1 Solution to Homework 1

Section 1.1

6.

$$y_{eq} + 2 = 0 \Rightarrow y_{eq} = -2.$$

If $y_0 = y_{eq}$, $y \equiv -2$. If $y_0 > y_{eq}$, $y(t) \to \infty$ as $t \to \infty$. If $y_0 < y_{eq}$, $y(t) \to -\infty$ as $t \to \infty$.

20. From Figure 1.1.10, y_{eq} is 0 and 3. So the differential equation is either (e) $y' = y(y-3)$ or (h) $y' = y(3-y)$. Since the slope is negative for $0 < y < 3$, the correct equation is (e) $y' = y(y-3)$.

22. Denote the radius, surface area and volume of the raindrop by R, S, and V respectively. By the assumption,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -kS,$$

where k is a constant rate of evaporation. V and S are not independent,

$$V = \frac{4\pi}{3}R^3, \quad S = 4\pi R^2 \Rightarrow S = 4\pi(\frac{3V}{4\pi})^{2/3} = \sqrt[3]{36\pi}V^{2/3}.$$

So the differential equation for V is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt[3]{36\pi}V^{2/3}.$$

26. Notice that $y = t - 3$ is a solution of the equation. Let $z = y - (t-3)$, then

$$z' = y' - 1 = -2 + t - y - 1 = -z.$$

So $z = Ae^{-t}$. In other words, if $y(0) = -3$, $y(t) = t - 3$; if $y(0) > -3$, $y(t)$ approaches $t - 3$ from above as $t \to \infty$; if $y(0) < -3$, $y(t)$ approaches $t - 3$ from below as $t \to \infty$.

Section 1.2

3. (a) $y_{eq} = \frac{b}{a}$. Let $z = y - y_{eq}$.

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = -a(y - y_{eq}) = -az.$$

So

$$z(t) = Ae^{-at} \Rightarrow y(t) = \frac{b}{a} + Ae^{-at},$$

where A is an arbitrary constant.

(b) If $y(0) = y_{eq}$, $y(t) \equiv y_{eq}$; if $y(0) > y_{eq}$, $y(t) \to y_{eq}$ from above as $t \to \infty$; if $y(0) < y_{eq}$, $y(t) \to y_{eq}$ from below as $t \to \infty$.

(c)
i. As a increases, y_{eq} decreases toward 0, and the convergence to y_{eq} is faster.

ii. As b increase, y_{eq} increases, while the convergence rate remains unchanged.
iii. As a and b increase proportionally, the convergence to y_{eq} is faster, while y_{eq} remains unchanged.

4. (a)

$$ay_e - b = 0 \Rightarrow y_e = \frac{b}{a}.$$

(b) $Y = y - y_e$,

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}y_e}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} = a(y - \frac{b}{a}) = a(y - y_e) = aY.$$

9. (a)

$$9.8 - \frac{v_{eq}}{5} = 0 \Rightarrow v_{eq} = 9.8 * 5 = 49.$$

Let $u = v - v_{eq}$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v - v_{eq}}{5} = -\frac{u}{5}.$$

$$u(t) = Ae^{-t/5} \Rightarrow v(t) = v_{eq} + Ae^{-t/5}.$$

$$v(0) = 0 \Rightarrow A = -v_{eq} \Rightarrow v(t) = v_{eq}(1 - e^{-t/5}).$$

Let the time for the object to reach 98% of its limiting velocity v_{eq} be T, then

$$1 - e^{-T/5} = 98\% \Rightarrow e^{-T/5} = 0.02 \Rightarrow T/5 = \ln 50 \Rightarrow T = 5\ln 50 \approx 19.56.$$

(b) The distance is

$$\begin{aligned}\int_0^T v(t)\mathrm{d}t &= \int_0^T v_{eq}(1 - e^{-t/5})\mathrm{d}t = v_{eq}[t + 5e^{-t/5}]_0^T = v_{eq}[T + 5(e^{-T/5} - 1)] \\ &= v_{eq}(T - 5 * 98\%) = 49(5\ln 50 - 4.9) \approx 718.3.\end{aligned}$$

Section 1.3

5. 2nd order nonlinear equation.

6. 3rd order linear equation.

20.

$$r(r-1)t^r - 4rt^r + 4t^r = 0 \Rightarrow t^r(r^2 - 5r + 4) = 0 \Rightarrow (r-1)(r-4) = 0.$$

The two roots of r are 1 and 4, so $y = t$ and $y = t^4$ solve the differential equation.

Section 2.1

15.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}.$$

$$p(t) = \frac{2}{t} \Rightarrow \mu(t) = \exp\int p(t)\mathrm{d}t = e^{2\ln|t|} = t^2.$$

$$g(t) = t - 1 + \frac{1}{t} \Rightarrow y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\mathrm{d}t = \frac{1}{t^2} \int (t^3 - t^2 + t)\mathrm{d}t = t^{-2}(\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C).$$

$$y(1) = \frac{1}{2} \Rightarrow \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{1}{2} \Rightarrow C = \frac{1}{12} \Rightarrow y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}.$$

20.

$$y' + (1 + \frac{1}{t})y = 1.$$

$$p(t) = 1 + \frac{1}{t} \Rightarrow \mu(t) = \exp \int p(t)\mathrm{d}t = e^{t + \ln|t|} = te^t. \quad (t > 0)$$

$$g(t) = 1 \Rightarrow y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\mathrm{d}t = \frac{1}{te^t} \int te^t\mathrm{d}t = \frac{1}{te^t}(e^t(t-1) + C) = 1 - \frac{1}{t} + \frac{C}{te^t}.$$

$$y(\ln 2) = 1 \Rightarrow C = 2 \Rightarrow y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}.$$

30.

$$y' - y = 1 + 3\sin t.$$

$$p(t) = -1 \Rightarrow \mu(t) = \exp \int p(t)\mathrm{d}t = e^{-t}.$$

$$y(t) = \frac{1}{e^{-t}} \int e^{-t}(1 + 3\sin t)\mathrm{d}t = e^t[e^{-t}(-1 - \frac{3}{2}(\cos t + \sin t)) + C] = -1 - \frac{3}{2}(\cos t + \sin t) + Ce^t.$$

It remains finite as $t \to \infty$ if and only if $C = 0$, which corresponds to

$$y_0 = -1 - \frac{3}{2}(1 + 0) = -\frac{5}{2}.$$

32.

$$y' + \frac{t}{2}y = 1.$$

$$p(t) = \frac{t}{2} \Rightarrow \mu(t) = \exp \int p(t)\mathrm{d}t = e^{t^2/4}.$$

$$y(t) = \frac{1}{e^{t^2/4}} \int e^{t^2/4}\mathrm{d}t = e^{-t^2/4}(\int_0^t e^{s^2/4}\mathrm{d}s + C) = e^{-t^2/4} \int_0^t e^{s^2/4}\mathrm{d}s + Ce^{-t^2/4}.$$

The second terms approaches 0 as $t \to \infty$. The first term becomes ∞/∞ as $t \to \infty$, so we can use the L'Hospital's rule to find its limit,

$$\lim_{t\to\infty} \frac{\int_0^t e^{s^2/4}\mathrm{d}s}{e^{t^2/4}} = \lim_{t\to\infty} \frac{e^{t^2/4}}{e^{t^2/4}\frac{t}{2}} = \lim_{t\to\infty} \frac{2}{t} = 0.$$

So all solutions approaches 0 as $t \to \infty$.

2 Solution to Homework 2

Section 2.2

6.

$$\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \int \frac{\mathrm{d}x}{x}.$$

$$\arcsin y = \ln|x| + C \Rightarrow y = \sin(\ln|x| + C).$$

19.

$$-\int \sin 2x \mathrm{d}x = \int \cos 3y \mathrm{d}y.$$

$$\frac{\cos 2x}{2} = \frac{\sin 3y}{3} + C.$$

$$y(\frac{\pi}{2}) = \frac{\pi}{3} \Rightarrow C = -\frac{1}{2} \Rightarrow \frac{\sin 3y}{3} = \frac{\cos 2x + 1}{2} = \cos^2 x \Rightarrow y = \frac{\pi - \arcsin(3\cos^2 x)}{3}.$$

$y(x)$ is defined in an interval around $\pi/2$ in which $\cos^2 x \leq 1/3$, ie., $[\arccos\frac{1}{\sqrt{3}}, \pi - \arccos\frac{1}{\sqrt{3}}]$.

21.

$$\int (3y^2 - 6y)\mathrm{d}y = \int (1 + 3x^2)\mathrm{d}x \Rightarrow y^3 - 3y^2 = x + x^3 + C.$$

$$y(0) = 1 \Rightarrow C = -2 \Rightarrow y^3 - 3y^2 + 2 = x + x^3.$$

$y' = (1+3x^2)/(3y(y-2))$. $y(0) = 1$ is between 0 and 2, as $y(x)$ varies between 0 and 2, $y' < 0$. If $y(x_+) = 0$ for certain $x_+ > 0$, and $y(x_-) = 2$ for certain $x_- < 0$, then y' diverges to $-\infty$ at both points, so the solution is invalid beyond the interval (x_-, x_+). From $y(x_+) = 0$, we have

$$2 = x_+ + x_+^3 \Rightarrow x_+ = 1.$$

From $y(x_-) = 2$, we have

$$2^3 - 3 * 2^2 + 2 = -2 = x_- + x_-^3 \Rightarrow x_- = -1.$$

So the validity interval of the solution is $(-1, 1)$.

24.

$$\int (3 + 2y)\mathrm{d}y = \int (2 - e^x)\mathrm{d}x \Rightarrow y^2 + 3y = 2x - e^x + C.$$

$$y(0) = 0 \Rightarrow C = 1 \Rightarrow y^2 + 3y = 2x + 1 - e^x \Rightarrow y = -\frac{3}{2} + \sqrt{2x + \frac{13}{4} - e^x}.$$

Since $y(0) = 0 > -3/2$, the maximum value of $y(x)$ is attained in the interior of the validity interval, and so $y' = 0$ at the maximum point.

$$y' = 0 \Rightarrow 2 - e^x = 0 \Rightarrow x = \ln 2.$$

31.

$$p = \frac{y}{x}, \quad \frac{\mathrm{d}(xp)}{\mathrm{d}x} = p^2 + p + 1 \Rightarrow x\frac{\mathrm{d}p}{\mathrm{d}x} = p^2 + 1.$$

$$\int \frac{\mathrm{d}p}{p^2+1} = \int \frac{\mathrm{d}x}{x} \Rightarrow \arctan p = \ln|x| + C.$$

$$p = \tan(\ln|x| + C) \Rightarrow y = x\tan(\ln|x| + C).$$

36.

$$p = \frac{y}{x}, \quad \frac{\mathrm{d}(xp)}{\mathrm{d}x} = p^2 + 3p + 1 \Rightarrow x\frac{\mathrm{d}p}{\mathrm{d}x} = p^2 + 2p + 1 = (p+1)^2.$$

$$\int \frac{\mathrm{d}p}{(p+1)^2} = \int \frac{\mathrm{d}x}{x} \Rightarrow -\frac{1}{p+1} = \ln|x| + C.$$

$$p + 1 = -\frac{1}{\ln|x| + C} \Rightarrow y = -\frac{x}{\ln|x| + C} - x.$$

Section 2.3

6. (a) For free fall from height h,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g, \quad v(0) = 0 \Rightarrow v(t) = gt.$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = v, \quad h(0) = 0 \Rightarrow h(t) = \int_0^t v(t)\mathrm{d}t = \frac{1}{2}gt^2 = \frac{v^2(t)}{2g}.$$

So

$$v = \sqrt{2gh}.$$

(b)

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\text{flow out}.$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{A(h)\mathrm{d}h}{\mathrm{d}t}.$$

$$\text{flow out } = \alpha a v = \alpha a\sqrt{2gh}.$$

(c)

$$\alpha = 0.6,\ a = \pi 0.1^2 = \pi/100\ m^2,\ A(h) = \pi 1^2 = \pi\ m^2,\ h(0) = 3\ m, g = 9.8\ m/s^2.$$

The solution to the differential equation is

$$\int \frac{A\mathrm{d}h}{\sqrt{2gh}} = -\int \alpha a \mathrm{d}t.$$

$$A\sqrt{\frac{2h}{g}} = -\alpha a t + C.$$

$$h(0) = 3 \Rightarrow C = A\sqrt{\frac{6}{g}}.$$

We need to find T at which $h(T) = 0$. So

$$0 = A\sqrt{\frac{2h(T)}{g}} = -\alpha aT + A\sqrt{\frac{6}{g}} \Rightarrow T = \frac{A}{\alpha a}\sqrt{\frac{6}{g}} \approx 130.41\ s.$$

24. (a)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\mu v^2.$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\mu v.$$

(b) Solving the equation for $v(x)$, we have

$$v(x) = Ae^{-\mu x}.$$

$$v(0) = 150 \Rightarrow A = 150.$$

$$v(2000) = 15 \Rightarrow 15 = 150e^{-\mu 2000} \Rightarrow \mu = \frac{\ln 10}{2000} \approx 0.00115\ \mathrm{ft}^{-1}.$$

(c) To find the slow down time T, we need to solve for $v(t)$ from the original equation.

$$-\int \frac{\mathrm{d}v}{v^2} = \int \mu \mathrm{d}t \Rightarrow \frac{1}{v} = \mu t + C.$$

$$v(0) = 150 \Rightarrow C = \frac{1}{150}.$$

$$v(T) = 15 \Rightarrow \frac{1}{15} = \mu T + \frac{1}{150} \Rightarrow T = \frac{1}{\mu}(\frac{1}{15} - \frac{1}{150}) \approx 52.12\ \mathrm{hr} \cdot \mathrm{mi}^{-1} \cdot \mathrm{ft} \approx 35.53\ \mathrm{sec}.$$

32. (a)

$$y' = \sqrt{\frac{k^2}{y} - 1}.$$

$y' \geq 0$ because the particle should always slide down to minimize the sliding time.

(b)

$$y = k^2 \sin^2 t \Rightarrow \mathrm{d}x = \frac{dy}{\sqrt{\frac{k^2}{y} - 1}} = \frac{2k^2 \sin t \cos t \mathrm{d}t}{\frac{\cos t}{\sin t}} = 2k^2 \sin^2 t \mathrm{d}t.$$

(c)

$$\theta = 2t, \quad y = k^2 \sin^2 \frac{\theta}{2} = k^2 \frac{1 - \cos\theta}{2}.$$

$$x + C = \int \mathrm{d}x = \int k^2 \sin^2 \frac{\theta}{2} \mathrm{d}\theta = \int k^2 \frac{1 - \cos\theta}{2} \mathrm{d}\theta = k^2 \frac{\theta - \sin\theta}{2}.$$

$$y(x=0) = 0 \Rightarrow C = 0.$$

(d) We need to determine θ first.

$$\frac{2}{1} = \frac{y_0}{x_0} = \frac{1-\cos\theta}{\theta - \sin\theta} \Rightarrow \theta \approx 1.401.$$

$$1 = x_0 = k^2 \frac{\theta - \sin\theta}{2} \Rightarrow k \approx 2.193.$$

Section 2.4

5. It's linear equation.

$$p(t) = \frac{2t}{4-t^2}, \quad g(t) = \frac{3t^2}{4-t^2}.$$

Since both $p(t)$ and $g(t)$ are continuous for $-2 < t < 2$, and $t_0 = 1$ is in this interval, so the solution $y(t)$ exists in the interval $(-2, 2)$.

27. (a)

$$n = 0: \quad y' + p(t)y = q(t).$$

$$\mu(t) = \exp \int p(t)\mathrm{d}t, \quad y(t) = \frac{1}{\mu(t)} \int \mu(t)q(t)\mathrm{d}t.$$

$$n = 1: \quad y' + p(t)y = q(t)y \Leftrightarrow y' = (q(t) - p(t))y.$$

$$y(t) = \exp \int (q(t) - p(t))\mathrm{d}t.$$

(b)

$$y^{-n}y' + p(t)y^{1-n} = q(t).$$

$$v = y^{1-n} \Rightarrow v' = (1-n)y^{-n}y' = (1-n)[q(t) - p(t)y^{1-n}] = (1-n)[q(t) - p(t)v].$$

$$v' + (1-n)p(t)v = (1-n)q(t).$$

It is a linear equation for $v(t)$. We can solve for $v(t)$ and then $y = v^{1/(1-n)}$.

33. For $0 \le t \le 1$,

$$y' + 2y = 0, \quad y(0) = 1 \Rightarrow y(t) = y(0)\exp(-\int_0^t 2\mathrm{d}t) = e^{-2t}.$$

From the solution above, $y(1) = e^{-2}$. For $t > 1$,

$$y' + y = 0, \quad y(1) = e^{-2} \Rightarrow y(t) = y(1)\exp(-\int_1^t 1\mathrm{d}t) = e^{-2}e^{1-t} = e^{-1-t}.$$

So $y(t) = e^{-2t}$ for $0 \le t \le 1$, $y(t) = e^{-1-t}$ for $t > 1$.

3 Solution to Homework 3

Section 2.6

7.

$$M_y = e^x \cos y - 2\sin x = N_x,$$

so it is an exact equation. Integrating

$$\psi_x = M(x,y) = e^x \sin y - 2y \sin x,$$

we have

$$\psi = e^x \sin y + 2y\cos x + h(y).$$

Setting $\psi_y = N(x,y)$ we have $h' = 0$ so we can let $h = 0$. Therefore the solution to the differential equation is

$$\psi(x,y) = e^x \sin y + 2y \cos x = C.$$

8.

$$M_y = e^x \cos y + 3 \neq -(3 - e^x \sin y) = N_x,$$

so it is not an exact equation.

21.

$$M_y = 1 \neq 2 = N_x,$$

but

$$(\mu M)_y = 2y = (\mu N)_x.$$

Integrating

$$\psi_x = \mu M = y^2,$$

we have

$$\psi = xy^2 + h(y).$$

Setting $\psi_y = \mu N$ we have

$$2xy + h'(y) = 2xy - y^2 e^y \Rightarrow h'(y) = -y^2 e^y \Rightarrow h = -\int y^2 e^y \mathrm{d}y = -e^y(y^2 - 2y + 2) + C.$$

The solution to the differential equation is

$$xy^2 - e^y(y^2 - 2y + 2) = C.$$

25.

$$M_y - N_x = 3x^2 + 2x + 3y^2 - 2x = 3(x^2 + y^2) = 3N.$$

So we can let μ be a function of x, and

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{M_y - N_x}{N}\mu = 3\mu \Rightarrow \mu = e^{3x}.$$

Integrating

$$\psi_x = \mu M = e^{3x}(3x^2y + 2xy + y^3),$$

we have

$$\psi = e^{3x}(x^2y + \frac{y^3}{3}).$$

Setting $\psi_y = \mu N$ we have $h'(y) = 0$ so we can let $h = 0$. The solution to the differential equation is

$$\psi = e^{3x}(x^2y + \frac{y^3}{3}) = C.$$

Section 2.7

2. Given $y' = 2y - 1$ and $y(0) = 1$, Euler method is

$$y_{n+1} = y_n + h(2y_n - 1), \quad y_0 = 1.$$

We calculate the solution at $t = 0.1$. For $h = 0.1$, it requires only 1 step.

$$y_1 = y_0 + h(2y_0 - 1) = 1 + 0.1 = 1.1.$$

For $h = 0.05$, it requires 2 steps.

$$y_1 = y_0 + h(2y_0 - 1) = 1 + 0.05 = 1.05,$$

$$y_2 = y_1 + h(2y_1 - 1) = 1.05 + 0.05 * 1.1 = 1.105.$$

For $h = 0.025$, it requires 4 steps.

$$y_1 = y_0 + h(2y_0 - 1) = 1 + 0.025 = 1.025,$$

$$y_2 = y_1 + h(2y_1 - 1) = 1.025 + 0.025 * 1.05 = 1.05125,$$

$$y_3 = y_2 + h(2y_2 - 1) = 1.05125 + 0.025 * 1.1025 = 1.0788125,$$

$$y_4 = y_3 + h(2y_3 - 1) = 1.0788125 + 0.025 * 1.157625 = 1.107753125.$$

The exact solution of the differential equation is

$$y(t) = \frac{e^{2x} + 1}{2} \Rightarrow y(0.1) = \frac{e^{0.2} + 1}{2} \approx 1.1107.$$

The error in $y(0.1)$ calculated by Euler method is smaller for smaller h.

6. If $y(0) = 0$, $y(t) \equiv 0$, which is an equilibrium solution.
If $y(0) > 0$, $y'(0) > 0$, $y(t)$ increases until $yt = 3$, after that $y' < 0$ and the solution converges to the hyperbola $y = 3/t$ from above.
If $y(0) < 0$, $y'(t) < 0$ for all $t > 0$, the solution diverges to $-\infty$ at finite t.

9. $y' > 0$ except for $t = y = 0$. All solutions diverge to ∞ at finite t.

Section 2.8

7.

$$\phi_{n+1}(t) = \int_0^t (s\phi_n(s) + 1)\mathrm{d}s, \quad \phi_0(t) = 0.$$

$$\phi_1(t) = \int_0^t 1\mathrm{d}s = t.$$

$$\phi_2(t) = \int_0^t (s \cdot s + 1)\mathrm{d}s = \frac{t^3}{3} + t.$$

$$\phi_3(t) = \int_0^t (s(\frac{s^3}{3} + s) + 1)\mathrm{d}s = \frac{t^5}{5 \cdot 3} + \frac{t^3}{3} + t.$$

$$\phi_4(t) = \int_0^t (s(\frac{s^5}{5 \cdot 3} + \frac{s^3}{3} + s) + 1)\mathrm{d}s = \frac{t^7}{7 \cdot 5 \cdot 3} + \frac{t^5}{5 \cdot 3} + \frac{t^3}{3} + t.$$

Upon inspection, it becomes apparent that

$$\phi_n(t) = t + \frac{t^3}{3} + \frac{t^5}{5 \cdot 3} + \cdots + \frac{t^{2n-1}}{(2n-1) \cdot (2n-3) \cdots 5 \cdot 3}.$$

As $n \to \infty$, $\phi_n(t)$ converges at any t.

10.

$$\phi_{n+1}(t) = \int_0^t (1 - \phi_n^3(s))\mathrm{d}s, \quad \phi_0(t) = 0.$$

$$\phi_1(t) = \int_0^t 1\mathrm{d}s = t.$$

$$\phi_2(t) = \int_0^t (1 - s^3)\mathrm{d}s = t - \frac{t^4}{4}.$$

$$\phi_3(t) = \int_0^t (1 - (s - \frac{s^4}{4})^3)\mathrm{d}s = t - \frac{t^4}{4} + \frac{3t^7}{28} - \frac{3t^{10}}{160} + \frac{t^{13}}{832}.$$

Although the solution exists for all $t > 0$, $\phi_n(t)$ appears to converge for small t but diverge for large t.

4 Solution to Homework 4

Section 3.1

20. The characteristic equation is

$$2r^2 - 3r + 1 = (r-1)(2r-1) = 0 \Rightarrow r = 1 \text{ or } r = \frac{1}{2}.$$

$$y(t) = c_1e^t + c_2e^{\frac{t}{2}} \Rightarrow y' = c_1e^t + \frac{c_2}{2}e^{\frac{t}{2}}.$$

$$y(0) = 2,\ y'(0) = \frac{1}{2} \Rightarrow c_1 + c_2 = 2,\ c_1 + \frac{c_2}{2} = \frac{1}{2} \Rightarrow c_1 = -1,\ c_2 = 3.$$

$$y(t) = -e^t + 3e^{\frac{t}{2}}.$$

To find the maximum value of $y(t)$, let

$$y'(t) = -e^t + \frac{3}{2}e^{\frac{t}{2}} = 0 \Rightarrow t = 2\ln\frac{3}{2} \Rightarrow y_M = y(t) = \frac{9}{4}.$$

To find the point where $y(t) = 0$, let

$$y(t) = -e^t + 3e^{\frac{t}{2}} = 0 \Rightarrow t = 2\ln 3.$$

21. The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r = 2 \text{ or } r = -1.$$

$$y(t) = c_1e^{2t} + c_2e^{-t} \Rightarrow y' = 2c_1e^{2t} - c_2e^{-t}.$$

$$y(0) = \alpha,\ y'(0) = 2 \Rightarrow c_1 + c_2 = \alpha,\ 2c_1 - c_2 = 2 \Rightarrow c_1 = \frac{2+\alpha}{3},\ c_2 = \frac{2\alpha - 2}{3}.$$

$$y(t) = \frac{2+\alpha}{3}e^{2t} + \frac{2\alpha-2}{3}e^{-t}.$$

$y(t) \to 0$ as $t \to \infty$ if and only if the coefficient of e^{2t} in $y(t)$ is 0, which requires

$$\frac{2+\alpha}{3} = 0 \Rightarrow \alpha = -2.$$

27. (a) If $y \equiv y_e$ is a solution, then

$$ay'' + by' + cy = cy_e = d \Rightarrow y_e = \frac{d}{c}.$$

(b)

$$Y = y - y_e \Rightarrow Y'' = y'',\ Y' = y' \Rightarrow aY'' + bY' + c(Y + y_e) = d \Rightarrow aY'' + bY' + cY = 0.$$

Section 3.2

25.

$$y_1'' - 2y_1' + y_1 = e^t - 2e^t + e^t = 0.$$
$$y_2'' - 2y_2' + y_2 = (te^t + 2e^t) - 2(te^t + e^t) + te^t = 0.$$

Since $y_2/y_1 = t$ is not a constant, y_1 and y_2 are linearly independent, and so constitute a fundamental set of solutions. Alternatively, we can verify the linear independency by checking the Wronskian determinant of y_1 and y_2 at 0,

$$W(0) = [y_1y_2' - y_2y_1'](0) = 1 \cdot 1 - 0 \cdot 1 = 1 \neq 0.$$

27.

$$(1 - x\cot x)y_1'' - xy_1' + y_1 = 0 - x + x = 0.$$
$$(1-x\cot x)y_2''-xy_2'+y_2 = (1-x\cot x)(-\sin x)-x\cos x+\sin x = -\sin x+x\cos x-x\cos x+\sin x = 0.$$

Since $y_2/y_1 = \sin x/x$ is not a constant for $0 < x < \pi$, y_1 and y_2 are linearly independent, and so constitute a fundamental set of solutions.

28. (a) The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0 \Rightarrow r = -1, 2.$$

So $y_1(t) = e^{-t}$ and $y_2(t) = e^{2t}$ are two solutions. Since $y_2/y_1 = e^{3t}$ is not a constant, y_1 and y_2 are linearly independent, and form a fundamental set of solutions. Alternatively, we can obtain the same conclusion by verifying that their Wronskian is nonzero.

$$W(y_1, y_2) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = 3e^t \neq 0.$$

(b) We notice that

$$y_3 = -2y_2,\ \ y_4 = y_1 + 2y_2,\ \ y_5 = 2y_1 - y_3 = 2y_1 + 4y_2.$$

Since y_3, y_4 and y_5 are linear combinations of y_1 and y_2, they are also solutions of the given linear equation.
(c) A pair of solutions is a fundamental set of solutions if and only if their Wronskian is nonzero.

$$W(y_1, y_3) = W(y_1, -2y_2) = -2W(y_1, y_2) \neq 0 \Rightarrow [y_1, y_3] \text{ is a fundamental set.}$$
$$W(y_2, y_3) = W(y_2, -2y_2) = -2W(y_2, y_2) = 0 \Rightarrow [y_2, y_3] \text{ is not a fundamental set.}$$
$$W(y_1, y_4) = W(y_1, y_1 + 2y_2) = 2W(y_1, y_2) \neq 0 \Rightarrow [y_1, y_4] \text{ is a fundamental set.}$$
$$W(y_4, y_5) = W(y_1 + 2y_2, 2y_1 + 4y_2) = 2W(y_4, y_4) = 0 \Rightarrow [y_4, y_5] \text{ is not a fundamental set.}$$

31.

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0.$$
$$p(x) = \frac{1}{x} \Rightarrow W = \exp(-\int p(x)\mathrm{d}x) = \exp(-\int \frac{\mathrm{d}x}{x}) = e^{-\ln x + C} = \frac{A}{x}.$$

34.

$$y'' + \frac{2}{t}y' + e^t y = 0.$$

$$p(t) = \frac{2}{t} \Rightarrow W = \exp(-\int p(t)\mathrm{d}t) = \exp(-\int \frac{2\mathrm{d}t}{t}) = e^{-2\ln t + C} = \frac{A}{t^2}.$$

$$W(y_1, y_2)(1) = \frac{A}{1^2} = 2 \Rightarrow A = 2 \Rightarrow W(y_1, y_2)(5) = \frac{A}{5^2} = \frac{2}{25}.$$

Section 3.3

19. The characteristic equation is

$$r^2 - 2r + 5 = (r-1)^2 + 4 = 0 \Rightarrow r = 1 \pm 2i.$$

$$y(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t.$$

$$y(\frac{\pi}{2}) = 0 \Rightarrow c_1 = 0 \Rightarrow y(t) = c_2 e^t \sin 2t \Rightarrow y'(t) = c_2 e^t(\sin 2t + 2\cos 2t).$$

$$y'(\frac{\pi}{2}) = 2 \Rightarrow c_2 = -e^{-\frac{\pi}{2}} \Rightarrow y(t) = -e^{t-\frac{\pi}{2}} \sin 2t.$$

36. Assume $y(t) = t^n$ is a solution to the equation, then

$$n(n-1)t^n + 4nt^n + 2t^n = t^n(n^2 + 3n + 2) = t^n(n+1)(n+2) = 0 \Rightarrow n = -1 \text{ or } n = -2.$$

So the general solution is

$$y(t) = \frac{c_1}{t} + \frac{c_2}{t^2}.$$

5 Solution to Homework 5

Section 3.4

4. The characteristic equation is

$$4r^2 + 12r + 9 = (2r+3)^2 = 0 \Rightarrow r_1 = r_2 = -\frac{3}{2}.$$

$$y(t) = e^{-\frac{3}{2}t}(c_1 + c_2 t).$$

10. The characteristic equation is

$$2r^2 + 2r + 1 = 2[(r+\frac{1}{2})^2 + \frac{1}{4}] = 0 \Rightarrow r_1 = \frac{-1 \pm i}{2}.$$

$$y(t) = e^{-\frac{t}{2}}(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}).$$

16. The characteristic equation is

$$r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2 = 0 \Rightarrow r_1 = r_2 = \frac{1}{2}.$$

$$y(t) = e^{\frac{t}{2}}(c_1 + c_2 t) \Rightarrow y'(t) = e^{\frac{t}{2}}(c_2 + \frac{c_1}{2} + \frac{c_2}{2}t).$$

$$y(0) = 2,\ y'(0) = b \Rightarrow c_1 = 2,\ c_2 + \frac{c_1}{2} = b \Rightarrow c_1 = 2,\ c_2 = b - 1.$$

$$y(t) = e^{\frac{t}{2}}[2 + (b-1)t].$$

The critical b that separates $y(t)$ between growing up and down as $t \to \infty$ satisfies

$$b - 1 = 0 \Rightarrow b = 1.$$

28.

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0.$$

$$y_1 = e^x,\ p(x) = -\frac{x}{x-1},\ y_2(x) = y_1(x)\int \frac{\exp(-\int p(x)\mathrm{d}x)}{y_1^2(x)}\mathrm{d}x.$$

$$y_2(x) = e^x \int \frac{\exp\int\frac{x\mathrm{d}x}{x-1}}{e^{2x}}\mathrm{d}x = e^x \int \frac{e^{x+\ln(x-1)}}{e^{2x}}\mathrm{d}x = e^x \int (x-1)e^{-x}\mathrm{d}x = e^x(-xe^{-x}) = -x.$$

41. Assume $y(t) = t^n$ is a solution to the equation, then

$$n(n-1)t^n - 3nt^n + 4t^n = t^n(n^2 - 4n + 4) = t^n(n-2)^2 = 0 \Rightarrow n_1 = n_2 = 2.$$

So the general solution is

$$y(t) = c_1 t^2 + c_2 t^2 \ln t.$$

Section 3.5

4. First solve the homogeneous equation.

$$y'' + 2y' = 0 \Rightarrow r(r+2) = 0 \Rightarrow y = c_1 + c_2e^{-2t}.$$

Next we find the particular solution to $y'' + 2y' = 3$ and $y'' + 2y' = 4\sin 2t$ and add them up. For $y'' + 2y' = 3$, since 0 is a single root of the characteristic equation, let $Y_1 = at$.

$$Y_1'' + 2Y_1' = 2a = 3 \Rightarrow a = \frac{3}{2} \Rightarrow Y_1 = \frac{3}{2}t.$$

For $y''+2y' = 4\sin 2t$, since $2i$ is not a root of the characteristic equation, let $Y_2 = a\sin 2t + b\cos 2t$.

$$Y_2'' + 2Y_2' = -4a\sin 2t - 4b\cos 2t + 4a\cos 2t - 4b\sin 2t = 4\sin 2t \Rightarrow a = b = -\frac{1}{2}.$$

So the general solution is

$$y = c_1 + c_2e^{-2t} + Y_1 + Y_2 = c_1 + c_2e^{-2t} + \frac{3}{2}t - \frac{\sin 2t + \cos 2t}{2}.$$

11. First solve the homogeneous equation.

$$y'' + y' + 4y = 0 \Rightarrow (r+\frac{1}{2})^2 + \frac{15}{4} = 0 \Rightarrow r = -\frac{1}{2} \pm \frac{\sqrt{15}}{2}i \Rightarrow y = e^{-\frac{t}{2}}(c_1\sin\frac{\sqrt{15}}{2}t + c_2\cos\frac{\sqrt{15}}{2}t).$$

Next we find the particular solution to $y''+y'+4y = e^t$ and $y''+y'+4y = -e^{-t}$ and add them up. For $y''+y'+4y = e^t$, since 1 is not a root of the characteristic equation, let $Y_1 = ae^t$.

$$Y_1'' + Y_1' + 4Y_1 = 2a = e^t(a + a + 4a) = e^t \Rightarrow a = \frac{1}{6} \Rightarrow Y_1 = \frac{1}{6}e^t.$$

For $y'' + y' + 4y = -e^{-t}$, since -1 is not a root of the characteristic equation, let $Y_2 = ae^{-t}$.

$$Y_2'' + Y_2' + 4Y_2 = e^{-t}(a - a + 4a) = -e^{-t} \Rightarrow a = -\frac{1}{4} \Rightarrow Y_2 = -\frac{1}{4}e^{-t}.$$

So the general solution is

$$y(t) = e^{-\frac{t}{2}}(c_1\sin\frac{\sqrt{15}}{2}t + c_2\cos\frac{\sqrt{15}}{2}t) + \frac{1}{6}e^t - \frac{1}{4}e^{-t}.$$

14. First solve the homogeneous equation.

$$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y = c_1\cos 2t + c_2\sin 2t.$$

Next we find the particular solution to $y'' + 4y = t^2$ and $y'' + 4y = 3e^t$ and add them up. For $y'' + 4y = t^2$, since 0 is not a root of the characteristic equation, let $Y_1 = at^2 + bt + c$.

$$Y_1'' + 4Y_1 = 2a + 4(at^2 + bt + c) = 4at^2 + 4bt + 4c + 2a = t^2 \Rightarrow a = \frac{1}{4},\ b = 0,\ c = -\frac{1}{8} \Rightarrow Y_1 = \frac{t^2}{4} - \frac{1}{8}.$$

For $y'' + 4y = 3e^t$, since 1 is not a root of the characteristic equation, let $Y_2 = ae^t$.

$$Y_2'' + 4Y_2 = e^t(a + 4a) = 3e^t \Rightarrow a = \frac{3}{5} \Rightarrow Y_2 = \frac{3}{5}e^t.$$

So the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t.$$

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{t}{2} + \frac{3}{5}e^t.$$

$$y(0) = 0,\ y'(0) = 2 \Rightarrow c_1 - \frac{1}{8} + \frac{3}{5} = 0,\ 2c_2 + \frac{3}{5} = 2 \Rightarrow c_1 = -\frac{19}{40},\ c_2 = \frac{7}{10}.$$

$$y(t) = -\frac{19}{40}\cos 2t + \frac{7}{10}\sin 2t + \frac{t^2}{4} - \frac{1}{8} + \frac{3}{5}e^t.$$

17. First solve the homogeneous equation.

$$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = \pm 2i \Rightarrow y = c_1 \cos 2t + c_2 \sin 2t.$$

Next we find the particular solution to $y'' + 4y = 3\sin 2t$. Since $2i$ is a root of the characteristic equation, let $Y = at\cos 2t + bt\sin 2t$.

$$Y'' + 4Y = -4a\sin 2t + 4b\cos 2t = 3\sin 2t \Rightarrow a = -\frac{3}{4},\ b = 0 \Rightarrow Y = -\frac{3}{4}t\cos 2t.$$

So the general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{3}{4}t\cos 2t.$$

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t.$$

$$y(0) = 2,\ y'(0) = -1 \Rightarrow c_1 = 2,\ 2c_2 - \frac{3}{4} = -1 \Rightarrow c_1 = 2,\ c_2 = -\frac{1}{8}.$$

$$y(t) = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t.$$

6 Solution to Homework 6

Section 3.6

5. First solve the homogeneous equation.

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_1 = \cos t,\ y_2 = \sin t.$$

The Wronskian determinant of y_1 and y_2 is

$$W = y_1 y_2' - y_2 y_1' = \cos t \cos t + \sin t \sin t = 1.$$

The general solution to $y'' + y = g(t) = \tan t$ is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t \\ &= -\cos t\int \frac{\sin^2 t}{\cos t}\mathrm{d}t + \sin t\int \sin t\mathrm{d}t = -\cos t\int(\sec t - \cos t)\mathrm{d}t + \sin t(-\cos t + c_2) \\ &= -\cos t[\ln(\sec t + \tan t) - \sin t - c_1] + \sin t(-\cos t + c_2) \\ &= -\cos t \ln(\sec t + \tan t) + c_1 \cos t + c_2 \sin t. \end{aligned}$$

10. First solve the homogeneous equation.

$$y'' - 2y' + y = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Rightarrow y_1 = e^t,\ y_2 = te^t.$$

The Wronskian determinant of y_1 and y_2 is

$$W = y_1 y_2' - y_2 y_1' = e^t e^t(t+1) - te^t e^t = e^{2t}.$$

The general solution to $y'' - 2y' + y = g(t) = e^t/(1+t^2)$ is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t \\ &= -e^t\int \frac{t}{1+t^2}\mathrm{d}t + te^t\int \frac{1}{1+t^2}\mathrm{d}t = -e^t[\frac{1}{2}\ln(1+t^2) - c_1] + te^t[\arctan t + c_2] \\ &= -\frac{1}{2}e^t\ln(1+t^2) + te^t\arctan t + c_1 e^t + c_2 te^t. \end{aligned}$$

13. The homogeneous equation is an Euler equation

$$t^2 y'' - 2y = 0$$

.

$$y = t^n \Rightarrow n(n-1) - 2 = (n-2)(n+1) = 0 \Rightarrow y_1 = t^2,\ y_2 = \frac{1}{t}.$$

The Wronskian determinant of y_1 and y_2 is

$$W = y_1 y_2' - y_2 y_1' = t^2(-\frac{1}{t^2}) - \frac{1}{t}2t = -3.$$

The general solution to

$$y'' - \frac{2}{t^2}y = g(t) = 3 - \frac{1}{t^2}$$

is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t \\ &= -t^2\int \frac{3t^2-1}{-3t^3}\mathrm{d}t + \frac{1}{t}\int \frac{3t^2-1}{-3}\mathrm{d}t = t^2[\ln t + \frac{1}{6t^2} + c_1] - \frac{1}{3t}[t^3 - t + c_2] \\ &= t^2\ln t + \frac{1}{2} + (c_1 - \frac{1}{3})t^2 - \frac{c_2}{3}\frac{1}{t} = t^2\ln t + \frac{1}{2} + c_1't^2 + c_2'\frac{1}{t}. \end{aligned}$$

14. It's easy to verify that $y_1(t) = t$ and $y_2(t) = te^t$ are two independent solutions of $t^2y'' - t(t+2)y' + (t+2)y = 0$. The Wronskian determinant of y_1 and y_2 is

$$W = y_1y_2' - y_2y_1' = te^t(t+1) - te^t = t^2e^t.$$

The general solution to

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = g(t) = 2t$$

is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t \\ &= -t\int \frac{te^t\cdot 2t}{t^2e^t}\mathrm{d}t + te^t\int \frac{t\cdot 2t}{t^2e^t}\mathrm{d}t = -t(2t - c_1) + te^t(-2e^{-t} + c_2) = -2t^2 - 2t + c_1t + c_2te^t. \end{aligned}$$

31.

$$y'' - (1+\frac{1}{t})y' + \frac{1}{t}y = te^{2t}.$$

$$p(t) = -(1+\frac{1}{t}), \quad g(t) = te^{2t}, \quad y_1(t) = 1+t.$$

The Wronskian determinant is

$$W(t) = \exp(-\int p(t)\mathrm{d}t) = te^t.$$

The other linearly independent solution is

$$y_2(t) = y_1\int \mathrm{d}t\frac{W(t)}{y_1^2(t)} = (1+t)\int \mathrm{d}t\frac{te^t}{(1+t)^2} = (1+t)\frac{e^t}{1+t} = e^t.$$

The general solution is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t = -(t+1)\int \frac{e^t\cdot te^{2t}}{te^t}\mathrm{d}t + e^t\int \frac{(t+1)\cdot te^{2t}}{te^t}\mathrm{d}t \\ &= -(t+1)(\frac{e^{2t}}{2} - c_1) + e^t(te^t + c_2) = \frac{t-1}{2}e^{2t} + c_1(t+1) + c_2e^t. \end{aligned}$$

32.

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1-t)e^{-t}.$$

$$p(t) = \frac{t}{1-t}, \quad g(t) = 2(1-t)e^{-t}, \quad y_1(t) = e^t.$$

The Wronskian determinant is

$$W(t) = \exp(-\int p(t)\mathrm{d}t) = e^t(1-t).$$

The other linearly independent solution is

$$y_2(t) = y_1 \int \mathrm{d}t \frac{W(t)}{y_1^2(t)} = e^t \int \mathrm{d}t \frac{e^t(1-t)}{e^{2t}} = e^t \cdot te^{-t} = t.$$

The general solution is

$$\begin{aligned} y(t) &= -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\mathrm{d}t + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\mathrm{d}t = -e^t \int \frac{2t(1-t)e^{-t}}{e^t(1-t)}\mathrm{d}t + t\int \frac{2(1-t)}{e^t(1-t)}\mathrm{d}t \\ &= e^t[(t+\frac{1}{2})e^{-2t} + c_1] + t[-2e^{-t} + c_2] = (\frac{1}{2} - t)e^{-t} + c_1e^t + c_2t. \end{aligned}$$

Section 3.7

13. The angular frequency of the undamped motion $u'' + u = 0$ is $\omega_0 = 1$. For damped motion, the characteristic equation

$$r^2 + \gamma r + 1 = 0 \Rightarrow r = -\frac{\gamma}{2} \pm i\sqrt{1 - \frac{\gamma^2}{4}}.$$

The quasi angular frequency is $\omega = \sqrt{1 - \frac{\gamma^2}{4}}$. Setting

$$T = \frac{2\pi}{\omega} = 1.5T_0 = \frac{3}{2}\frac{2\pi}{\omega_0} = 3\pi,$$

we have

$$\gamma = \frac{2}{3}\sqrt{5}.$$

14. The balance of the spring tension with the gravity on the mass gives

$$mg = kL \Rightarrow \frac{k}{m} = \frac{g}{L}.$$

Let u be the distance of the mass from the equilibrium position, then

$$mu'' + ku = 0 \Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}.$$

24.

$$\omega = \sqrt{\frac{2k}{3}},\ \pi = T = \frac{2\pi}{\omega} \Rightarrow \omega = 2 \Rightarrow k = 6.$$

$$u(t) = A\sin(\omega t + \phi) = 3\sin(2t + \phi).$$

$$u(0) = 3\sin\phi = 2 \Rightarrow \sin\phi = \frac{2}{3}.$$

$$v = u'(0) = 6\cos\phi = \pm 6\sqrt{1 - \sin^2\phi} = \pm 2\sqrt{5}.$$

Section 3.8

11. (a) The equation of motion is

$$mx'' + \gamma x' + kx = F = F_0 \cos\omega t.$$

In the units of ft, lb and sec, $m = 8/g = 8/32 = 1/4$, $\gamma = 0.25$, $k = mg/L = 8/6 = 4/3$, $F_0 = 4$, $\omega = 2$. Let the steady state response be

$$x = R\cos(\omega t - \delta),$$

then

$$\begin{aligned} F_0\cos\omega t = mx'' + \gamma x' + kx &= R[(k - m\omega^2)\cos(\omega t - \delta) - \gamma\omega\sin(\omega t - \delta)] \\ = \ & R\sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2}\cos(\omega t - \delta + \operatorname{arccot}\frac{k - m\omega^2}{\gamma\omega}) \end{aligned}$$

so

$$R = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\gamma\omega)^2}},\ \delta = \operatorname{arccot}\frac{k - m\omega^2}{\gamma\omega}.$$

(b) We seek the m that maximizes R with F_0, ω, k and γ fixed. Obviously it happens when

$$k - m\omega^2 = 0 \Rightarrow m = \frac{k}{\omega^2}.$$

15. For $0 \le t \le \pi$,

$$u'' + u = F_0 t,\ u(0) = 0,\ u'(0) = 0 \quad\Rightarrow\quad u(t) = F_0 t - F_0\sin t.$$

$$u(\pi) = F_0\pi,\ u'(\pi) = 2F_0.$$

For $\pi \le t \le 2\pi$,

$$u'' + u = F_0(2\pi - t),\ u(\pi) = F_0\pi,\ u'(\pi) = 2F_0 \quad\Rightarrow\quad u(t) = F_0(2\pi - t) - 3F_0\sin t.$$

$$u(2\pi) = 0,\ u'(2\pi) = -4F_0.$$

For $2\pi \le t$,

$$u'' + u = 0,\ u(2\pi) = 0,\ u'(2\pi) = -4F_0 \quad\Rightarrow\quad u(t) = -4F_0\sin t.$$

7 Solution to Homework 7

Section 5.1

4. Apply the ratio test:

$$\lim_{n\to\infty}\frac{|2^{n+1}x^{n+1}|}{|2^n x^n|}=2|x|.$$

Hence the series converges absolutely for $2|x|<1$, or $|x|<1/2$. The radius of convergence is $\rho=1/2$.

8. Apply the ratio test:

$$\lim_{n\to\infty}\frac{|(n+1)!x^{n+1}n^n|}{|(n+1)^{n+1}n!x^n|}=\lim_{n\to\infty}\frac{n^n}{(n+1)^n}|x|=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^n}|x|=\frac{|x|}{e}.$$

Hence the series converges absolutely for $|x|/e<1$, or $|x|<e$. The radius of convergence is $\rho=e$.

13.

$$(\ln x)'=\frac{1}{x},\ (\ln x)''=-\frac{1}{x^2},\ (\ln x)'''=\frac{2}{x^3},\ (\ln x)''''=-\frac{3!}{x^4},\ldots,\ (\ln x)^{(n)}=(-1)^{n-1}\frac{(n-1)!}{x^n}.$$

$$\ln 1=0,\ (\ln x)'(1)=1,\ (\ln x)''(1)=-1,\ldots,\ (\ln x)^{(n)}(1)=(-1)^{n-1}(n-1)!.$$

$$\ln x=\sum_{n=1}^{\infty}\frac{(\ln x)^{(n)}(1)}{n!}(x-1)^n=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}.$$

Apply the ratio test:

$$\lim_{n\to\infty}\frac{|n(-1)^n(x-1)^{n+1}|}{|(n+1)(-1)^{n-1}(x-1)^n|}=|x-1|\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=|x-1|.$$

Hence the series converges absolutely for $|x-1|<1$. The radius of convergence is $\rho=1$.

23.

$$x\sum_{n=1}^{\infty}na_nx^{n-1}+\sum_{k=0}^{\infty}a_kx^k=\sum_{n=1}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}na_nx^n+\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}(n+1)a_nx^n.$$

26.

$$\begin{aligned}&\sum_{n=1}^{\infty}na_nx^{n-1}+x\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=0}^{\infty}a_nx^{n+1}\\ =\ &\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n+\sum_{n=1}^{\infty}a_{n-1}x^n=a_1+\sum_{n=1}^{\infty}[(n+1)a_{n+1}+a_{n-1}]x^n.\end{aligned}$$

8 Solution to Homework 8

Section 5.2

1. Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

$$y'' - y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - a_n]x^n = 0 \Rightarrow a_{n+2} = \frac{a_n}{(n+1)(n+2)},\ n \geq 0.$$

$$a_2 = \frac{a_0}{2!},\ a_4 = \frac{a_0}{4!}, \ldots,\ a_{2n} = \frac{a_0}{(2n)!}.$$

$$a_3 = \frac{a_1}{3!},\ a_5 = \frac{a_1}{5!}, \ldots,\ a_{2n+1} = \frac{a_1}{(2n+1)!}.$$

The general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

in which

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \quad y_2(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

21. (a) Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n.$$

$$y'' - 2xy' + \lambda y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (2n-\lambda)a_n]x^n = 0 \Rightarrow a_{n+2} = \frac{2n-\lambda}{(n+1)(n+2)}a_n,\ n \geq 0.$$

$$a_2 = \frac{-\lambda}{2!}a_0,\ a_4 = \frac{-\lambda(4-\lambda)}{4!}a_0, \ldots,\ a_{2n} = \frac{-\lambda(4-\lambda)\ldots(4n-4-\lambda)}{(2n)!}a_0.$$

$$a_3 = \frac{2-\lambda}{3!}a_1,\ a_5 = \frac{(2-\lambda)(6-\lambda)}{5!}a_1, \ldots,\ a_{2n+1} = \frac{(2-\lambda)(6-\lambda)\ldots(4n-2-\lambda)}{(2n+1)!}a_1.$$

The general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

in which

$$y_1(x) = 1 + \frac{-\lambda}{2!}x^2 + \frac{-\lambda(4-\lambda)}{4!}x^4 + \cdots + \frac{-\lambda(4-\lambda)\ldots(4n-4-\lambda)}{(2n)!}x^{2n} + \cdots.$$

$$y_2(x) = x + \frac{2-\lambda}{3!}x^3 + \frac{(2-\lambda)(6-\lambda)}{5!}x^5 + \cdots + \frac{(2-\lambda)(6-\lambda)\ldots(4n-2-\lambda)}{(2n+1)!}x^{2n+1} + \cdots.$$

(b) If $\lambda = 2k$, where k is a nonnegative number, then depending on whether k is even or odd, either y_1 or y_2 becomes a polynomial.

$$\lambda = 0: \ y_1 = 1; \qquad \lambda = 2: \ y_2 = x; \qquad \lambda = 4: \ y_1 = 1 - 2x^2;$$

$$\lambda = 6: \ y_2 = x - \frac{2}{3}x^3; \qquad \lambda = 8: \ y_1 = 1 - 4x^2 + \frac{4}{3}x^4; \qquad \lambda = 10: \ y_2 = x - \frac{4}{3}x^3 + \frac{4}{15}x^5.$$

(c) $H_k(x)$ is obtained by multiplying the polynomial corresponding to $\lambda = 2k$ by an appropriate constant so that the coefficient of x^k is 2^k.

$$H_0 = 1; \qquad H_1 = 2x; \qquad H_2 = 4x^2 - 2;$$

$$H_3 = 8x^3 - 12x; \qquad H_4 = 16x^4 - 48x^2 + 12; \qquad H_5 = 32x^5 - 160x^3 + 120x.$$

Section 5.3

10. (a) Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$(1-x^2)y'' = (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n.$$

$$xy' = \sum_{n=1}^{\infty} na_n x^n = \sum_{n=0}^{\infty} na_n x^n.$$

$$y'' - xy' + \alpha^2 y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (n(n-1) + n - \alpha^2)a_n]x^n = 0$$

$$a_{n+2} = \frac{n^2 - \alpha^2}{(n+1)(n+2)}a_n, \ n \geq 0.$$

$$a_2 = \frac{-\alpha^2}{2!}a_0, \ a_4 = \frac{-\alpha^2(4-\alpha^2)}{4!}a_0, \ldots, \ a_{2n} = \frac{-\alpha^2(4-\alpha^2)\ldots((2n-2)^2 - \alpha^2)}{(2n)!}a_0.$$

$$a_3 = \frac{1-\alpha^2}{3!}a_1, \ a_5 = \frac{(1-\alpha^2)(9-\alpha^2)}{5!}a_1, \ldots, \ a_{2n+1} = \frac{(1-\alpha^2)(9-\alpha^2)\ldots((2n-1)^2 - \alpha^2)}{(2n+1)!}a_1.$$

The general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

in which

$$y_1(x) = 1 + \frac{-\alpha^2}{2!}x^2 + \frac{-\alpha^2(4-\alpha^2)}{4!}x^4 + \cdots + \frac{-\alpha^2(4-\alpha^2)\ldots((2n-2)^2-\alpha^2)}{(2n)!}x^{2n} + \cdots.$$

$$y_2(x) = x + \frac{1-\alpha^2}{3!}x^3 + \frac{(1-\alpha^2)(9-\alpha^2)}{5!}x^5 + \cdots + \frac{(1-\alpha^2)(9-\alpha^2)\ldots((2n-1)^2-\alpha^2)}{(2n+1)!}x^{2n+1} + \cdots.$$

(b) If α is an even number, y_1 becomes a polynomial; if α is an odd number, y_2 becomes a polynomial.
(c)

$$\alpha = 0: y_1 = 1; \quad \alpha = 1: y_2 = x; \quad \alpha = 2: y_1 = 1 - 2x^2; \quad \alpha = 3: y_2 = x - \frac{4}{3}x^3.$$

22. Let $y = \sum_{n=0}^{\infty} a_n x^n$,

$$(1-x^2)y'' = (1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n]x^n.$$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n = \sum_{n=0}^{\infty} n a_n x^n.$$

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - (n(n-1) + 2n - \alpha(\alpha+1))a_n]x^n = 0$$

$$a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+1)(n+2)}a_n = -\frac{(\alpha-n)(n+\alpha+1)}{(n+1)(n+2)}a_n, \ n \geq 0.$$

$$a_2 = -\frac{\alpha(\alpha+1)}{2!}a_0, a_{2n} = (-1)^n\frac{\alpha(\alpha+1)(\alpha-2)(3+\alpha)\dots(\alpha-2n+2)(2n-1+\alpha)}{(2n)!}a_0.$$

$$a_3 = -\frac{(\alpha-1)(2+\alpha)}{3!}a_1, a_{2n+1} = (-1)^n\frac{(\alpha-1)(2+\alpha)(\alpha-3)(4+\alpha)\dots(\alpha-2n+1)(2n+\alpha)}{(2n+1)!}a_1.$$

The general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x),$$

in which

$$y_1(x) = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{\alpha(\alpha+1)(\alpha-2)(3+\alpha)\dots(\alpha-2n+2)(2n-1+\alpha)}{(2n)!}x^{2n}.$$

$$y_2(x) = x + \sum_{n=1}^{\infty}(-1)^n\frac{(\alpha-1)(2+\alpha)(\alpha-3)(4+\alpha)\dots(\alpha-2n+1)(2n+\alpha)}{(2n+1)!}x^{2n+1}.$$

23. If $\alpha = 2k$, where k is a nonnegative integer, y_1 becomes a polynomial of order α.

$$\alpha = 0 : y_1 = 1; \quad \alpha = 2 : y_1 = 1 - 3x^2; \quad \alpha = 4 : y_1 = 1 - 10x^2 + \frac{35}{3}x^4.$$

If $\alpha = 2k+1$, where k is a nonnegative integer, y_2 becomes a polynomial of order α.

$$\alpha = 1 : y_2 = x; \quad \alpha = 3 : y_2 = x - \frac{5}{3}x^3; \quad \alpha = 5 : y_2 = x - \frac{14}{3}x^3 + \frac{21}{5}x^5.$$

Section 5.4

9.

$$x^2y'' - 5xy' + 9y = 0.$$

The indicial equation is

$$F(r) = r(r-1) - 5r + 9 = (r-3)^2 = 0 \Rightarrow r_1 = r_2 = 3.$$

The general solution is

$$y(x) = x^3(c_1 + c_2 \ln|x|).$$

20.

$$y'' + \frac{2}{x^3(1-x)(1+x)}y' + \frac{4}{x^2(1-x)(1+x)}y = 0.$$

The three singular points are 0, 1 and -1. Near $x = 0$,

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to 0}\frac{2}{x^2} \quad \text{does not exist.}$$

So $x = 0$ is an irregular singular point. Near $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to 1}\frac{-2}{x^3(1+x)} = -1.$$

$$\lim_{x\to 1}(x-1)^2q(x) = \lim_{x\to 1}(x-1)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to 1}\frac{4(1-x)}{x^2(1+x)} = 0.$$

So $x = 1$ is a regular singular point. Near $x = -1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to -1}\frac{2}{x^3(1-x)} = -1.$$

$$\lim_{x\to -1}(x+1)^2q(x) = \lim_{x\to -1}(x+1)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to -1}\frac{4(1+x)}{x^2(1-x)} = 0.$$

So $x = -1$ is a regular singular point.

22.

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0.$$

The only singular points is 0. Near $x = 0$,

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{x} = 1.$$

$$\lim_{x\to 0} x^2q(x) = \lim_{x\to 0} x^2(1 - \frac{\nu^2}{x^2}) = \lim_{x\to 0}(x^2 - \nu^2) = -\nu^2.$$

So $x = 0$ is a regular singular point.

33.

$$y'' + \frac{x}{\sin x}y' + \frac{4}{\sin x}y = 0.$$

The singular points are the zeros of $\sin x$, ie., $x = n\pi$, where n is any integer. Near $x = n\pi$,

$$\lim_{x\to n\pi}(x-n\pi)p(x) = \lim_{x\to n\pi}\frac{x(x-n\pi)}{\sin x} = \lim_{z=x-n\pi\to 0}\frac{(z+n\pi)z}{\sin(z+n\pi)} = \lim_{z\to 0}(-1)^n(z+n\pi)\frac{z}{\sin z} = (-1)^n n\pi.$$

$$\lim_{x\to n\pi}(x-n\pi)^2q(x) = \lim_{x\to n\pi}\frac{4(x-n\pi)^2}{\sin x} = \lim_{z=x-n\pi\to 0}\frac{4z^2}{\sin(z+n\pi)} = \lim_{z\to 0}(-1)^n\frac{4z^2}{\sin z} = 0.$$

So all singular points $x = n\pi$ are regular singular points.

9 Solution to Homework 9

Section 5.5

8.

$$2x^2y'' + 3xy' + (2x^2 - 1)y = 0.$$

It's easy to verify that $x = 0$ is a regular singular point. The associated Euler equation is

$$2x^2y'' + 3xy' - y = 0.$$

The indicial equation is

$$F(r) = 2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1) = 0 \Rightarrow r_1 = \frac{1}{2},\ r_2 = -1.$$

Since $r_1 - r_2 = 3/2$ is not an integer, the solution can be written as

$$y(x) = x^r \sum_{n=0}^{\infty} a_n(r)x^n,$$

where $r = r_1$ or r_2.

$$x^2y'' = x^r x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n-2} = x^r \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^n.$$

$$xy' = x^r x \sum_{n=0}^{\infty} (n+r)a_n x^{n-1} = x^r \sum_{n=0}^{\infty} (n+r)a_n x^n.$$

$$(2x^2 - 1)y = x^r [\sum_{n=2}^{\infty} (2a_{n-2} - a_n)x^n - (a_0 + a_1x)].$$

$$2x^2y'' + 3xy' + (2x^2 - 1)y = x^r \{\sum_{n=2}^{\infty} [F(n+r)a_n + 2a_{n-2}]x^n + F(r)a_0 + F(r+1)a_1x\} = 0.$$

$F(r_1) = F(r_2) = 0$, so a_0 is arbitrary, we set $a_0 = 1$. $F(r_1 + 1) \neq 0$, $F(r_2 + 1) \neq 0$, so $a_1 = 0$.

$$a_n = -\frac{2}{F(r+n)}a_{n-2} = \frac{-1}{(r+n-r_1)(r+n-r_2)}a_{n-2},\ n \geq 2.$$

$$a_{2n}(r) = \frac{(-1)^n}{(r+2-r_1)(r+2-r_2)\dots(r+2n-r_1)(r+2n-r_2)}.$$

$$a_{2n}(r_1) = \frac{(-1)^n}{n!(4+2(r_1-r_2))\dots(4n+2(r_1-r_2)} = \frac{(-1)^n}{n!(4+3)\dots(4n+3)}.$$

$$y_1(x) = x^{1/2}[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(4+3)\dots(4n+3)}x^{2n}].$$

$$a_{2n}(r_2) = \frac{(-1)^n}{n!(4+2(r_2-r_1))\dots(4n+2(r_2-r_1)} = \frac{(-1)^n}{n!(4-3)\dots(4n-3)}.$$

$$y_2(x) = x^{-1}[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(4-3)\dots(4n-3)} x^{2n}].$$

The general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

9.

$$x^2y'' - (x^2+3x)y' + (x+3)y = 0.$$

It's easy to verify that $x = 0$ is a regular singular point. The corresponding Euler equation is

$$x^2y'' - 3xy' + 3y = 0.$$

The indicial equation is

$$F(r) = r(r-1) - 3r + 3 = r^2 - 4r + 3 = (r-3)(r-1) = 0 \Rightarrow r_1 = 3,\ r_2 = 1.$$

Since $r_1 - r_2 = 2$ is an integer, here we only look for the solution in the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n(r) x^n,$$

where $r = r_1 = 3$.

$$x^2y'' = x^r x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n-2} = x^r \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^n.$$

$$(x^2+3x)y' = x^r [\sum_{n=1}^{\infty} ((n-1+r)a_{n-1} + 3(n+r)a_n) x^n + 3ra_0].$$

$$(x+3)y = x^r [\sum_{n=1}^{\infty} (a_{n-1} + 3a_n) x^n + 3a_0].$$

$$x^2y'' - (x^2+3x)y' + (x+3)y = x^r \{\sum_{n=1}^{\infty} [F(n+r)a_n - (n-2+r)a_{n-1}]x^n + F(r)a_0\} = 0.$$

$F(r_1) = 0$, so a_0 is arbitrary, we set $a_0 = 1$. For $n \geq 1$,

$$a_n = -\frac{n-2+r}{F(r+n)} a_{n-1} = \frac{r+n-2}{(r+n-r_1)(r+n-r_2)} a_{n-1}.$$

$$a_n(r) = \frac{(r-1)\dots(r+n-2)}{(r+1-r_1)(r+1-r_2)\dots(r+n-r_1)(r+n-r_2)}.$$

$$a_n(r_1) = \frac{2}{n!(n+2)}.$$

Therefore

$$y_1(x) = x^3[1 + \sum_{n=1}^{\infty} \frac{2}{n!(n+2)} x^n].$$

Section 5.6

13.

$$x^2y'' + xy' - xy = 0.$$

It's easy to verify that $x = 0$ is a regular singular point. The corresponding Euler equation is

$$x^2y'' + xy' = 0.$$

The indicial equation is

$$F(r) = r(r-1) + r = r^2 = 0 \Rightarrow r_1 = r_2 = 0.$$

Since $r_1 = r_2$, we first find the solution in the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$, where $r = r_1 = 0$.

$$x^2y'' = x^r \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^n, \quad xy' = x^r \sum_{n=0}^{\infty} (n+r)a_n x^n, \quad xy = x^r \sum_{n=1}^{\infty} a_{n-1} x^n.$$

$$x^2y'' + xy' - xy = x^r\{\sum_{n=1}^{\infty} [F(n+r)a_n - a_{n-1}]x^n + F(r)a_0\} = 0.$$

$F(r_1) = 0$, so a_0 is arbitrary, we set $a_0 = 1$. For $n \geq 1$,

$$a_n = \frac{1}{F(r+n)} a_{n-1} = \frac{1}{(r+n)^2} a_{n-1} \Rightarrow a_n(r) = \frac{1}{(r+1)^2 \ldots (r+n)^2}.$$

For $y_1(x)$, $r = r_1 = 0$,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(n!)^2}.$$

For $y_2(x)$, since $r_2 = r_1$,

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n x^n.$$

$$b_n = a_n'(r_1) = \left[\frac{1}{(r+1)^2 \ldots (r+n)^2} \left(\frac{-2}{r+1} + \ldots + \frac{-2}{r+n} \right) \right]_{r=0} = -2 \frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{(n!)^2}.$$

Therefore,

$$y_2(x) = y_1(x) \ln x - 2 \sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \ldots + \frac{1}{n}}{(n!)^2} x^n.$$

16.

$$x^2y'' + xy = 0.$$

It's easy to verify that $x = 0$ is a regular singular point. The corresponding Euler equation is

$$x^2y'' = 0.$$

The indicial equation is

$$F(r) = r(r-1) = 0 \Rightarrow r_1 = 1, \ r_2 = 0.$$

Since $N = r_1 - r_2 = 1$, we first find the solution in the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$, where $r = r_1 = 1$.

$$x^2 y'' = x^r \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^n, \quad xy = x^r \sum_{n=1}^{\infty} a_{n-1} x^n.$$

$$x^2 y'' + xy = x^r \{ \sum_{n=1}^{\infty} [F(n+r) a_n + a_{n-1}] x^n + F(r) a_0 \} = 0.$$

$F(r_1) = 0$, so a_0 is arbitrary, we set $a_0 = 1$. For $n \geq 1$,

$$a_n = \frac{-1}{F(r+n)} a_{n-1} = \frac{-1}{(r+n)(r+n-1)} a_{n-1} \Rightarrow a_n(r) = \frac{(-1)^n}{r(r+1)^2 \ldots (r+n-1)^2 (r+n)}.$$

For $y_1(x)$, $r = r_1 = 1$,

$$y_1(x) = x[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(n+1)(n!)^2}].$$

For $y_2(x)$, since $N = r_1 - r_2 = 1$,

$$y_2(x) = a y_1(x) \ln x + x^{r_2} \sum_{n=0}^{\infty} c_n x^n.$$

$$a = \lim_{r \to r_2} (r - r_2) a_N(r) = \lim_{r \to 0} r a_1(r) = \lim_{r \to 0} r \frac{-1}{r(r+1)} = -1.$$

$c_0 = 1$. For $n \geq 1$,

$$\begin{aligned} c_n &= ((r - r_2) a_n(r))'_{(r_2)} = (\frac{(-1)^n}{(r+1)^2 \ldots (r+n-1)^2 (r+n)})'_{(0)} \\ &= \left[\frac{(-1)^n}{(r+1)^2 \ldots (r+n-1)^2 (r+n)} (\frac{-2}{r+1} + \ldots + \frac{-2}{r+n-1} + \frac{-1}{r+n}) \right]_{r=4} \\ &= (-1)^{n-1} \frac{2(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}) + \frac{1}{n}}{n!(n-1)!}. \end{aligned}$$

Therefore,

$$y_2(x) = -y_1(x) \ln x + 1 + \sum_{n=6}^{\infty} (-1)^{n-1} \frac{2(1 + \frac{1}{2} + \ldots + \frac{1}{n-1}) + \frac{1}{n}}{n!(n-1)!} x^n.$$