Section 5-1 First Derivatives and Graphs

Goal: To use the first derivative to analyze graphs

**Theorem 1:** Increasing and Decreasing Functions

For the interval \((a, b)\), if \( f'(x) > 0 \), then \( f(x) \) is increasing and the graph of \( f \) rises. If \( f'(x) < 0 \), then \( f(x) \) is decreasing and the graph of \( f \) falls.

**Definition:** Critical Values

The values of \( x \) in the domain of \( f \) where \( f'(x) = 0 \) or where \( f'(x) \) does not exist are called the critical values of \( f \).

**Procedure:** First Derivative Test for Local Extrema

Let \( c \) be a critical value of \( f \). Construct a sign chart for \( f'(x) \) close to and on either side of \( c \). Then \( f(c) \) will be

a. A **local minimum** if \( f'(x) \) changes from negative to positive.
b. A **local maximum** if \( f'(x) \) changes from positive to negative.
c. Neither a local minimum nor a local maximum if \( f'(x) \) does not change signs.

1. Given the function \( f(x) = x^3 - 48x + 21 \) find:

   a. \( f'(x) \)

   \[ f'(x) = 3x^2 - 48 \]
b. The critical values of the function

\[ f'(x) = 3x^2 - 48 \]
0 = 3x^2 - 48
48 = 3x^2
16 = x^2
±4 = x

c. The partition numbers for the first derivative

The partition numbers are when the first derivative equals 0 or undefined, therefore, the partition numbers are 4 and –4.

2. Given the function \( f(x) = \frac{9}{x+3} \) find:

a. \( f'(x) \)

\[ f(x) = \frac{9}{x+3} \]
\[ f'(x) = 9(x+3)^{-1} \]
\[ f'(x) = -9(x+3)^{-2} \quad \text{or} \quad \frac{9}{(x+3)^2} \]

b. The critical values of the function

\[ f'(x) = \frac{-9}{(x+3)^2} \]

\( f'(x) \) is always negative and cannot equal zero. \( f'(x) \) is undefined at \( x = -3 \). However, \( x = -3 \) is not in the domain of \( f \), therefore there is no critical value.

c. The partition numbers for the first derivative

Partition numbers are found by setting the first derivative equal to zero (which can’t happen) and where it is undefined. It is undefined at \( x = -3 \), so the partition number is –3.
3 – 5 For the functions given, find:

a. The intervals on which the function is increasing
b. The intervals on which the function is decreasing
c. The local extrema

3. \( f(x) = 3x^2 - 24x + 13 \)

First find the derivative and the critical values as follows:

\[
\begin{align*}
\frac{d}{dx} f(x) &= 6x - 24 \\
0 &= 6x - 24 \\
24 &= 6x \\
4 &= x
\end{align*}
\]

Choose a number on each side of the critical value to find the value of the first derivative:

<table>
<thead>
<tr>
<th>Test number: 3</th>
<th>Test number: 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(3) = 6(3) - 24 )</td>
<td>( f'(5) = 6(5) - 24 )</td>
</tr>
<tr>
<td>( f'(3) = 18 - 24 )</td>
<td>( f'(5) = 30 - 24 )</td>
</tr>
<tr>
<td>( f'(3) = -6 )</td>
<td>( f'(5) = 6 )</td>
</tr>
</tbody>
</table>

a. The test number 5 has a first derivative that is a positive value, therefore, the function is increasing on the interval \((4, \infty)\).
b. The test number 4 has a first derivative that is a negative value, therefore, the function is decreasing on the interval \((-\infty, 4)\).
c. Because the function changes from a decreasing function to an increasing function at the critical value 4, the point \((4, -35)\) is a local minimum.
4. \( f(x) = x^3 - 27x + 12 \)

First find the derivative and the critical values as follows:

\[
 f'(x) = 3x^2 - 27 \\
0 = 3x^2 - 27 \\
27 = 3x^2 \\
9 = x^2 \\
\pm 3 = x
\]

The partition numbers (also critical values of \( f \)) are \( x = -3 \) and \( x = 3 \). Choose a number within each interval to find the value of the first derivative:

<table>
<thead>
<tr>
<th>Test number: -4</th>
<th>Test number: 0</th>
<th>Test number: 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-4) = 3(-4)^2 - 27 )</td>
<td>( f'(0) = 3(0)^2 - 27 )</td>
<td>( f'(4) = 3(4)^2 - 27 )</td>
</tr>
<tr>
<td>( f'(-4) = 3(16) - 27 )</td>
<td>( f'(0) = 3(0) - 27 )</td>
<td>( f'(4) = 3(16) - 27 )</td>
</tr>
<tr>
<td>( f'(-4) = 48 - 27 )</td>
<td>( f'(0) = 0 - 27 )</td>
<td>( f'(4) = 48 - 27 )</td>
</tr>
<tr>
<td>( f'(-4) = 21 )</td>
<td>( f'(0) = -27 )</td>
<td>( f'(4) = 21 )</td>
</tr>
</tbody>
</table>

a. The test numbers \(-4\) and \(4\) have a first derivative that is a positive value, therefore, the function is increasing on the intervals \((-\infty, -3)\) and \((3, \infty)\).

b. The test number \(0\) has a first derivative that is a negative value, therefore, the function is decreasing on the interval \((-3, 3)\).

c. Because the function changes from an increasing function to a decreasing function at the critical value of \(-3\) the point \((-3, 66)\) is a local maximum. Because the function changes from a decreasing function to an increasing function at the critical value \(3\), the point \((3, -42)\) is a local minimum.
5. \( f(x) = x^4 - 4x^3 + 3 \)

First find the derivative and the critical values as follows:

\[
\begin{align*}
 f'(x) &= 4x^3 - 12x^2 \\
 0 &= 4x^3 - 12x^2 \\
 0 &= 4x^2(x - 3) \\
 x &= 3, 0
\end{align*}
\]

The partition numbers (also critical values of \( f \)) are \( x = 0 \) and \( x = 3 \). Choose a number within each interval to find the value of the first derivative:

<table>
<thead>
<tr>
<th>Test number: –1</th>
<th>Test number: 2</th>
<th>Test number: 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-1) = 4(-1)^3 - 12(-1)^2 )</td>
<td>( f'(2) = 4(2)^3 - 12(2)^2 )</td>
<td>( f'(4) = 4(4)^3 - 12(4)^2 )</td>
</tr>
<tr>
<td>( f'(-1) = 4(-1) - 12(1) )</td>
<td>( f'(2) = 4(8) - 12(4) )</td>
<td>( f'(4) = 4(64) - 12(16) )</td>
</tr>
<tr>
<td>( f'(-1) = -4 - 12 )</td>
<td>( f'(2) = 32 - 48 )</td>
<td>( f'(4) = 256 - 192 )</td>
</tr>
<tr>
<td>( f'(-1) = -16 )</td>
<td>( f'(2) = -16 )</td>
<td>( f'(4) = 64 )</td>
</tr>
</tbody>
</table>

a. The test number 4 has a first derivative that is a positive value, therefore, the function is increasing on the interval \((3, \infty)\).

b. The test numbers –1 and 2 have a first derivative that is a negative value, therefore, the function is decreasing on the intervals \((-\infty, 0)\) and \((0, 3)\).

c. The critical value 0 does not give a local extrema because the first derivative does not change sign at \( x = 0 \). Because the function changes from a decreasing function to an increasing function at the critical value 3, the point \((3, -24)\) is a local minimum.
6–7 For the functions given, find:

a. The intervals on which the function is increasing
b. The intervals on which the function is decreasing
c. Sketch the graph of the function

6. \( f(x) = 2x^3 + 3x^2 - 12x \)

First find the derivative and the critical values as follows:

\[
\begin{align*}
  f'(x) &= 6x^2 + 6x - 12 \\
  0 &= 6x^2 + 6x - 12 \\
  0 &= 6(x^2 + x - 2) \\
  0 &= 6(x + 2)(x - 1) \\
  -2, 1 &= x
\end{align*}
\]

The partition numbers (also critical values of \( f \)) are \( x = -2 \) and \( x = 1 \). Choose a number within each interval to find the value of the first derivative:

<table>
<thead>
<tr>
<th>Test number: (-3)</th>
<th>Test number: 0</th>
<th>Test number: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-3) = 6(-3)^2 + 6(-3) - 12 )</td>
<td>( f'(0) = 6(0)^2 + 6(0) - 12 )</td>
<td>( f'(2) = 6(2)^2 + 6(2) - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 6(9) + 6(-3) - 12 )</td>
<td>( f'(0) = 6(0) + 6(0) - 12 )</td>
<td>( f'(2) = 6(4) + 6(2) - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 54 - 18 - 12 )</td>
<td>( f'(0) = 0 - 0 - 12 )</td>
<td>( f'(2) = 24 + 12 - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 24 )</td>
<td>( f'(0) = -12 )</td>
<td>( f'(2) = 24 )</td>
</tr>
</tbody>
</table>

a. The test numbers \(-3\) and 2 have a first derivative that is a positive value, therefore, the function is increasing on the intervals \((−∞, −2)\) and \((1, ∞)\).

b. The test number 0 has a first derivative that is a negative value, therefore, the function is decreasing on the interval \((-2, 1)\).

c.
7. \( f(x) = x^3 - 12x + 4 \)

First find the derivative and the critical values as follows:

\[
\begin{align*}
f'(x) &= 3x^2 - 12 \\
0 &= 3x^2 - 12 \\
12 &= 3x^2 \\
4 &= x^2 \\
\pm 2 &= x
\end{align*}
\]

The partition numbers (also critical values of \( f \)) are \( x = -2 \) and \( x = 2 \). Choose a number within each interval to find the value of the first derivative:

<table>
<thead>
<tr>
<th>Test number: –3</th>
<th>Test number: 0</th>
<th>Test number: 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-3) = 3(-3)^2 - 12 )</td>
<td>( f'(0) = 3(0)^2 - 12 )</td>
<td>( f'(3) = 3(3)^2 - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 3(9) - 12 )</td>
<td>( f'(0) = 3(0) - 12 )</td>
<td>( f'(3) = 3(9) - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 27 - 12 )</td>
<td>( f'(0) = 0 - 12 )</td>
<td>( f'(3) = 27 - 12 )</td>
</tr>
<tr>
<td>( f'(-3) = 15 )</td>
<td>( f'(0) = -12 )</td>
<td>( f'(3) = 15 )</td>
</tr>
</tbody>
</table>

a. The test numbers –3 and 3 have a first derivative that is a positive value, therefore, the function is increasing on the intervals \((-\infty, -2)\) and \((2, \infty)\).

b. The test number 0 has a first derivative that is a negative value, therefore, the function is decreasing on the interval \((-2, 2)\).
Section 5-2 Second Derivatives and Graphs

Goal: To use the second derivative to analyze graphs

Notation: Second Derivative

For \( y = f(x) \), the second derivative of \( f \), provided that it exists, is \( f''(x) = \frac{d}{dx} f'(x) \).

Definition: Concavity

The graph of a function \( f \) is **concave upward** on the interval \((a, b)\) if \( f'(x) \) is increasing on \((a, b)\) and is concave downward on the interval \((a, b)\) if \( f'(x) \) is decreasing on \((a, b)\).

Summary: Concavity

For the interval \((a,b)\), if \( f''(x) > 0 \), then \( f'(x) \) is increasing and the graph of \( f \) is **concave upward**. If \( f''(x) < 0 \), then \( f'(x) \) is decreasing and the graph of \( f \) is **concave downward**.

Theorem: Inflection Point

If \( y = f(x) \) is continuous on \((a, b)\) and has an **inflection point** at \( x = c \), then either \( f''(c) = 0 \) or \( f''(c) \) does not exist.

Procedure: Graphing Strategy (first version)

Step 1 Analyze \( f(x) \). Find the domain and the intercepts
Step 2 Analyze \( f'(x) \). Find the partition numbers for, and the critical values of, \( f'(x) \), and determine local extrema.
Step 3 Analyze \( f''(x) \). Find the partition numbers for \( f''(x) \), and determine concavity.
Step 4 Sketch the graph of the function \( f \).
1. Given the function \( f(x) = 2x^3 - 21x + 13 \), find \( f''(x) \).

\[
\begin{align*}
  f'(x) &= 6x^2 - 21 \\
  f''(x) &= 12x
\end{align*}
\]

2. Given the function \( f(x) = x^4 + 3x^{5/4} + 1 \), find \( f''(x) \).

\[
\begin{align*}
  f'(x) &= 4x^3 + \frac{15}{4}x^{1/4} \\
  f''(x) &= 12x^2 + \frac{15}{16}x^{-3/4}
\end{align*}
\]

3. Given the function \( f(x) = x^3 - 6x^2 + 13x - 1 \), find the \( x \) and \( y \) coordinates of all inflection points.

Inflection points are found by setting the second derivative equal to zero.

\[
\begin{align*}
  f'(x) &= 3x^2 - 12x + 13  \\
  f''(x) &= 6x - 12 \\
  f''(x) &= 0 \\
  f''(x) &= 12 = 6x \\
  2 &= x
\end{align*}
\]

\[
\begin{align*}
  f(2) &= x^3 - 6x^2 + 13x - 1 \\
  f(2) &= (2)^3 - 6(2)^2 + 13(2) - 1 \\
  f(2) &= 8 - 24 + 26 - 1 \\
  f(2) &= 9
\end{align*}
\]

Now check the value of the second derivative on both sides of 2.

\[
\begin{align*}
  f''(1) &= 6(1) - 12 \\
  f''(1) &= -6
\end{align*}
\]

\[
\begin{align*}
  f''(3) &= 6(3) - 12 \\
  f''(3) &= 6
\end{align*}
\]

Because the second derivative changes from negative to positive, the point (2, 9) is an inflection point.
4 – 6 For the functions given, find:

a. The intervals on which the function is concave upward 

b. The intervals on which the function is concave downward 

c. The inflection points

4. \( f(x) = x^4 - 24x^2 \)

First find the second derivative and the partition numbers as follows:

\[
\begin{align*}
\frac{d}{dx} f(x) &= 4x^3 - 48x \\
\frac{d^2}{dx^2} f(x) &= 12x^2 - 48 \\
0 &= 12x^2 - 48 \\
48 &= 12x^2 \\
4 &= x^2 \\
\pm 2 &= x
\end{align*}
\]

The partition numbers are \( x = -2 \) and \( x = 2 \). Choose a number on each side of the partition numbers. Find the value of the second derivative at these test numbers

<table>
<thead>
<tr>
<th>Test number: -3</th>
<th>Test number: 0</th>
<th>Test number: 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(-3) = 12(-3)^2 - 48 )</td>
<td>( f''(0) = 12(0)^2 - 48 )</td>
<td>( f''(3) = 12(3)^2 - 48 )</td>
</tr>
<tr>
<td>( f''(-3) = 12(9) - 48 )</td>
<td>( f''(0) = 12(0) - 48 )</td>
<td>( f''(3) = 12(9) - 48 )</td>
</tr>
<tr>
<td>( f''(3) = 108 - 48 )</td>
<td>( f''(0) = 0 - 48 )</td>
<td>( f''(3) = 108 - 48 )</td>
</tr>
<tr>
<td>( f''(-3) = 60 )</td>
<td>( f''(0) = -48 )</td>
<td>( f''(3) = 60 )</td>
</tr>
</tbody>
</table>

a. The test numbers \(-3\) and \(3\) have a second derivative that is a positive value, therefore the function is concave upward on the intervals \((-\infty, -2)\) and \((2, \infty)\).

b. The test number \(0\) has a second derivative that is a negative value, therefore the function is concave downward on the interval \((-2, 2)\).

c. The partition numbers \(-2\) and \(2\) are both inflection points because the function changes concavity at each number. The inflection points are \((-2, -80)\) and \((2, -80)\).
5. \( f(x) = 2x^3 - 6x^2 + 12x - 9 \)

First find the second derivative and the partition numbers as follows:

\[
\begin{align*}
   f'(x) &= 6x^2 - 12x + 12 \\
   f''(x) &= 12x - 12 \\
   0 &= 12x - 12 \\
   12 &= 12x \\
   1 &= x
\end{align*}
\]

The partition number is \( x = 1 \). Choose a number on each side. Find the value of the second derivative at these test numbers

<table>
<thead>
<tr>
<th>Test number: 0</th>
<th>Test number: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(0) = 12(0) - 12 )</td>
<td>( f''(2) = 12(2) - 12 )</td>
</tr>
<tr>
<td>( f''(0) = 0 - 12 )</td>
<td>( f''(2) = 24 - 12 )</td>
</tr>
<tr>
<td>( f''(0) = -12 )</td>
<td>( f''(2) = 12 )</td>
</tr>
</tbody>
</table>

a. The test number 2 has a second derivative that is a positive value, therefore the function is concave upward on the interval \((1, \infty)\).

b. The test number 0 has a second derivative that is a negative value, therefore the function is concave downward on the interval \((-\infty, 1)\).

c. The partition value 1 is an inflection point because the function changes concavity at this number. The inflection point is \((1, -1)\).
6. \( f(x) = 9e^{2x} - e^{3x} \)

First find the second derivative and the partition numbers as follows:

\[
\begin{align*}
    f'(x) &= 18e^{2x} - 3e^{3x} \\
    f''(x) &= 36e^{2x} - 9e^{3x} \\
    0 &= 36e^{2x} - 9e^{3x} \\
    9e^{3x} &= 36e^{2x} \\
    e^x &= 4 \\
    x &= \ln 4
\end{align*}
\]

The partition number is \( x = \ln 4 \). Choose a number on each side. Find the value of the second derivative at these test numbers.

<table>
<thead>
<tr>
<th>Test number: 0</th>
<th>Test number: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(0) = 36e^{2(0)} - 9e^{3(0)} )</td>
<td>( f''(2) = 36e^{2(2)} - 9e^{3(2)} )</td>
</tr>
<tr>
<td>( f''(0) = 36e^{0} - 9e^{0} )</td>
<td>( f''(2) = 36e^{4} - 9e^{6} )</td>
</tr>
<tr>
<td>( f''(0) = 36 - 9 )</td>
<td>( f''(2) = 1965.53 - 3630.86 )</td>
</tr>
<tr>
<td>( f''(0) = 27 )</td>
<td>( f''(2) = -1665.33 )</td>
</tr>
</tbody>
</table>

a. The test number 0 has a second derivative that is a positive value, therefore the function is concave upward on the interval \((-\infty, \ln 4)\).
b. The test number 2 has a second derivative that is a negative value, therefore the function is concave downward on the interval \((\ln 4, \infty)\).
c. The partition number \( \ln 4 \) is an inflection point because the function changes concavity at this number. The inflection point is \((\ln 4, 80)\).
7–9 Apply the four-step graphing procedure and sketch the graph of the following functions.

7. \( f(x) = (x + 2)(x^2 - x - 4) \)

**Step 1:** Analyze \( f(x) \). The domain of the function is all real numbers. To find the intercepts, first find \( f(0) \). Next, find the solutions of \( f(x) = 0 \).

\[
\begin{align*}
  f(0) &= (0 + 2)((0)^2 - 0 - 4) \\
  f(0) &= (2)(-4) \\
  f(0) &= -8
\end{align*}
\]

\[ x + 2 = 0 \quad \text{or} \quad x^2 - x - 4 = 0 \]

\[ x = -2 \quad \text{or} \quad x = \frac{1 \pm \sqrt{17}}{2} \]

Therefore, the \( x \) intercepts are \( \left(\frac{1+\sqrt{17}}{2}, 0\right) \), \( \left(\frac{1-\sqrt{17}}{2}, 0\right) \) and \( (-2, 0) \), and the \( y \) intercept is \( (0, -8) \).

**Step 2:** Analyze the first derivative.

\[
\begin{align*}
  f'(x) &= (x + 2)(2x - 1) + (x^2 - x - 4)(1) \\
  f'(x) &= 2x^2 + 3x - 2 + x^2 - x - 4 \\
  0 &= 3x^2 + 2x - 6 \\
  \frac{-1 \pm 2\sqrt{5}}{3} &= x
\end{align*}
\]

The partition numbers are \( x = \frac{-1+2\sqrt{5}}{3} \) and \( x = \frac{-1-2\sqrt{5}}{3} \). Choose a number within each interval. Find the value of the first derivative at these test numbers

<table>
<thead>
<tr>
<th>Test number: (-2)</th>
<th>Test number: 0</th>
<th>Test number: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-2) = 3(-2)^2 + 2(-2) - 6 )</td>
<td>( f'(0) = 3(0)^2 + 2(0) - 6 )</td>
<td>( f'(2) = 3(2)^2 + 2(2) - 6 )</td>
</tr>
<tr>
<td>( f'(-2) = 3(4) - 4 - 6 )</td>
<td>( f'(0) = 3(0) - 0 - 6 )</td>
<td>( f'(2) = 3(4) + 4 - 6 )</td>
</tr>
<tr>
<td>( f'(-2) = 2 )</td>
<td>( f'(0) = -6 )</td>
<td>( f'(2) = 10 )</td>
</tr>
</tbody>
</table>

Therefore, the function is increasing on the intervals \((-\infty, \frac{-1-2\sqrt{5}}{3})\) and \((\frac{-1+2\sqrt{5}}{3}, \infty)\), decreasing on the interval \((\frac{-1-2\sqrt{5}}{3}, \frac{-1+2\sqrt{5}}{3})\), and has a local maximum at \( x = \frac{-1-2\sqrt{5}}{3} \) and a local minimum at \( x = \frac{-1+2\sqrt{5}}{3} \).
Step 3: Analyze the second derivative

\[ f'(x) = 3x^2 + 2x - 6 \]
\[ f''(x) = 6x + 2 \]
\[ 0 = 6x + 2 \]
\[ -2 = 6x \]
\[ x = -\frac{1}{3} \]

The partition number is \( x = -\frac{1}{3} \). Choose a number on each side of the partition number. Find the value of the second derivative at these numbers.

Test number: –1
\[ f''(-1) = 6(-1) + 2 = -4 \]

Test number: 0
\[ f''(0) = 6(0) + 2 = 2 \]

Therefore, the function is concave upward on the interval \( (-\frac{1}{3}, \infty) \), concave downward on the interval \( (-\infty, -\frac{1}{3}) \), and has an inflection point at \( x = -\frac{1}{3} \).

Step 4: Sketch the function
8. \( f(x) = 5x^4 - 10x^3 \)

**Step 1:** Analyze \( f(x) \). The domain of the function is all real numbers. To find the intercepts, first find \( f(0) \). Next, find the solution of \( f(x) = 0 \).

\[
\begin{align*}
 f(0) &= 5(0)^4 - 10(0)^3 = 0 = 5x^4 - 10x^3 \\
 f(0) &= 0 - 0 = 0 = 5x^3(x - 2) \\
 f(0) &= 0 = x = 0, 2 \\

\end{align*}
\]

Therefore, the \( x \) intercepts are \((0, 0)\) and \((2, 0)\), and the \( y \) intercept is \((0, 0)\).

**Step 2:** Analyze the first derivative.

\[
\begin{align*}
 f'(x) &= 20x^3 - 30x^2 \\
 0 &= 10x^2(2x - 3) \\
 0, \frac{3}{2} &= x \\

\end{align*}
\]

The partition numbers are \( x = 0 \) and \( x = \frac{3}{2} \). Choose a number within each interval. Find the value of the first derivative at these test numbers.

<table>
<thead>
<tr>
<th>Test number: (-1)</th>
<th>Test number: (1)</th>
<th>Test number: (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(-1) = 20(-1)^3 - 30(-1)^2 )</td>
<td>( f'(1) = 20(1)^3 - 30(1)^2 )</td>
<td>( f'(2) = 20(2)^3 - 30(2)^2 )</td>
</tr>
<tr>
<td>( f'(-1) = 20(-1) - 30(1) )</td>
<td>( f'(1) = 20(1) - 30(1) )</td>
<td>( f'(2) = 20(8) - 30(4) )</td>
</tr>
<tr>
<td>( f'(-1) = -50 )</td>
<td>( f'(1) = -10 )</td>
<td>( f'(2) = 40 )</td>
</tr>
</tbody>
</table>

Therefore, the function is increasing on the interval \( \left( \frac{3}{2}, \infty \right) \) and decreasing on the intervals \((-\infty, 0)\) and \((0, \frac{3}{2})\). The function has a local minimum at \( x = \frac{3}{2} \).
Step 3: Analyze the second derivative

\[ f'(x) = 20x^3 - 30x^2 \]
\[ f''(x) = 60x^2 - 60x \]

\[ 0 = 60x^2 - 60x \]
\[ 0 = 60x(x - 1) \]

\[ 0, 1 = x \]

The partition numbers are \( x = 0 \) and \( x = 1 \). Choose a number on each side of the partition number. Find the value of the second derivative at these test numbers.

<table>
<thead>
<tr>
<th>Test number: –1</th>
<th>Test number: 0.5</th>
<th>Test number: 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(-1) = 60(-1)^2 - 60(-1) )</td>
<td>( f''(0.5) = 60(0.5)^2 - 60(0.5) )</td>
<td>( f''(2) = 60(2)^2 - 60(2) )</td>
</tr>
<tr>
<td>( f''(-1) = 60(1) + 60 )</td>
<td>( f''(0.5) = 60(0.25) - 30 )</td>
<td>( f''(2) = 60(4) - 120 )</td>
</tr>
<tr>
<td>( f''(-1) = 120 )</td>
<td>( f''(0.5) = -15 )</td>
<td>( f''(2) = 120 )</td>
</tr>
</tbody>
</table>

Therefore, the function is concave upward on the intervals \((-\infty, 0)\) and \((1, \infty)\) and concave downward on the interval \((0, 1)\), and has an inflection point at \( x = 0 \).

Step 4: Sketch the function
9. \( f(x) = 8e^x - e^{2x} \)

**Step 1:** Analyze \( f(x) \). The domain of the function is all real numbers. To find the intercepts, first find \( f(0) \). Next, find the solutions of \( f(x) = 0 \).

\[
\begin{align*}
f(0) &= 8e^0 - e^{2(0)} \\
f(0) &= 8(1) - 1 \\
f(0) &= 7
\end{align*}
\]

Therefore, the \( x \) intercept is \((\ln 8, 0)\), and the \( y \) intercept is \((0, 7)\).

**Step 2:** Analyze the first derivative.

\[
\begin{align*}
f'(x) &= 8e^x - 2e^{2x} \\
0 &= 8e^x - 2e^{2x} \\
0 &= 2e^x(4 - e^x) \\
\ln 4 &= x
\end{align*}
\]

The partition number is \( x = \ln 4 \). Choose a number within each interval. Find the value of the first derivative at these test numbers.

Test number: 0 \hspace{1cm} Test number: 2

\[
\begin{align*}
f'(0) &= 8e^0 - 2e^{2(0)} \\
f'(0) &= 8(1) - 2(1) \\
f'(0) &= 4 \\
f'(2) &= 8e^2 - 2e^{2(2)} \\
f'(2) &= 8(7.39) - 2(54.60) \\
f'(2) &= 50.08
\end{align*}
\]

Therefore, the function is increasing on the interval \((-\infty, \ln 4)\) and decreasing on the interval \((\ln 4, \infty)\). The function has a local maximum at \( x = \ln 4 \).
Step 3: Analyze the second derivative

\[ f'(x) = 8e^x - 2e^{2x} \]
\[ f''(x) = 8e^x - 4e^{2x} \]

0 = 8e^x - 4e^{2x}
0 = 4e^x(2 - e^x)
\ln 2 = x

The partition number is \( x = \ln 2 \). Choose a number on each side of the partition number. Find the value of the second derivative at these test numbers.

Test number: 0  
Test number: 1

\[ f''(0) = 8e^0 - 4e^{2(0)} \quad f''(1) = 8e^1 - 4e^{2(1)} \]
\[ f''(0) = 8(1) - 4(1) \quad f''(1) = 8(2.72) - 4(7.39) \]
\[ f''(0) = 4 \quad f''(1) = -7.81 \]

Therefore, the function is concave upward on the interval \((-\infty, \ln 2)\), concave downward on the interval \((\ln 2, \infty)\), and has an inflection point at \( x = \ln 2 \).

Step 4: Sketch the function

![Graph of the function](image)
10. A company estimates that it will sell $N(x)$ units of a product after spending $x$ thousand dollars on advertising, as given by

$$N(x) = -0.25x^4 + 23x^3 - 360x^2 + 45,000 \quad 25 \leq x \leq 54$$

When is the rate of change of sales increasing and when is it decreasing? What is the point of diminishing returns and the maximum rate of change of sales?

To find when the rate of change is increasing and decreasing, find the second derivative and use the second derivative test.

$$N'(x) = -x^3 + 69x^2 - 720x$$

$$N''(x) = -3x^2 + 138x - 720$$

$$0 = -3(x^2 - 46x + 240)$$

$$0 = -3(x - 40)(x - 6)$$

$$6, 40 = x$$

The partition numbers are $x = 6$ and $x = 40$. Choose a number on each side of the partition numbers within the given domain. Find the value of the second derivative at these test numbers.

<table>
<thead>
<tr>
<th>Test number: 39</th>
<th>Test number: 41</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(39) = -3(39)^2 + 138(39) - 720$</td>
<td>$f''(41) = -3(41)^2 + 138(41) - 720$</td>
</tr>
<tr>
<td>$f''(39) = -4563 + 5382 - 720$</td>
<td>$f''(41) = -5043 + 5658 - 720$</td>
</tr>
<tr>
<td>$f''(39) = 99$</td>
<td>$f''(41) = -105$</td>
</tr>
</tbody>
</table>

a. The test number 39 has a second derivative that is a positive value, therefore the function is concave upward on the interval $(25, 40)$ and so the sales are also increasing over this interval.

b. The test number 41 has a second derivative that is a negative value, therefore the function is concave downward on the interval $(40, 54)$ and so the sales are also decreasing over this interval.

c. The partition number 40 is the point of diminishing returns because it is an inflection point. The maximum rate of change is $N'(40) = 17,600$. 


Section 5-3 L’Hôpital’s Rule

Goal: To find limits of functions that are in the form zero over zero or infinity over infinity

Theorems: L’Hôpital’s Rule for 0/0 Indeterminate forms

1. (Version 1) For \( c \) a real number, if \( \lim_{x \to c} f(x) = 0 \) and \( \lim_{x \to c} g(x) = 0 \), then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

providing that the second limit exists or is \( \pm\infty \).

2. (Version 2) The first version of L’Hôpital’s rule remains valid if the symbol \( x \to c \) is replaced everywhere it occurs with one of the following symbols:

\( x \to c^+ \quad x \to c^- \quad x \to \infty \quad x \to -\infty \)

3. Versions 1 and 2 of L’Hôpital’s rule for the indeterminate form 0/0 are also valid if the limit of \( f \) and the limit of \( g \) are both infinite.

Use L’Hôpital’s rule to find the limit in Problems 1 and 2.

1. \[
\lim_{x \to 5} \frac{x^3 - 125}{x^2 + 2x - 35} = \lim_{x \to 5} \frac{3x^2}{2x + 2} = \frac{3(5)^2}{2(5) + 2} = \frac{75}{12} = 75/12
\]

2. \[
\lim_{x \to \infty} \frac{2x^2 + 7}{6x^3 + 5} = \lim_{x \to \infty} \frac{4x}{18x^2} = \lim_{x \to \infty} \frac{4}{36x} = 0
\]
3. Explain why L’Hôpital’s rule does not apply to the following problem. If the limit exists, find it by other means.

\[
\lim_{{x \to 2}} \frac{{x^3 - 9}}{{x^2 - 3}}
\]

L’Hôpital’s rule does not apply because when the value is substituted into the function, the result is not 0/0.

\[
\lim_{{x \to 2}} \frac{{x^3 - 9}}{{x^2 - 3}} = \frac{{(2)^3 - 9}}{{(2)^2 - 3}} = \frac{-1}{1} = -1
\]

4. Use L’Hôpital’s rule to find the following limit and then verify that limit by a second method.

\[
\lim_{{x \to 3}} \frac{{x^2 + 4x - 21}}{{x^2 - x - 6}} = \lim_{{x \to 3}} \frac{{2x + 4}}{{2x - 1}} = \frac{{2(3) + 4}}{{2(3) - 1}} = \frac{10}{5} = 2
\]

\[
\lim_{{x \to 3}} \frac{{x^2 + 4x - 21}}{{x^2 - x - 6}} = \lim_{{x \to 3}} \frac{{x - 3)(x + 7)}}{{x^2 - x - 6}} = \lim_{{x \to 3}} \frac{{x + 7}}{{x^2 - x - 6}} = \frac{3 + 7}{3 + 2} = 2
\]

Find each limit in Problems 5–8. Note that L’Hôpital’s rule does not apply to every problem and some problems will require more than one application of L’Hôpital’s rule.

5. \[
\lim_{{x \to 1}} \frac{{x^3 + 7x - 6}}{{x^2 - x}}
\]

If the value –1 is substituted into the function, the result will be 0/0. Therefore, we use L’Hôpital’s rule.

\[
\lim_{{x \to 1}} \frac{{x^3 + 7x - 6}}{{x^2 - x}} = \lim_{{x \to 1}} \frac{{3x^2 + 7}}{{2x - 1}} = \frac{{3(-1)^2 + 7}}{{2(-1) - 1}} = \frac{10}{-3} = -\frac{10}{3}
\]
6. \[ \lim_{x \to 2} \frac{x^4 - 6x - 3}{x^3 + 2x - 6} \]

The value can be substituted into the function and not result in 0/0. Therefore, use simple substitution.

\[ \lim_{x \to 2} \frac{x^4 - 6x - 3}{x^3 + 2x - 6} = \frac{(2)^4 - 6(2) - 3}{(2)^3 + 2(2) - 6} = \frac{1}{6} \]

7. \[ \lim_{x \to \infty} \frac{7x^3 + 4x - 5}{9x^4 - 6x + 7} \]

If the value \( \infty \) is substituted into the function, the result will be \( \infty / \infty \). Therefore, we use L’Hôpital’s rule.

\[ \lim_{x \to \infty} \frac{7x^3 + 4x - 5}{9x^4 - 6x + 7} = \lim_{x \to \infty} \frac{21x^2 + 4}{36x^3 - 6} = \lim_{x \to \infty} \frac{42x}{108x^2} = \lim_{x \to \infty} \frac{42}{216x} = 0 \]

8. \[ \lim_{x \to -3} \frac{x + 3}{x^2 - 2x - 15} \]

If the value \(-3\) is substituted into the function, the result will be 0/0. Therefore, we use L’Hôpital’s rule.

\[ \lim_{x \to -3} \frac{x + 3}{x^2 - 2x - 15} = \lim_{x \to -3} \frac{1}{2x - 2} = \frac{1}{2(-3) - 2} = -\frac{1}{8} \]

**Alternative Solution:**

\[ \lim_{x \to -3} \frac{x + 3}{x^2 - 2x - 15} = \lim_{x \to -3} \frac{x + 3}{(x + 3)(x - 5)} = \lim_{x \to -3} \frac{1}{x - 5} = \frac{1}{-3 - 5} = -\frac{1}{8} \]
Section 5-4 Curve-Sketching Techniques

Goal: To graph functions using the final version of the graphing strategy.

Procedure: Graphing Strategy (Final Version)

Step 1 Analyze $f(x)$.
   a. Find the domain.
   b. Find the intercepts.
   c. Find the asymptotes.

Step 2 Analyze $f'(x)$. Find the partition numbers for, and the critical values of, $f'(x)$. Determine the intervals on which $f$ is increasing and decreasing, and find the local extrema.

Step 3 Analyze $f''(x)$. Find the partition numbers for $f''(x)$. Determine the intervals on which the graph of $f$ is concave upward or downward, and find the inflection points.

Step 4 Sketch the graph of the function.

In problems 1–3, summarize the pertinent information obtained by applying the graphing strategy and sketch the graph.

1. $f(x) = \frac{x - 4}{x + 4}$

   Step 1: Analyze the function.

   a. The domain is all real numbers except where $x + 4 = 0 \Rightarrow x = -4$.
   b. The $x$ intercepts are found by setting the function (or the numerator) equal to zero and the $y$ intercept is found by setting the value of $x$ equal to zero.

   $x - 4 = 0 \Rightarrow x = 4$
   $f(0) = \frac{0 - 4}{0 + 4} = \frac{-4}{4} = -1$

   Therefore, the $x$ intercept is (4, 0), and the $y$ intercept is (0, -1).

   c. The vertical asymptote will be at the point of discontinuity or $x = -4$.

   The horizontal asymptote is found by taking the limit of the function as it goes to infinity. Since the exponents are the same in the numerator and denominator, it is found by calculating the ratio of the coefficients. Therefore, the horizontal asymptote is $y = 1$. 

5-25
Step 2: Analyze the first derivative.

\[ f'(x) = \frac{(x + 4)(1) - (x - 4)(1)}{(x + 4)^2} \]

\[ f'(x) = \frac{8}{(x + 4)^2} \]

There is only one partition number, which is the value that makes the denominator zero, or \( x = -4 \). Pick a test number on each side of the partition number and find the value of the first derivative at these test numbers.

Test number: \(-5\)  Test number: \(-3\)

\[ f'(-5) = \frac{8}{(-5 + 4)^2} \quad \quad f'(-3) = \frac{8}{(-3 + 4)^2} \]

\[ f'(-5) = \frac{8}{(-1)^2} = 8 \quad \quad f'(-3) = \frac{8}{(1)^2} = 8 \]

Therefore, the function is increasing on the intervals \((-\infty, -4)\) and \((-4, \infty)\).

Step 3: Analyze the second derivative

\[ f''(x) = 8(x + 4)^{-2} \]

\[ f''(x) = -16(x + 4)^{-3} \]

\[ f''(x) = \frac{-16}{(x + 4)^3} \]

There is only one partition number, which is the value that makes the denominator zero, or \( x = -4 \). Pick a test number on each side of the partition number and find the value of the second derivative at these test numbers.

Test number: \(-5\)  Test number: \(-3\)

\[ f''(-5) = \frac{-16}{(-5 + 4)^3} \quad \quad f''(-3) = \frac{-16}{(-3 + 4)^3} \]

\[ f''(-5) = \frac{-16}{(-1)^3} = 16 \quad \quad f''(-3) = \frac{-16}{(1)^3} = -16 \]

Therefore, the function is concave upward on the interval \((-\infty, -4)\) and concave downward on the interval \((-4, \infty)\). However, the function does not have an inflection point because \( f(-4) \) is undefined.
2. \( f(x) = \frac{-4x}{x-3} \)

**Step 1:** Analyze the function.

a. The domain is all real numbers except where \( x - 3 = 0 \Rightarrow x = 3 \).

b. The \( x \) intercepts are found by setting the function (or the numerator) equal to zero and the \( y \) intercept is found by setting the value of \( x \) equal to zero.

\[
\begin{align*}
-4x &= 0 \\
x &= 0 \\
f(0) &= \frac{-4(0)}{(0-3)} = \frac{0}{-3} = 0
\end{align*}
\]

Therefore, the \( x \) intercept is \((0, 0)\), and the \( y \) intercept is \((0, 0)\).

c. The vertical asymptote will be at the point of discontinuity or \( x = 3 \).

The horizontal asymptote is found by taking the limit of the function as it goes to infinity. Since the exponent in the numerator is the same as the denominator, the horizontal asymptote is \( y = \frac{-4}{1} = -4 \).

**Step 2:** Analyze the first derivative.

\[
\begin{align*}
f(x) &= \frac{-4x}{x-3} \\
f'(x) &= \frac{(x-3)(-4) - (-4x)(1)}{(x-3)^2} \\
f'(x) &= \frac{12}{(x-3)^2}
\end{align*}
\]

There is only one partition number; the value that makes the denominator zero is \( x = 3 \). Pick a test number in each interval. Find the value of the first derivative at these test numbers.
Test number: 2

\[ f'(2) = \frac{12}{(2-3)^2} \]
\[ f'(2) = \frac{12}{1} = 12 \]

Test number: 4

\[ f'(4) = \frac{12}{(4-3)^2} \]
\[ f'(4) = \frac{12}{1} = 12 \]

Therefore, the function is increasing on the intervals \((-\infty, 3)\) and \((3, \infty)\). Because the function does not decrease, there is no local extrema.

**Step 3:** Analyze the second derivative

\[ f'(x) = \frac{12}{(x-3)^2} \]

\[ f''(x) = \frac{(x-3)^2(0) - (12)(2(x-3)(1))}{(x-3)^4} \]

\[ f''(x) = \frac{-24}{(x-3)^3} \]

There is only one partition number; the value that makes the denominator zero is \(x = 3\). Pick a test number in each interval. Find the value of the second derivative at these test numbers.

Test number: 2

\[ f''(2) = \frac{-24}{(2-3)^3} \]
\[ f''(2) = \frac{-24}{-1} = 24 \]
\[ f''(2) = 24 \]

Test number: 4

\[ f''(4) = \frac{-24}{(4-3)^3} \]
\[ f''(4) = \frac{-24}{1} = -24 \]
\[ f''(4) = -24 \]

Therefore, the function is concave upward on the interval \((-\infty, 3)\) and concave downward on the interval \((3, \infty)\). However, the function does not have an inflection point because \(f(3)\) is undefined.
Step 4: Sketch the graph

3. \( f(x) = e^{-3x^2} \)

Step 1: Analyze the function.

a. The domain is all real numbers.
b. There are no \( x \) intercepts and the \( y \) intercept is found by setting the value of \( x \) equal to zero.

\[
f(0) = e^{-3(0)^2} = e^0 = 1
\]

Therefore, there is no \( x \) intercept, and the \( y \) intercept is \((0, 1)\).
c. There is no vertical asymptote because the domain is all real numbers.

The horizontal asymptote is found by taking the limit of the function as it goes to infinity. Therefore, the horizontal asymptote is \( y = 0 \).

Step 2: Analyze the first derivative.

\[
f'(x) = -6xe^{-3x^2}
\]

There is only one partition number; and also a value that makes the function zero or \( x = 0 \). Pick a test number on each side of the critical value and find the value of the first derivative at these test numbers.

Test number: \(-1\) \hspace{2cm} Test number: 1

\[
\begin{align*}
f'(-1) &= -6(-1)e^{-3(-1)^2} \\
f'(-1) &= 6e^{-3} \\
f'(-1) &= 0.299
\end{align*}
\]

\[
\begin{align*}
f'(1) &= -6(1)e^{-3(1)^2} \\
f'(1) &= -6e^{-3} \\
f'(1) &= -0.299
\end{align*}
\]

Therefore, the function is increasing on the interval \((-\infty, 0)\) and decreasing on the interval \((0, \infty)\), and has a local maximum at \( x = 0 \).
Step 3: Analyze the second derivative

\[
\begin{align*}
f'(x) &= -6xe^{-3x^2} \\
f''(x) &= -6x(-6xe^{-3x^2}) + e^{-3x^2}(-6) \\
f''(x) &= 6e^{-3x^2}(6x^2 - 1)
\end{align*}
\]

There are two partition numbers. Set the second derivative equal to zero to find the partition numbers \( x = \pm 0.41 \). Pick a test number in each interval of the partition number and find the value of the second derivative at these test numbers.

Test number: -1
\[
\begin{align*}
f''(-1) &= 6e^{-3(-1)^2}(6(-1)^2 - 1) \\
f''(-1) &= 6e^{-3}(5) \\
f''(-1) &= 1.49
\end{align*}
\]

Test number: 0
\[
\begin{align*}
f''(0) &= 6e^{-3(0)^2}(6(0)^2 - 1) \\
f''(0) &= 6e^{-3}(0) \\
f''(0) &= 6
\end{align*}
\]

Test number: 1
\[
\begin{align*}
f''(1) &= 6e^{-3(1)^2}(6(1)^2 - 1) \\
f''(1) &= 6e^{-3}(5) \\
f''(1) &= 1.49
\end{align*}
\]

Therefore, the function is concave upward on the intervals \((-\infty, -0.41)\) and \((-0.41, \infty)\). The function is concave downward on the interval \((-0.41, 0.41)\). The function has inflection points at \( x = -0.41 \) and \( x = 0.41 \).

Step 4: Sketch the graph
Section 5-5 Absolute Maxima and Minima

Goal: To find absolute maxima or minima on open and closed intervals

**Definition:** Absolute Maxima and Minima

If \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \), then \( f(c) \) is called the **absolute maximum**.

If \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \), then \( f(c) \) is called the **absolute minimum**.

**Theorems:**

1. A function \( f \) that is continuous on a closed interval \([a, b]\) has both an absolute maximum and an absolute minimum on that interval.

2. Absolute extrema (if they exist) must always occur at critical values or at endpoints.

3. Second Derivative Test
   Let \( f \) be continuous on an interval \( I \) with only one critical value \( c \) in \( I \).

   If \( f'(c) = 0 \) and \( f''(c) > 0 \), then \( f(c) \) is the absolute minimum of \( f \) on \( I \).

   If \( f'(c) = 0 \) and \( f''(c) < 0 \), then \( f(c) \) is the absolute maximum of \( f \) on \( I \).

**Procedure:** Finding absolute extrema on closed intervals

1. Check to make certain that \( f \) is continuous over \([a, b]\).
2. Find the critical values in the interval \((a, b)\).
3. Evaluate \( f \) at the endpoints \( a \) and \( b \) and at the critical values found in step 2.
4. The absolute maximum is the largest value found in step 3.
5. The absolute minimum is the smallest value found in step 3.
1. Find the absolute maximum and the absolute minimum for the given function on the given interval.

\[ f(x) = x^2 - 4x + 7 \quad [-1, 8] \]

First find the critical value(s) for the function.

\[ f(x) = x^2 - 4x + 7 \]
\[ f'(x) = 2x - 4 \]
\[ 0 = 2x - 4 \]
\[ 4 = 2x \]
\[ 2 = x \]

Calculate the function value at the endpoints and the critical value.

\[ f(-1) = (-1)^2 - 4(-1) + 7 \]
\[ f(2) = (2)^2 - 4(2) + 7 \]
\[ f(8) = (8)^2 - 4(8) + 7 \]
\[ f(-1) = 1 + 4 + 7 \]
\[ f(2) = 4 - 8 + 7 \]
\[ f(8) = 64 - 32 + 7 \]
\[ f(-1) = 12 \]
\[ f(2) = 3 \]
\[ f(8) = 39 \]

Therefore, the absolute maximum is at the point \((8, 39)\) and the absolute minimum is at the point \((2, 3)\).

In Problems 2–3, find the absolute maximum and the absolute minimum, if either exists, for each function.

2. \[ f(x) = 5x^3 - 6x^4 \]

First find the critical value(s) for the function.

\[ f(x) = 5x^3 - 6x^4 \]
\[ 3x^2 = 0 \]
\[ 5 - 8x = 0 \]
\[ f'(x) = 15x^2 - 24x^3 \]
\[ x^2 = 0 \]
\[ -8x = -5 \]
\[ 0 = 3x^2(5 - 8x) \]
\[ x = 0 \]
\[ x = \frac{5}{8} \]

Then use the second-derivative test at the critical values.

\[ f''(x) = 30x - 72x^2 \]
\[ f''(0) = 30(0) - 72(0)^2 \]
\[ f''(\frac{5}{8}) = 30(\frac{5}{8}) - 72(\frac{5}{8})^2 \]
\[ f''(0) = 0 \]
\[ f''(\frac{5}{8}) = -\frac{75}{8} \]
The second-derivative test fails at 0 and shows that there is a local maximum at \( \frac{5}{8} \).

Calculate the function value at the critical values.

\[
\begin{align*}
f(0) &= 5(0)^3 - 6(0)^4 \\
f\left(\frac{5}{8}\right) &= 5\left(\frac{5}{8}\right)^3 - 6\left(\frac{5}{8}\right)^4 \\
f(0) &= 0 - 0 \\
f\left(\frac{5}{8}\right) &= 5\left(\frac{125}{512}\right) - 6\left(-\frac{625}{4096}\right) \\
f\left(\frac{5}{8}\right) &= \frac{625}{512} - \frac{3750}{4096} \\
f\left(\frac{5}{8}\right) &= \frac{625}{2048}
\end{align*}
\]

Therefore, the local maximum is at the point \( \left(\frac{5}{8}, \frac{625}{2048}\right) \) and the local minimum is at the point \((0, 0)\). Also, \( f\left(\frac{5}{8}\right) = \frac{625}{2048} \) is the absolute maximum; there is no absolute minimum.

3. \( f(x) = \frac{x^2 - 4}{x^2 + 1} \)

First find the critical value(s) for the function.

\[
\begin{align*}
f(x) &= \frac{x^2 - 4}{x^2 + 1} \\
f'(x) &= \frac{(x^2 + 1)(2x) - (x^2 - 4)(2x)}{(x^2 + 1)^2} \\
f''(x) &= \frac{10x}{(x^2 + 1)^2} \\
0 &= 10x \\
0 &= x
\end{align*}
\]

To determine if this is a maximum or a minimum, we find the second derivative and use the second-derivative test.
\[
\frac{df}{dx} = \frac{10x}{x^4 + 2x^2 + 1}
\]
\[
\frac{d^2f}{dx^2} = \frac{(x^4 + 2x^2 + 1)(10) - (10x)(4x^3 + 4x)}{(x^4 + 2x^2 + 1)^2}
\]
\[
\frac{d^3f}{dx^3} = \frac{-30x^4 - 20x^2 + 10}{(x^4 + 2x^2 + 1)^2}
\]
\[
f''(0) = \frac{-30(0)^4 - 20(0)^2 + 10}{((0)^4 + 2(0)^2 + 1)^2} = 10 > 0
\]

Therefore, \((0, -4)\) is a local minimum point because the second derivative is positive. Also, \(f(0) = -4\) is the absolute minimum; there is no absolute maximum.

In Problems 4–6, find the indicated extremum of each function on the given interval.

4. Absolute minimum value on \((0, \infty)\) for \(f(x) = 6 + 2x + \frac{7}{x}\).

First find the critical value(s) for the function.

\[
f(x) = 6 + 2x + 7x^{-1}
\]
\[
f'(x) = 2 - 7x^{-2}
\]
\[
f''(x) = 2 - \frac{7}{x^2}
\]
\[
0 = 2 - \frac{7}{x^2}
\]
\[
x = \sqrt{\frac{7}{2}}
\]

There is only one critical value. To determine if this is a minimum, we will find the second derivative and use the second-derivative test.

\[
f'(x) = 2 - 7x^{-2}
\]
\[
f''(x) = 14x^{-3}
\]
\[
f''(1.87) = 14(1.87)^{-3}
\]
\[
f''(1.87) \approx 2.14
\]

The second derivative of the critical value is positive, therefore, the absolute minimum is \(f(x) = f(\sqrt{\frac{7}{2}}) = 13.48\).
5. Absolute minimum value on \([0, \infty)\) for \(f(x) = 5x^2 - 4x + 13\).

First find the critical value(s) for the function.

\[
\begin{align*}
f(x) &= 5x^2 - 4x + 13 \\
f'(x) &= 10x - 4 \\
0 &= 10x - 4 \\
-10x &= -4 \\
x &= 0.4
\end{align*}
\]

There is only one critical value. To determine if this is a minimum, we will find the second derivative and use the second-derivative test.

\[
\begin{align*}
f'(x) &= 10x - 4 \\
f''(x) &= 10 \\
f''(0.4) &= 10
\end{align*}
\]

The second derivative of the critical value is positive, therefore, the absolute minimum is \(f(x) = f(0.4) = 12.2\).
6. Absolute maximum value on \((0, \infty)\) for \(f(x) = x^6 - 6x^4\).

First find the critical value(s) for the function.

\[
f(x) = x^6 - 6x^4
\]

\[
f'(x) = 6x^5 - 24x^3
\]

\[
0 = 6x^3(x^2 - 4)
\]

\[
6x^3 = 0 \quad \text{or} \quad x^2 - 4 = 0
\]

\[
x = 0 \quad x^2 = 4 \quad x = \pm 2
\]

Only \(x = 2\) is in the interval \((0, \infty)\). To determine if the critical value \(x = 2\) is a maximum, we find the second derivative and use the second-derivative test.

\[
f''(x) = 30x^4 - 72x^2
\]

\[
f''(2) = 30(2)^4 - 72(2)^2
\]

\[
f''(2) = 480 - 288
\]

\[
f''(2) = 192
\]

The second derivative of the critical value is positive, therefore, the absolute maximum is \(f(x) = f(2) = -32\).
Section 5-6 Optimization

Goal: To solve application problems that involve maximum and minimum values

Procedure: Strategy for Solving Optimization Problems

1. Introduce variables, look for relationships among the variables, and construct a mathematical model of the form
   Maximize (or minimize) \( f(x) \) on the interval \( I \).
2. Find the critical values of \( f(x) \).
3. Use the procedure developed in Section 5–5 to find the absolute maximum (or minimum) value of \( f(x) \) on the interval \( I \) and the value(s) of \( x \) where this occurs.
4. Use the solution to the mathematical model to answer all the questions asked in the problem.

Solve the following problems using the four-step procedure outlined above.

1. Find two numbers whose sum is 36 and whose product is maximum.

   **Step 1:** Maximize \( A = xy \) subject to \( x + y = 36 \)

   **Step 2:**
   \[
   x + y = 36 \\
   x = 36 - y
   \]

   \[
   A(y) = xy \\
   A(y) = (36 - y)y \\
   A(y) = 36y - y^2
   \]

   Find the first derivative:
   \[
   A'(y) = 36 - 2y \\
   0 = 36 - 2y \\
   2y = 36 \\
   y = 18
   \]

   \[
   x = 36 - y = 36 - 18 = 18
   \]
Step 3: Use the second-derivative test to determine whether the critical value produces a maximum value.

\[ A'(y) = 36 - 2y \]
\[ A''(y) = -2 \]

Since the second derivative is always negative, the value –2 gives a maximum.

Step 4: The numbers 18 and 18 have a sum 36 and gives the maximum product 324.

2. A fence is to be built to enclose a rectangular area of 900 square feet. The fence along three sides is to be made of material that costs $7 per foot. The material for the fourth side costs $21 per foot. Find the dimensions of the rectangle that will allow for the most economical fence to be built.

Step 1: Minimize \( C = 14x + 28y \) subject to \( xy = 900 \)

Step 2:
\[
\begin{align*}
xy &= 900 \\
x &= \frac{900}{y} \\
x &= 900y^{-1}
\end{align*}
\]

\[
\begin{align*}
C(y) &= 14x + 28y \\
C(y) &= 14(900y^{-1}) + 28y \\
C(y) &= 12600y^{-1} + 28y
\end{align*}
\]

Find the first derivative:
\[
C'(y) = -12600y^{-2} + 28
\]
\[
0 = -12600y^{-2} + 28
\]
\[
12600y^{-2} = 28
\]
\[
28y^2 = 12600
\]
\[
y^2 = 450
\]
\[
y = 3\sqrt{50} \approx 21.21
\]
\[
x = \frac{900}{y} = \frac{900}{3\sqrt{50}} \approx 424.43
\]
Step 3: Use the second-derivative test to determine whether the critical value produces a maximum value.

\[ C'(y) = -12,600y^{-2} + 28 \]
\[ C''(y) = 25,200y^{-3} \]
\[ C''(3\sqrt{50}) = 25,200(3\sqrt{50})^{-3} \approx 2.64 \]

Since the second-derivative has a positive value, the critical value \(3\sqrt{50}\) produces a minimum value.

Step 4: A rectangular field that has a width of 21.21 feet (at a cost of $7.00 per foot) and a length of 42.43 feet (at a cost of $21.00 per foot) will minimize the cost of the fence, which is approximately $1,485.

3. A commercial apple grower estimates from past records that if 40 trees are planted per acre, then each tree will yield an average of 60 pounds of apples per season. If, for each additional tree planted per acre (up to 30), the average yield is reduced by 1 pounds, how many trees should be planted per acre to obtain the maximum yield per acre? What is the maximum yield?

Step 1: Maximize \(P(x) = (40 + x)(60 - x)\) subject to \(x \leq 30\)

Step 2: \(P(x) = (40 + x)(60 - x)\)
\[ P(x) = 2400 + 20x - x^2 \]

Find the first derivative: \(P(x) = 2400 + 20x - x^2\)
\[ P'(x) = 20 - 2x \]
\[ 0 = 20 - 2x \]
\[ 2x = 20 \]
\[ x = 10 \]

Step 3: Use the second-derivative test to determine whether the critical value produces a maximum value.
\[ P'(x) = 20 - 2x \]
\[ P''(x) = -2 \]

Since the second derivative is always negative, the critical value gives an absolute maximum.
Step 4: The variable $x$ is the number of additional trees that should be planted. Therefore, 10 additional trees should be planted per acre for a total of 50 trees per acre. The production will be reduced by 10 pounds per tree for a production level of 50 pounds per tree and a total production of 2500 pounds per acre.

4. A 400-room hotel in New York City is filled to capacity every night at $100 a room. For each $2 increase in rent, 4 fewer rooms are rented. If each rented room costs $12 to service per day, how much should the management charge for each room to maximize gross profit? What is the maximum gross profit?

Step 1: Maximize $P(x) = R(x) - C(x)$ subject to $x \leq 100$

Step 2: 

$P(x) = (400 - 4x)(100 + 2x) - 12(400 - 4x)$

$P(x) = 35,200 + 448x - 8x^2$

Find the first derivative: $P'(x) = 448 - 16x$

$0 = 448 - 16x$

$16x = 448$

$x = 28$

Step 3: Use the second-derivative test to determine whether the critical value produces a maximum value.

$P''(x) = -16$

Since the second derivative is always negative, the critical value gives an absolute maximum.

Step 4: The variable $x$ is the number of $2 increases that the management should add to the price of the room. Therefore, management should increase the price of the room by $56, which becomes $156 per room. This increase result’s in a decrease of 112 rooms being rented, or 288 rooms. The profit will be 288 rooms rented for $156 per room minus the $12 per room spent to clean them. The maximum profit would be $41,472.