MATH555 Fall 2009
Differential Equations
Exam I

1. (*20 points*) Solve the given initial value problem

$$ty' + y = 2t + 1, \quad y(1) = 1.$$

Solution:

$$y' + \frac{1}{t}y = 2 + \frac{1}{t}.$$

$$p(t) = \frac{1}{t} \Rightarrow \mu(t) = \exp \int p(t)\mathrm{d}t = e^{\ln|t|} = t. \quad (t > 0)$$

$$g(t) = 2 + \frac{1}{t} \Rightarrow y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\mathrm{d}t = \frac{1}{t}\int (2t+1)\mathrm{d}t = t^{-1}(t^2 + t + C) = t + 1 + \frac{C}{t}.$$

$$y(1) = 1 \Rightarrow 1 + 1 + C = 1 \Rightarrow C = -1 \Rightarrow y(t) = t + 1 - \frac{1}{t}.$$

2. (*20 points*) Find the solution of the given initial value problem in explicit form, and determine the interval in which the solution is defined.

$$\mathrm{d}x = \cos^2 x \cos y \mathrm{d}y, \quad y(0) = 0.$$

Solution:

$$\int \sec^2 x \mathrm{d}x = \int \cos y \mathrm{d}y.$$

$$\tan x = \sin y + C.$$

$$y(0) = 0 \Rightarrow C = 0 \Rightarrow y = \arcsin(\tan x).$$

$y(x)$ is defined in an interval around 0 in which $-1 \leq \tan x \leq 1$, ie.,

$$[-\frac{\pi}{4}, \frac{\pi}{4}].$$

3. (*20 points*) Solve the given initial value problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4y-3x}{2x-y}, \quad y(1) = -2.$$

Solution:

Let $p = y/x$,

$$\frac{\mathrm{d}(xp)}{\mathrm{d}x} = \frac{4p-3}{2-p} \Rightarrow x\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{4p-3}{2-p} - p = \frac{(p-1)(p+3)}{2-p}.$$

$$\int \frac{\mathrm{d}p(2-p)}{(p-1)(p+3)} = \int \frac{\mathrm{d}x}{x} \Rightarrow \frac{1}{4}\int (\frac{1}{p-1} - \frac{5}{p+3})\mathrm{d}p = \frac{\ln|p-1| - 5\ln|p+3|}{4} = \ln|x| + C.$$

$$\frac{p-1}{(p+3)^5} = Ax^4 \Rightarrow y - x = A(y+3x)^5.$$

$$y(1) = -2 \Rightarrow A = -3 \Rightarrow y - x + 3(y+3x)^5 = 0.$$

4. (*20 points*) Find an integrating factor and solve the given initial value problem.

$$(2x^2y + 2xy^2 + y)\mathrm{d}x + (x + 2y)\mathrm{d}y = 0, \quad y(0) = 1.$$

Solution:

$$M_y - N_x = 2x^2 + 4xy + 1 - 1 = 2x(x + 2y) = 2xN.$$

So we can let μ be a function of x, and

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N}\mu = 2x\mu \Rightarrow \mu = e^{x^2}.$$

Integrating

$$\psi_x = \mu M = e^{x^2}(2x^2y + 2xy^2 + y),$$

we have

$$\psi = e^{x^2}(xy + y^2) + h(y).$$

Setting $\psi_y = \mu N$ we have $h'(y) = 0$ so we can let $h = 0$. The solution to the differential equation is

$$\psi = e^{x^2}(xy + y^2) = C.$$

$$y(0) = 1 \Rightarrow C = 1 \Rightarrow e^{x^2}(xy + y^2) = 1.$$

5. A raindrop falling in the air is subject to gravity (downward) and drag (upward). The drag is proportional to the square of the speed of the raindrop. The downward acceleration is given by

$$a = g - kv^2,$$

where g and k are constants. The raindrop starts from still.
(a) (*10 points*) Find the terminal (equilibrium) velocity v_T of the rain drop, and the time for the velocity of the raindrop to reach half of v_T.
(b) (*10 points*) Denote by x the distance travelled by the raindrop. Use the relation $dv/dt = v(dv/dx)$ to write the equation of motion in terms of v and x. Find the distance travelled by the raindrop by the time its velocity reaches half of v_T.

Solution:
(a) Denote the terminal velocity by v_T. At terminal velocity, the raindrop no longer accelerates, ie. $a = 0$, so

$$g - kv_T^2 = 0 \Rightarrow v_T = \sqrt{\frac{g}{k}}.$$

We solve

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2 = k(v_T^2 - v^2), \quad v(0) = 0.$$

$$kt + C = k\int \mathrm{d}t = \int \frac{\mathrm{d}v}{v_T^2 - v^2} = \frac{1}{2v_T}\int \mathrm{d}v(\frac{1}{v_T + v} + \frac{1}{v_T - v}) = \frac{1}{2v_T}\ln\frac{v_T + v}{v_T - v}.$$

$$v(0) = 0 \Rightarrow C = 0.$$

Let the time for the velocity of the raindrop to reach half of v_T be T, then

$$v(T) = \frac{v_T}{2} \Rightarrow kT = \frac{1}{2v_T}\ln\frac{1.5v_T}{0.5v_T} = \frac{\ln 3}{2v_T} \Rightarrow T = \frac{\ln 3}{2kv_T} = \frac{\ln 3}{2\sqrt{gk}}.$$

(b) The equation for $x(v)$ is

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x} = g - kv^2 = k(v_T^2 - v^2), \quad x(0) = 0.$$

The solution is

$$kx + C = k\int \mathrm{d}x = \int \frac{v\mathrm{d}v}{v_T^2 - v^2} = -\frac{1}{2}\ln|v_T^2 - v^2|.$$

$$x(0) = 0 \Rightarrow C = -\frac{1}{2}\ln v_T^2 \Rightarrow x = \frac{1}{2k}\ln\frac{v_T^2}{v_T^2 - v^2}.$$

Therefore when $v = v_T/2$,

$$x = \frac{1}{2k}\ln\frac{1}{1 - \frac{1}{4}} = \frac{1}{2k}\ln\frac{4}{3}.$$

6. (*Bonus problem*)

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y, \quad y(0) = 1.$$

(a) (*10 points*) Solve the given initial value problem and find $y(1)$. Then apply Euler method to the problem with $y_0 = y(0)$ and $h = 1/n$, where n is a positive integer. Derive the formula for y_n, which corresponds to the approximation of $y(1)$. Show that it converge to the exact $y(1)$ as $n \to \infty$.
(b) (*10 points*) Apply the method of successive approximations to the problem with $\phi_0 = 1$. Derive the formula for $\phi_n(t)$. Show that $\phi_n(1)$ also converges to the exact $y(1)$ as $n \to \infty$.

Solution:
(a) The exact solution is

$$y(t) = e^t \Rightarrow y(1) = e.$$

Using Euler method,

$$y_{n+1} = y_n + hy_n = (1+h)y_n.$$

Using the relation recursively,

$$y_n = (1+h)y_{n-1} = (1+h)^2 y_{n-2} = \cdots = (1+h)^n y_0 = (1+h)^n = (1+\frac{1}{n})^n.$$

And we know from the definition of e that

$$\lim_{n\to\infty} (1+\frac{1}{n})^n = e = y(1).$$

(b)

$$\phi_{n+1}(t) = y(0) + \int_0^t \phi_n(s)\mathrm{d}s = 1 + \int_0^t \phi_n(s)\mathrm{d}s.$$

$\phi_0(t) = 1$,

$$\phi_1(t) = 1 + \int_0^t 1\mathrm{d}s = 1 + t,$$

$$\phi_2(t) = 1 + \int_0^t (1+s)\mathrm{d}s = 1 + t + \frac{t^2}{2!},$$

$$\phi_3(t) = 1 + \int_0^t (1 + s + \frac{s^2}{2!})\mathrm{d}s = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}.$$

Upon inspection, it becomes apparent that

$$\phi_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}.$$

So

$$\phi_n(1) = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

It is exactly the truncated Maclaurin series of e^t evaluated at $t = 1$, which converges to e^1 as $n \to \infty$.