Connected sum construction of constant Q-curvature manifolds in higher dimensions

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§1 Conformal transformation law of scalar curvature

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$$\Box_g u = \lambda u^{\frac{n+2}{n-2}}, \qquad (3)$$

  for some constant λ.

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- For example:
  On $(\mathbb{R}^n, |dx|^2)$, $P^n = (-\Delta)^2$.
  On (M^n, g), g is Einstein, $P^n = (-\Delta)^2 + \alpha\Delta + \beta$, where $\alpha, \beta \in \mathbb{R}$.

§1 An existence result of constant Q-curvature: a special case

- **Qing-Raske('06)**: For $n \geq 5$, there exists a metric in the conformal class of g with positive constant Q-curvature if (M^n, g) is a closed locally conformally flat manifold with positive Yamabe constant and its Poincaré exponent less than $\frac{n-4}{2}$.

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  Mazzeo-Pollack-Uhlenbeck('95) did the gluing for noncompact, positive constant scalar curvature manifolds.

§2 Main theorem

Nondegeneracy condition:
A n-dimensional compact manifold (M^n, g) satisfies the nondegeneracy condition if the linearized operator

$$L_g := P_g^n - \frac{n+4}{2} Q_g^n : W^{4,2}(M) \to L^2(M)$$

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Theorem (L.)

Given a compact manifold (M, g_2) of dimension $n \geq 6$ with constant Q-curvature ν. Let (N, g_1) be a compact manifold of dimension n. Assume
(i) (M, g_2) satisfies the nondegeneracy condition.
(ii) The Paneitz operator $P_{g_1}^n$ has positive Green's function $G(x, y)$ on N.
Then the connected sum $\widetilde{M} = M \# N$ admits a smooth metric $\tilde{g}$ with constant Q- curvature ν.

§2 Gluing construction procedure

Step 1: Remove a small disk centered at p on (N, g_1) and a small disk centered at q on (M, g_2).

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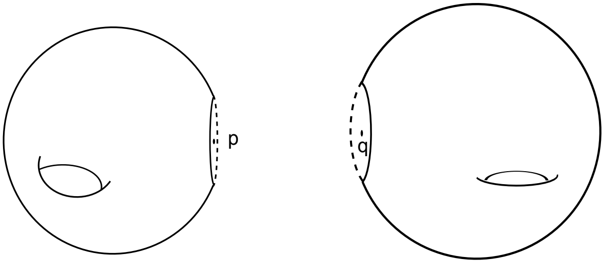

$(N \setminus B_p, g_1)$ $(M \setminus B_q, g_2)$

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Step 2: Obtain an asymptotically flat manifold $(N \setminus \{p\}, g_N)$ by defining $g_N := G_p^{\frac{4}{n-4}} g_1$. Moreover, we have $Q^n_{g_N} \equiv 0$ on $N \setminus \{p\}$.

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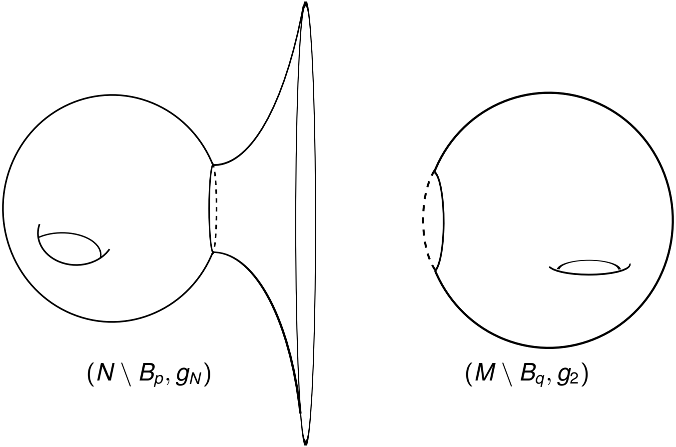

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Step 3: Scale down the metric g_N to $a^2 b^2 g_N$ by two small parameters $a, b > 0$. And glue (identify) the two annuli together.

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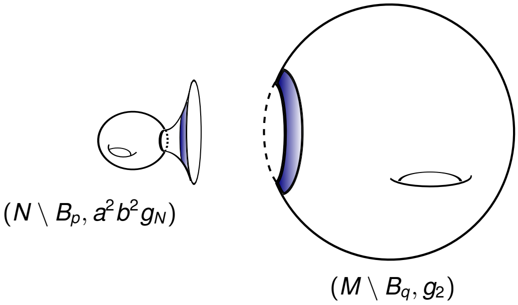

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Step 4: Define an "approximate" metric $g_{a,b}$ on the connected sum $\widetilde{M} = M \# N$ by smoothly joining $a^2 b^2 g_N$ and g_2 using cut-off functions.

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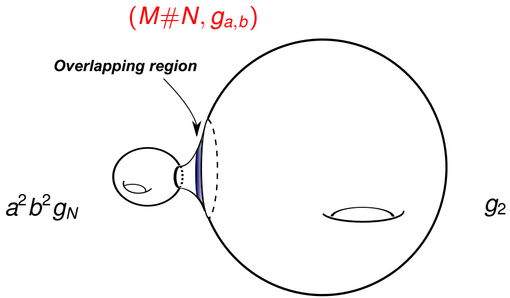

Goal: To find a conformal metric $\tilde{g} = (1+\phi)^{\frac{4}{n-4}} g_{a,b}$ s.t. $Q_{\tilde{g}} \equiv \nu \equiv \text{const}$.

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(3) Fixed point argument.

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Example

For P a compact positive Einstein manifold, N a compact negative Einstein manifold, and F a compact flat manifold, the manifold

$$M^n = \underbrace{P \times P \times \cdots \times P}_{k-copies} \times \underbrace{N \times N \times \cdots \times N}_{k-copies} \times \underbrace{F \times F \times \cdots \times F}_{l-copies}.$$

has negative constant Q-curvature and positive scalar curvature.

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Corollary

Suppose the dimension $n \geq 6$. Assume (M, g_2) has negative constant Q-curvature ν and positive scalar curvature. Let (N, g_1) be a compact positive Einstein manifold. Then there exists a metric with negative constant Q-curvature ν on the connected sum $\widetilde{M} = N \# M$.