MATH555
Differential Equations
Practice Exam I

1. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 2\cos t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

Solution:

$$y' - y = 1 + 2\cos t.$$

$$p(t) = -1 \Rightarrow \mu(t) = \exp \int p(t)\mathrm{d}t = e^{-t}.$$

$$y(t) = \frac{1}{e^{-t}} \int e^{-t}(1 + 2\cos t)\mathrm{d}t = e^{t}[e^{-t}(-1 - \cos t + \sin t) + C] = -1 - \cos t + \sin t + Ce^{t}.$$

It remains finite as $t \to \infty$ if and only if $C = 0$, which corresponds to

$$y_0 = -1 - 1 + 0 = -2.$$

2. Find the solution of the given initial value problems in explicit form, and determine the interval in which the solution is defined and *correct*.

$$x^2y' + \sqrt{1-y^2} = 0, \quad y(1) = 0.$$

Solution:

$$\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} = -\int \frac{\mathrm{d}x}{x^2}.$$

$$\arcsin y = \frac{1}{x} + C \Rightarrow y = \sin(\frac{1}{x} + C).$$

$$y(1) = 0 \Rightarrow 0 = \sin(1+C) = 0 \Rightarrow C = -1 \Rightarrow y = \sin(\frac{1}{x} - 1).$$

$\sin(\frac{1}{x} - 1)$ is defined for all x in the interval $(0, \infty)$. However, it is not the correct solution for all $x > 0$. From the obtained solution,

$$y' = \cos(\frac{1}{x} - 1)(-\frac{1}{x^2}).$$

From the differential equation, $y' \leq 0$ for all $x > 0$. So

$$\cos(\frac{1}{x} - 1) \geq 0 \Rightarrow -\frac{\pi}{2} \leq \frac{1}{x} - 1 \leq \frac{\pi}{2} \Rightarrow x \in [\frac{2}{2+\pi}, \infty).$$

It is the interval on which the obtained solution is correct.

3. Solve the given initial value problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3}{x^3 + xy^2}, \quad y(1) = 1.$$

Solution:
Let $p = y/x$,

$$\frac{\mathrm{d}(xp)}{\mathrm{d}x} = \frac{p^3}{1+p^2} \Rightarrow x\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p^3}{1+p^2} - p = -\frac{p}{1+p^2}.$$

$$\int \frac{\mathrm{d}p(1+p^2)}{p} = -\int \frac{\mathrm{d}x}{x} \Rightarrow \ln|p| + \frac{p^2}{2} = -\ln|x| + C.$$

$$xpe^{\frac{p^2}{2}} = A.$$

$$y(1) = 1 \Rightarrow A = e^{\frac{1}{2}} \Rightarrow y\exp(\frac{y^2 - x^2}{2x^2}) = 1.$$

4. Find an integrating factor and solve the given initial value problem.

$$2xy\mathrm{d}x + (3x^2 + 4y)\mathrm{d}y = 0, \quad y(0) = 1.$$

Solution:

$$M_y - N_x = 2x - 6x = -4x = -\frac{2}{y}M.$$

So we can let μ be a function of y, and

$$\frac{\mathrm{d}\mu}{\mathrm{d}y} = -\frac{M_y - N_x}{M}\mu = \frac{2}{y}\mu \Rightarrow \mu = y^2.$$

Integrating

$$\psi_x = \mu M = 2xy^3,$$

we have

$$\psi = x^2y^3 + h(y).$$

Setting $\psi_y = \mu N$ we have $h'(y) = 4y^3$ so we can let $h(y) = y^4$. The solution to the differential equation is

$$\psi = x^2y^3 + y^4 = C.$$

$$y(0) = 1 \Rightarrow C = 1 \Rightarrow x^2y^3 + y^4 = 1.$$

5. A raindrop falling in the air is subject to gravity (downward) and drag (upward). Assume that the drag is proportional to the speed of the raindrop. The downward acceleration is given by

$$a = g - kv,$$

where g and k are constants. The raindrop starts from still.
(a) Find the terminal (equilibrium) velocity v_T of the rain drop, and the time for the velocity of the raindrop to reach half of v_T.
(b) Denote by x the distance travelled by the raindrop. Use the relation $dv/dt = v(dv/dx)$ to write the equation of motion in terms of v and x. Find the distance travelled by the raindrop by the time its velocity reaches half of v_T.

Solution:
(a) Denote the terminal velocity by v_T. At terminal velocity, the raindrop no longer accelerates, ie. $a = 0$, so

$$g - kv_T = 0 \Rightarrow v_T = \frac{g}{k}.$$

We solve

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv = k(v_T - v), \quad v(0) = 0.$$

$$kt + C = k\int \mathrm{d}t = \int \frac{\mathrm{d}v}{v_T - v} = -\ln|v_T - v| \Rightarrow v(t) = v_T - Ae^{-kt}.$$

$$v(0) = 0 \Rightarrow A = v_T \Rightarrow v(t) = v_T(1 - e^{-kt}).$$

Let the time for the velocity of the raindrop to reach half of v_T be T, then

$$v(T) = \frac{v_T}{2} \Rightarrow kT = \ln 2 \Rightarrow T = \frac{\ln 2}{k}.$$

(b) The equation for $x(v)$ is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x} = g - kv = k(v_T - v), \quad x(0) = 0.$$

The solution is

$$kx + C = k\int \mathrm{d}x = \int \frac{v\mathrm{d}v}{v_T - v} = -v_T \ln|v_T - v| - v.$$

$$x(0) = 0 \Rightarrow C = -v_T \ln v_T \Rightarrow x = -\frac{v_T}{k}[\frac{v}{v_T} + \ln(1 - \frac{v}{v_T})].$$

Therefore when $v = v_T/2$,

$$x = -\frac{v_T}{k}(\frac{1}{2} + \ln\frac{1}{2}) = \frac{g}{k^2}(\ln 2 - \frac{1}{2}).$$

6. Solve the given initial value problem.

$$y' = \frac{2y}{x} + y^3 + x^2 y. \quad y(1) = 1.$$

Solution:

We know the solution to the equatoon

$$y' = \frac{2y}{x}.$$

is $y = Cx^2$. Now we use the method of variation of parameters. Let

$$y = z(x)x^2.$$

$$x^2 z' = y' - \frac{2y}{x} = y^2 + x^2 y \Rightarrow z' = x^2(z^2 + z).$$

$$\int \frac{\mathrm{d}z}{z(z+1)} = \ln|\frac{z}{z+1}| = \int x^2 \mathrm{d}x = \frac{x^3}{3} + C.$$

$$z = \frac{1}{Ae^{-\frac{x^3}{3}} - 1} \Rightarrow y = \frac{x^2}{Ae^{-\frac{x^3}{3}} - 1}.$$

$$y(1) = 1 \Rightarrow A = 2e^{\frac{1}{3}} \Rightarrow y = \frac{x^2}{2e^{\frac{1-x^3}{3}} - 1}.$$