) = x$ is indeed a solution to the homogeneous equation. Rewrite the equation as

$$y'' - \frac{2x}{1-x^2}y' + \frac{2}{1-x^2}y = \frac{4}{1-x^2}.$$

$$p(x) = -\frac{2x}{1-x^2}, \quad g(x) = \frac{4}{1-x^2}.$$

The Wronskian determinant is

$$W(x) = \exp(-\int p(x)\mathrm{d}x) = \exp\int \frac{2x\mathrm{d}x}{1-x^2} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}.$$

The other linearly independent solution is

$$y_2(x) = y_1 \int \mathrm{d}x \frac{W(x)}{y_1^2(x)} = x\int \frac{\mathrm{d}x}{(1-x^2)x^2} = \frac{x}{2}\ln\frac{1+x}{1-x} - 1.$$

(b) It's easy to see that $Y(x) = 2$ is a particular solution to the nonhomogeneous equation, so the general solution is

$$y(x) = c_1 x + c_2[x\ln\frac{1+x}{1-x} - 2] + 2.$$

Alternatively, a particular solution to the nonhomogeneous equation can be obtained by the following formula.

$$\begin{aligned}
Y(x) &= -y_1(x)\int \frac{y_2(x)g(x)}{W(x)}\mathrm{d}x + y_2(x)\int \frac{y_1(x)g(x)}{W(x)}\mathrm{d}x \\
&= -x\int (2x\ln\frac{1+x}{1-x} - 4)\mathrm{d}x + (\frac{x}{2}\ln\frac{1+x}{1-x} - 1)\int 4x\mathrm{d}x \\
&= -x[(x^2-1)\ln\frac{1+x}{1-x} - 2x] + [x\ln\frac{1+x}{1-x} - 2]x^2 = x\ln\frac{1+x}{1-x}.
\end{aligned}$$

So the general solution can also be written as

$$y(x) = c_1 x + c_2[x\ln\frac{1+x}{1-x} - 2] + x\ln\frac{1+x}{1-x}.$$

5. (*Bonus problem*) (*20 points*)
Find the general solution to the given equation

$$t^2 y'' + 3ty' + y = t^2.$$

*Solution:*
First solve the homogeneous equation, which is an Euler equation. Assume $y(t) = t^n$ is a solution to the given equation, then

$$n(n-1) + 3n + 1 = n^2 + 2n + 1 = (n+1)^2 = 0 \Rightarrow n_1 = n_2 = -1.$$

$$y_1(t) = \frac{1}{t}, \quad y_2(t) = \frac{\ln t}{t}.$$

Next we find a particular solution to $t^2y''+3ty'+y = t^2$. Since $t^2$ is not a solution to the homogeneous equation, let $Y = at^2$. We have

$$t^2 Y'' + 3tY' + Y = (2+6+1)at^2 = 9at^2 = t^2.$$

So $a = 1/9$. The general solution is

$$y(t) = \frac{t^2}{9} + \frac{c_1 + c_2 \ln t}{t}.$$