UNIQUENESS AND HÖLDER TYPE STABILITY OF CONTINUATION FOR THE LINEAR THERMOELASTICITY SYSTEM WITH RESIDUAL STRESS

NANHEE KIM
Wichita State University
1845 Fairmount
Wichita, KS 67260-0033, USA

ABSTRACT. By introducing some auxiliary functions, an elasticity system with thermal effects becomes a coupled hyperbolic-parabolic system. Using this reduced system, we obtain a Carleman estimate with two large parameters for the linear thermoelasticity system with residual stress which is the basic tool for showing stability estimates in the lateral Cauchy problem.

1. Introduction. We consider the linear elasticity system with thermal effects, that is, the linear thermoelasticity system with residual stress. This is an anisotropic system. Thermal effects within an elastic solid produce heat transfer by conduction and the flow of thermal energy establishes a temperature field in the material. According to the theory of thermoelasticity the elastic strain field is coupled with the temperature field. Most solids exhibit a volumetric change with temperature variation, and thus the presence of a temperature distribution generally induces stresses created from boundary or internal constraints. The thermal component is natural for residual stress modeling used in many engineering and geophysical applications. This residual stress might be induced by thermal changes due to cooling or heating, so the temperature changes cannot be ignored. Thermal stresses exist whenever temperature gradients are present in a material body. In general, residual stress is anisotropic [15]. Therefore its possible source cannot be assumed to be isotropic. Hence we consider a general parabolic second order partial differential operator to model this thermal effect.

In this paper we obtain Carleman estimates with a second large parameter $\gamma$ and a general weight function $\psi$ for a second order parabolic partial differential operator and the linear thermoelasticity system with residual stress. Based on this Carleman estimate, we prove Hölder stability for the Cauchy problem of thermoelasticity system with residual stress.

Here we let $x \in \mathbb{R}^3$ and $(x, t) \in \Omega = G \times (-T, T) \subset \mathbb{R}^4$. The residual stress is modeled by a symmetric second-rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^2(\bar{\Omega})$ which is divergence free, $\nabla \cdot R = 0$. Let $u(x, t) = (u_1, u_2, u_3)^\top : \Omega \to \mathbb{R}^3$ be the displacement vector in $\Omega$. The temperature is given by $\theta = \theta(x, t)$.

2000 Mathematics Subject Classification. Primary: 35L51, 35M30; Secondary: 35K10, 58J35.
Key words and phrases. Partial differential equations, Carleman estimate, Cauchy problem, Thermoelasticity, residual stress.
We now introduce the most general operator of linear thermoelasticity system with residual stress.

\[
A_T(u; \theta) = A_R u + B_{1;1}(u, \theta),
\]

\[
P_T(u; \theta) = \partial_t \theta - \sum_{j,k=1}^3 a_{jk} \partial_j \partial_k \theta + \sum_{j=1}^3 a_j \partial_j \theta + a \theta + B_{2;1}(u, \nabla \cdot u, \nabla \times u)
\]

where \(B_{1;1}\) is a first order matrix linear partial differential operator which does not involve \(\partial_\theta\) with \(C^1(\bar{\Omega})\)-coefficients, \(B_{2;1}\) is a first order scalar linear partial differential operator with \(L^\infty(\Omega)\)-coefficients, and coefficients in \(B_{1;1}\) and in \(B_{2;1}\) are the coupling parameters of the linear thermoelasticity system. Coefficients \(a_j, a \in L^\infty(\Omega)\), and the coefficients \(a_{jk} = a_{kj} \in C^1(\bar{\Omega})\) satisfy the uniform ellipticity condition:

\[
\sum_{j,k=1}^3 a_{jk}(x, t) \xi_j \xi_k \geq \varepsilon_0|\xi|^2 \text{ for all } \xi \neq 0, \; \xi \in \mathbb{R}^3, \; \varepsilon_0 > 0, \; (x, t) \in \bar{\Omega},
\]

and \(A_R u\) is given by

\[
A_R u = \rho \partial_t^2 u - \mu \Delta u - (\lambda + \mu) \nabla (\nabla \cdot u) - (\nabla \cdot u) \nabla \lambda - 2\varepsilon(u) \nabla \mu - \nabla \cdot ((\nabla u) R)
\]

where \(\rho \in C^1(\bar{\Omega})\) and \(\lambda, \mu \in C^2(\bar{\Omega})\) are density and Lamé parameters depending only on \(x\), with \(\varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^\top)\). Let \(\Box(\mu; R) = \partial_t^2 - \sum_{j,k=1}^3 \frac{\mu_{jk} + r_{jk}}{\rho} \partial_j \partial_k\).

The results of Carleman estimates for scalar partial differential operators are complete in many cases, while there are still challenges in some cases and the situation with systems is quite different. Specially, there are difficulties obtaining Carleman estimates for coupled systems using the concept of pseudo-convexity condition. While the thermoelasticity system cannot be principally diagonalized, it has triangular structure which makes it possible to obtain Carleman estimates and uniqueness of the continuation [1], [3], [7]. In the paper [1], Carleman estimates with boundary terms for the heat equations and the wave equations have been derived and the observability estimates for the coupled system have been obtained. Also, as an application, the constant coefficient case of linear thermoelasticity system was discussed. Carleman estimates with a second large parameter for thermoelasticity systems under additional assumptions were obtained and uniqueness of the continuation and controllability results were discussed in [3], [7]. The controllability problem of thermoelastic plates by boundary control with a variable thermal coefficient without any smallness assumption was studied in [4]. And a thermoelastic plate problem, where the elastic equation does not account for rotational forces, was discussed in [14]. In the recent papers [11], [13], Carleman estimates with two large parameters have been obtained for the linear elasticity system with anisotropic residual stress, and [9] for the vectorial anisotropic transversely isotropic elasticity system with residual stress with special conditions. Isakov, [8], proved Carleman estimates with a second large parameter for partial differential operators using a technique involving functions with compact support and principal symbols, while Bellassoued and Yamamoto, [2], have proved Carleman estimates with two large parameters for thermoelasticity system using integration by parts with functions without compact support. Carleman estimate for the thermoelastic system with memory effect which is a special type with the hyperbolic memory kernel under additional conditions have been obtained in [16].

We here derive global uniqueness of the continuation results in \(\Omega_0 \subset \Omega\) under \(K\)-pseudo-convexity conditions on a general weight function \(\psi\) defining \(\Omega_0\) with
ψ

A function

Definition 1.1.

in a bounded domain Ω of the space \( \mathbb{R}^n \) respectively. And \( \| \cdot \|_\sigma \) and let \( | \cdot | \)
with \( D \) and \( \nabla \) dependence is indicated. We recall that \( K \)
in their corresponding (see below) norms of a differential operator, on the constant \( \varepsilon \)
components, \( H = \| \cdot \|_\Omega \) is the norm of the Sobolev space \( H_{\Omega} \), \( x = (x, t) \in \Omega \subset \mathbb{R}^n \), and \( b_j, b \in L^\infty(\Omega) \). The principal symbol of this
operator is

\[
A(x; \xi) = - \sum_{j,k=0}^n b^{jk}(x) \xi_j \xi_k.
\]

(5)

We use the following convention and standard notations. We understand \( x = (x_1, x_2, \ldots, x_n) \) and \( t \) respectively as the spatial and the time variables. With \( \partial_j = \frac{\partial}{\partial x_j} \) and \( \partial_t = \frac{\partial}{\partial t} \), \( \nabla = (\partial_1, \ldots, \partial_n) \) for spatial variable and \( \nabla_{x,t} = (\partial_0, \partial_1, \ldots, \partial_n) \) for both time and spatial variables. Let \( \partial = (\partial_0, \partial_1, \ldots, \partial_n) \), with \( D = -i\partial \) and let \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \) be a multi-index with integer components, \( \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \), \( |\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n \), denote \( |\alpha| \) as the order of differential operator. The operators \( D^\alpha \) and \( \partial^\alpha \) are defined similarly. The vector \( \nu \)
is the outward normal to the boundary of a domain. We use generic constants \( C \)
(different at different places) depending only on the upper bound, \( M \), of coefficients in their corresponding (see below) norms of a differential operator, on the constant \( K \), on the function \( \psi \), on the value \( \varepsilon_0 \), and on the domain \( \Omega \). Any additional
dependence is indicated. We recall that

\[
\| u \|_{(k)}(\Omega) = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha u|^2 \right)^{\frac{1}{2}}
\]

is the norm of the Sobolev space \( H^k(\Omega) \), and we use the norms \( \| \cdot \|_\infty(\Omega) \) and \( | \cdot |_{k}(\Omega) \) in the space \( L^\infty(\Omega) \) and \( C^k(\Omega) \), the space of \( k \) times continuously differential functions, respectively. And \( \| \cdot \|_{(0)} \) is the \( L^2 \)-norm. Define the weight function

\[
\varphi = e^{\gamma \psi}
\]

and let \( \sigma = \gamma \varphi, \Omega_\delta = \Omega \cap \{ \psi(x) > \delta \}. \)

Now we define \( K \)-pseudo-convex \( \psi, [13] \).

**Definition 1.1.** A function \( \psi \) is called \( K \)-pseudo-convex on \( \Omega \) with respect to \( A \) if \( \psi \in C^2(\Omega) \), \( A(x, \nabla \psi(x)) \neq 0, x \in \Omega \), and

\[
\sum_{j,k=0}^n \partial_j \partial_k \psi(x) \frac{\partial A}{\partial \xi_j} \frac{\partial A}{\partial \xi_k}(x; \xi) + \sum_{j,k=0}^n \left( \frac{\partial^2 A}{\partial \xi_j \partial \xi_k} - \frac{\partial A}{\partial \xi_j} \frac{\partial A}{\partial \xi_k} \right) \partial_j \partial_k \psi(x)(x; \xi) \geq K |\xi|^2
\]

for some positive constant \( K \) and for any \( \xi \in \mathbb{R}^{n+1} \) and any point \( x \) of \( \bar{\Omega} \) provided

\[
A(x; \xi) = 0, \quad \sum_{j=0}^n \frac{\partial A}{\partial \xi_j}(x; \xi) \partial_j \psi(x) = 0.
\]
Here the constant $K$ in pseudo-convexity (7) depends only on an operator $A$. Hence the constant $C$ in the stability based on Carleman estimates depends only on some constant $K$ in the condition of pseudo-convexity (7).

It is not easy to find functions $\psi$ which are pseudo-convex with respect to a general anisotropic operator, in particular, to the hyperbolic operator $A = \partial_t^2 - \sum_{j,k=1}^n b^{jk} \partial_j \partial_k$. In the isotropic case explicit and verifiable conditions for $\psi(x,t) = |x - \beta|^2 - s^2 t^2$ were found by Isakov in 1980 and their simplifications are given in [8] (Section 3.4). In the general anisotropic case the known conditions of pseudo-convexity are not so easy to verify. It imposes some restricted conditions on second partial derivatives of $b^{jk}$. Moreover a useful concept of pseudo-convexity is not available for systems, hence Carleman estimates had been obtained only in some particular cases. The operator of linear thermoelasticity can be extended to principally triangular hyperbolic operators and parabolic operator. It allows us to obtain a Carleman estimate for thermoelasticity system by using Carleman estimates for second order scalar partial differential operators with two large parameters.

In Theorem 1.2 we assume, in addition, that the coefficients of a general operator $A$ admit the following bound
\[
|b^{jk}|_2(\Omega) + \|b^j\|_\infty(\Omega) + \|b\|_\infty(\Omega) \leq M.
\]
This assumption is needed to guarantee that constants $C$ as used in the theorem do not depend on a particular $A$. It is not needed for the definition of $K$-pseudo-convexity in Definition 1.1 where it suffices that $b^{jk} \in C^1$.

**Theorem 1.2.** Let $\psi \in C^2(\bar{\Omega})$ be $K$-pseudo-convex with respect to $A$ in $\bar{\Omega}$. Then there are constants $C, C_0(\gamma)$ such that
\[
\int_{\Omega} \sigma^{3-2|\alpha|} e^{2\tau \varphi} |\partial^\alpha u|^2 \leq C \int_{\Omega} e^{2\tau \varphi} |Au|^2
\]
for all $u \in C^0_0(\Omega), |\alpha| \leq 1, C < \gamma, \text{ and } C_0(\gamma) < \tau$.

The proof of Theorem 1.2 is given in [12].

Now we state a weak form of Theorem 1.2, where we assume, in addition, that the coefficients of $A$ admit the bound
\[
|b^{jk}|_2(\Omega) + \|b^j\|_\infty(\Omega) + \|\partial_j b^j\|_\infty(\Omega) + \|b\|_\infty(\Omega) \leq M.
\]

In [13] the following Theorem and its proof are shown.

**Theorem 1.3.** Let $A$ be a linear partial differential operator of second order with the principal coefficients in $C^2(\bar{\Omega})$ and with the coefficients of the first order derivatives in $C^1(\bar{\Omega})$. Let $\psi$ be a $K$-pseudo-convex $C^3(\bar{\Omega})$-function with respect to $A$ in $\bar{\Omega}$. Let $Av = f_0 + \sum_{j=1}^n \partial_j f_j$ in $\Omega$. Then there are constants $C, C_0(\gamma)$ such that
\[
\int_{\Omega} \sigma^{2\tau \varphi} v^2 \leq C \int_{\Omega} e^{2\tau \varphi} \left( \frac{1}{\sigma^2} f_0^2 + \sum_{j=1}^n f_j^2 \right) \text{ for all } v \in H^2_0(\Omega)
\]
provided $C < \gamma, C_0(\gamma) < \tau$.

The following Theorem introduces a Carleman estimate for the second order parabolic operator with a second large parameter $\gamma$ and a general function $\psi$. Now consider the second order parabolic partial differential operator
\[
P = \partial_t - \sum_{j,k=1}^n a^{jk} \partial_j \partial_k + \sum_{j=1}^n a^j \partial_j + a
\]
in a bounded domain $\Omega$ of the space $\mathbb{R}^{n+1}$ with the real-valued coefficients $a^{jk}(x) \in C^1(\Omega)$ and $a^i, a \in L^\infty(\Omega)$. Moreover we assume that the coefficients of $P$ admit the bound
\[ |a^{jk}|_2(\Omega) + |a|^\infty(\Omega) + |a|^\infty(\Omega) \leq M. \]
The dual variables of $x = (x, t)$ are denoted with $\zeta = (\zeta', \zeta_0)$ and $\xi = (\xi', \xi_0)$. The principal symbol of this operator is
\[ P(x; \zeta) = i\zeta_0 + \sum_{j,k=1}^n a^{jk}(x)\zeta_j\zeta_k. \quad (11) \]

**Theorem 1.4.** Let $0 \leq \psi \in C^2(\Omega)$, $|\nabla \psi(x)| \neq 0$. Then for a parabolic operator $P$ there are constants $C(\varepsilon_0), C_0(\varepsilon_0, \gamma)$ such that
\[ \gamma \left( \int_\Omega \sigma^{-1}e^{2\tau\varphi}|\partial_x u|^2 + \int_\Omega \sigma^{3-2\alpha}e^{2\tau\varphi}|\partial^3 u|^2 \right) \leq C \int_\Omega e^{2\tau\varphi}|Pu|^2 \quad (12) \]
for all $u \in C_0^2(\Omega)$, $|\alpha| \leq 2$ with $\alpha_0 = 0$, provided $C < \gamma$ and $C_0 < \tau$.

The Carleman estimate is an $L^2$-weighted estimate with large parameter $\tau$ was first introduced by Carleman in 1939 to prove the uniqueness of the continuation for elliptic systems with nonanalytic coefficients on the plane. During several decades there have been many achievements of the results from Carleman estimates for the second order partial differential equations including elliptic, parabolic, hyperbolic, and Schrödinger equations, and also for isotropic systems of partial differential equations. It was crucial to use Carleman type estimates with two large parameters $\tau$ and $\gamma$ to prove the uniqueness of continuation for some anisotropic systems. This idea was first introduced by Isakov in [6] to the classical elasticity system.

The following Theorem 1.5, Carleman estimates for the linear thermoelasticity systems with residual stress, is useful tool for stability estimates of the lateral Cauchy problem and for solving inverse problems.

We assume that
\[ |\rho^{-1}|_2(\Omega) + |\lambda|_2(\Omega) + |\mu|_2(\Omega) + |r_{jk}|_2(\Omega) + |a_{jk}|_2(\Omega) + \|a\|_\infty(\Omega) + \|a\|_\infty(\Omega) \leq M. \]

We now obtain the Carleman estimate based on a global $K$-pseudo-convexity condition of $\psi$ and $|\nabla \psi| \neq 0$.

**Theorem 1.5.** Let $0 \leq \psi \in C^2(\Omega)$, $|\nabla \psi(x)| > \varepsilon_0 > 0$, be $K$-pseudo-convex with respect to $\Box(\mu; R)$ and $\Box(\lambda + 2\mu; R)$ in $\Omega$. Then there are constants $C, C_0(\gamma)$ such that
\[ \int_\Omega \left( \sigma(|\nabla_x u|^2 + |\nabla_x, \text{div} u|^2 + |\nabla_x, \text{curl} u|^2 + |\nabla \theta|^2) \right. 
\[ + \sigma^3(|\text{div} u|^2 + |\text{curl} u|^2 + |\theta|^2)) e^{2\gamma\varphi} \leq C \int_\Omega \left( |A_T(u; \theta)|^2 + |A_{T\theta}(u; \theta)|^2 + \gamma^{-\frac{1}{2}}|P_T(u; \theta)|^2 \right) e^{2\gamma\varphi} \quad (13) \]
for all $u \in H_0^2(\Omega), \theta \in H_0^2(\Omega), C < \gamma, C_0(\gamma) < \tau$.

Using Theorem 1.3 one can obtain a weak form of Carleman estimate for thermoelasticity system, that is, it is the simple version of Theorem 1.5 without $\nabla A_T(u; \theta)$ on the right side and some additional derivatives on the left side of (13). Then we have better estimates of Hölder stability with reduced regularities in data.

Let us consider the following Cauchy problem:
\[ A_T(u; \theta) = f_1, \quad P_T(u; \theta) = f_2 \quad \text{in} \quad \Omega = G \times (-T, T), \quad (14) \]
\[ u = g_0, \quad \partial_x u = g_1, \quad \theta = h_0, \quad \partial_x \theta = h_1 \] on \( \Gamma \subset \partial G \times (-T, T) \),

where \( \Gamma \in C^3 \). By standard arguments in [8] (Chapter 3), Carleman estimate (13) implies the following conditional Hölder stability estimate for (14) in \( \Omega_\delta = \Omega \cap \{ \psi(x) > \delta \} \), and hence uniqueness in \( \Omega_0 \).

**Theorem 1.6.** Suppose that all coefficients \( \lambda, \mu, \rho, R, a_j, a_j, a \) in (14) are in \( C^2(\bar{\Omega}) \).

Let \( \psi \in C^2(\Omega) \), \( |\nabla \psi(x)| > \varepsilon_0 > 0 \), be \( K \)-pseudo-convex with respect to \( \Box(\mu; R) \) and \( \Box(\lambda + 2\mu; R) \) in \( \bar{\Omega} \). Assume that \( \Omega_0 \subset \Omega \cup \Gamma \).

Then there exist \( C(\delta), \kappa(\delta) \in (0, 1) \) such that for a solution \( (u; \theta) \in (H^3(\Omega); H^2(\Omega)) \) to the Cauchy problem (14) one has

\[
\|u\|^2_{L^2(\Omega_\delta)} + \|\theta\|^2_{L^2(\Omega_\delta)} \leq C(F^2 + M_1^{1-\kappa} F^2) \tag{15}
\]

where \( F = \|f_1\|_{L^2(\Omega_0)} + \|f_2\|_{L^2(\Omega_0)} + \|g_0\|_{L^2(\Gamma)} + \|g_1\|_{L^2(\Gamma)} + \|h_0\|_{L^2(\Gamma)} + \|h_1\|_{L^2(\Gamma)} \), \( M_1 = \|u\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} \).

### 2. Proof of Carleman estimate for the second order parabolic operator with a second large parameter.

In this section we prove Theorem 1.4 with a general function \( \psi \). The proof is based on some results on differential forms. In the following \( \zeta = \xi + i\tau \nabla x, \iota \phi(x) \) with

\[
P^{(k)}(x; \zeta) = \frac{\partial P(x; \zeta)}{\partial \zeta_k}
\]

and

\[
P_{(k)}(x; \zeta) = \frac{\partial P(x; \zeta)}{\partial x_k}.
\]

We will introduce the differential quadratic form

\[
\mathcal{F}(x, \tau, D, \bar{D})v \bar{\nu} = |P(x, D + i\tau \nabla x, \iota \phi(x))v|^2 - |P(x, D - i\tau \nabla x, \iota \phi(x))v|^2. \tag{16}
\]

This differential quadratic form is of order (3, 2) in \( (\tau, D) \) since the coefficients of the principal part of \( P \) are real valued. By Lemma 8.2.2 in Hörmander’s book [5] there exists differential quadratic form \( \mathcal{G}(x, \tau, D, \bar{D}) \) of order (2, 1) in \( (\tau, D) \), such that

\[
\int_\Omega \mathcal{G}(x, D, \bar{D}) v \bar{\nu} = \int_\Omega \mathcal{F}(x, D, \bar{D}) v \bar{\nu} \tag{17}
\]

and its symbol

\[
\mathcal{G}(x, \tau, \xi, \bar{\xi}) = \frac{1}{2} \sum_{k=0}^n \frac{\partial^2}{\partial x_k \partial \zeta_k} \mathcal{F}(x, \tau, \zeta, \bar{\zeta}), \quad \zeta = \xi + i\eta, \quad \text{at } \eta = 0
\]

where

\[
\mathcal{F}(x, \tau, \zeta, \bar{\zeta}) = P(x, \zeta + i\tau \nabla x, \iota \phi(x))P(x, \bar{\zeta} - i\tau \nabla x, \iota \phi(x)) - P(x, \zeta - i\tau \nabla x, \iota \phi(x))P(x, \bar{\zeta} + i\tau \nabla x, \iota \phi(x)).
\]

**Lemma 2.1.** We have

\[
\mathcal{G}(x, \tau, \xi, \bar{\xi}) = 2\tau \sum_{j,k=0}^n \frac{\partial P}{\partial \zeta_j} \frac{\partial P}{\partial \bar{\zeta}_k} \partial_j \partial_k \phi + 23 \sum_{k=0}^n \partial_k P \frac{\partial P}{\partial \zeta_k} + 23 \sum_{k=0}^n P \left( \frac{\partial^2 P}{\partial \zeta_k \partial x_k} - i\tau \sum_{j=0}^n \frac{\partial^2 P}{\partial \zeta_j \partial \zeta_k} \partial_j \partial_k \phi \right) \tag{18}
\]

where \( P, \partial_x P, \ldots \) are taken at \( (x, \zeta(\phi(x))) \).
Proof is given in [12].

Using Lemma 2.1 we now calculate the symbol of the differential quadratic form \( \mathcal{G} \) on the characteristic set \( \{ (x; \zeta) | P(x; \zeta) = 0 \} \). With \( \varphi(x) = e^{\psi(x)} \) the standard calculation gives
\[
\partial_j \varphi = \gamma \varphi \partial_j \psi, \quad \partial_j \partial_k \varphi = \gamma \varphi \partial_j \partial_k \psi + \gamma^2 \varphi \partial_j \psi \partial_k \psi, \quad j, k = 0, \ldots, n. \tag{19}
\]

Here we let \( \sigma = \gamma \tau \varphi \). \( \zeta = (\zeta', \zeta_0) \) and \( \xi = (\xi', \xi_0) \) denote the dual variables of \( x = (x, t) \). The characteristic equation
\[
P(x; \zeta) = i\zeta_0 + \sum_{j,k=1}^n a^{jk}(x) \xi_j \xi_k = 0
\]
gives two identities
\[
\sum_{j,k=1}^n a^{jk}(x) \xi_j \xi_k = \sigma \partial_0 \psi(x) + \sigma^2 \sum_{j,k=1}^n a^{jk}(x) \partial_j \psi(x) \partial_k \psi(x)
\]
and
\[
\xi_0 = -2\sigma \sum_{j,k=1}^n a^{jk}(x) \xi_j \partial_k \psi(x).
\]

These two identities with ellipticity condition for \( a^{jk} \) give two bounds
\[
|\xi'|^2 \leq C\sigma^2, \quad |\xi_0| \leq C\sigma^2. \tag{20}
\]

From (18) on the characteristic set we calculate
\[
\mathcal{G}(x, \tau, \xi, \xi)
\]
\[
= 2\sigma (\partial_0 \partial_0 \psi + 4\sigma \sum_{j,m=1}^n a^{jm} \partial_m \psi \partial_0 \partial_j \psi + 4\sigma^2 \sum_{j,k,l,m=1}^n a^{jm} a^{kl} \partial_m \psi \partial_l \psi \partial_j \partial_k \psi)
\]
\[
+ 2\gamma (\sigma (\partial_0 \psi)^2 + 4\sigma^2 \sum_{j,m=1}^n a^{jm} \partial_0 \psi \partial_j \psi \partial_m \psi)
\]
\[
+ 8\gamma \sigma (\sum_{j,m=1}^n a^{jm} \xi_m \partial_j \psi)^2 + 8\gamma \sigma^3 (\sum_{j,m=1}^n a^{jm} \partial_j \psi \partial_m \psi)^2 \tag{21}
\]
\[
- 2 \sum_{j,m=1}^n \partial_0 a^{jm} \xi_j \xi_m + 2\sigma^2 \sum_{j,k,m=1}^n \partial_0 a^{jk} \partial_j \psi \partial_k \psi + 4\sigma^3 \sum_{j,k,l,m=1}^n a^{kl} a^{jm} \partial_k \partial_l \psi \partial_j \partial_m \psi
\]
\[
+ 2\sigma (4 \sum_{j,k,l,m=1}^n a^{jm} a^{kl} \xi_m \xi_l \partial_j \partial_k \psi + 2 \sum_{j,k,l,m=1}^n a^{kl} \partial_k a^{jm} (2\xi_l \xi_m \partial_j \psi - \xi_j \xi_m \partial_k \psi))
\]
\[
\geq -C\sigma^3 - C\gamma \sigma + C^{-1} \gamma \sigma^3 - C\sigma^3 + C\sigma
\]
since \( |\nabla \psi(x)| \neq 0 \), in Theorem 1.4. Hence
\[
\mathcal{G}(x, \tau, \xi, \xi) \geq C^{-1} \gamma \sigma^3.
\]

This implies, by using the bounds (20),
\[
\tau \varphi \mathcal{G}(x, \tau, \xi, \xi) \geq C^{-1} \sigma^4 \geq C^{-1} (|\xi'|^4 + |\xi_0|^2)
\]
on the characteristic set \( P(x; \zeta) = 0 \). By the homogeneity of the real part of the symbol \( P(x; \zeta) \) and by the continuity of the differential quadratic form \( \mathcal{G} \) we obtain
\[
C^{-1} (|\xi'|^4 + |\xi_0|^2) \leq \tau \varphi \mathcal{G}(x, \tau, \xi, \xi) + C |P(x; \zeta)|^2. \tag{22}
\]
Now we fix $x$ at $x_0 = (x_0, t_0)$ in (22). By using Parseval’s identity,

$$(\tau^2|\nabla_{x,t} \varphi(x_0)|^2)^{m-|\alpha|} \int |\partial^{\alpha} v|^2 dx \leq (2\pi)^{-n} \int |\xi|^2^{2m}(\varphi(x_0)|\hat{v}(\xi)|^2 d\xi),$$

multiplying the inequality (22) by $|\hat{v}(\xi)|^2$, $v \in C_0^2(\Omega_\varepsilon)$, and integrating over the whole space in the $\xi'$ and $\xi_0$-variables we yield

$$C^{-1} \left( \int |\partial^\alpha v|^2 + \sum_{|\alpha| \leq 2, \alpha_0 = 0} \int (\gamma \tau \varphi(x_0))^{4-2|\alpha|}|\partial^{\alpha} v|^2 \right)$$

$$\leq \tau \varphi(x_0) \int G(x_0, \tau, D, D) v \hat{v} + C \int |P(x_0, \xi + i\tau \nabla_{x,t} \varphi(x_0))|^2 |\hat{v}(\xi)|^2 d\xi.$$  (23)

The first term on the right side of (23) has the bound based on (21) such that

$$\mathcal{G}(x_0, \tau, D, D) v \hat{v} \leq \mathcal{G}(x, \tau, D, D) v \hat{v} + \varepsilon_1(\delta; \gamma) \sum_{|\alpha| \leq 1, \alpha_0 = 0} \tau^{3-2|\alpha|}|\partial^{\alpha} v|^2$$  (24)

where a function $\varepsilon_1(\delta; \gamma) \rightarrow 0$ as $\delta \rightarrow 0$ and $\gamma$ is fixed and for all $v \in C_0^2(\Omega_\varepsilon \cap B(x_0; \delta))$. Moreover, since $|P(x, D + i\tau \nabla_{x,t} \varphi(x))|^2$ is a differential quadratic form of the order $(4, 2)$ in $(D, \tau)$, the second term on the right side of (23) also has the bound

$$\int |P(x_0, D + i\tau \nabla_{x,t} \varphi(x_0))|^2 v^2 \leq \int |P(x, D + i\tau \nabla_{x,t} \varphi(x))|^2 v^2$$

$$+ \varepsilon_2(\delta; \gamma) \sum_{|\alpha| \leq 2, \alpha_0 = 0} \int \tau^{4-2|\alpha|}|\partial^{\alpha} v|^2$$  (25)

for all $v \in C_0^2(\Omega_\varepsilon \cap B(x_0; \delta))$. Using (24) and (25) the bound (23) yields

$$C^{-1} \left( \int |\partial^\alpha v|^2 + \sum_{|\alpha| \leq 2, \alpha_0 = 0} \int (\gamma \tau \varphi(x_0))^{4-2|\alpha|}|\partial^{\alpha} v|^2 \right)$$

$$\leq \tau \varphi(x_0) \int G(x, \tau, D, D) v \hat{v} + \varepsilon_1(\delta; \gamma) \sum_{|\alpha| \leq 1, \alpha_0 = 0} \int (\tau \varphi(x_0))^{4-2|\alpha|}|\partial^{\alpha} v|^2$$

$$+ C \int |P(x, D + i\tau \nabla_{x,t} \varphi(x))|^2 v^2 + C \varepsilon_2(\delta; \gamma) \sum_{|\alpha| \leq 2, \alpha_0 = 0} \int (\tau \varphi(x_0))^{4-2|\alpha|}|\partial^{\alpha} v|^2$$  (26)

for $v \in C_0^2(\Omega_\varepsilon \cap B(x_0; \delta))$. Choosing $\delta > 0$ small and $\tau$ large enough so that

$$\varepsilon_1(\delta; \gamma) + C \varepsilon_2(\delta; \gamma) < (2C)^{-1}$$

we absorb the second and fourth term on the right side of the inequality (26) to arrive at the inequality

$$C^{-1} \left( \int |\partial^\alpha v|^2 + \sum_{|\alpha| \leq 2, \alpha_0 = 0} \int (\gamma \tau \varphi(x_0))^{4-2|\alpha|}|\partial^{\alpha} v|^2 \right)$$

$$\leq \tau \varphi(x_0) \int G(x, \tau, D, D) v \hat{v} + C \int |P(x, D + i\tau \nabla_{x,t} \varphi(x))|^2 v^2.$$
As above dividing by \( \tau \varphi(x_0) \) and then choosing large \( \tau > C(\gamma) \) one can replace \( \varphi(x_0) \) on the left side of this inequality by \( \varphi(x) \). Using (16) and (17) we conclude that

\[
\gamma \left( \int \sigma^{-1} |\partial_\alpha v|^2 + \sum_{|\alpha|\leq 2, \alpha_0=0} \int (\gamma \tau \varphi(x))^{3-2|\alpha|} |\partial_\alpha v|^2 \right) \leq C \int |P(x, D+i\tau \nabla_{x,t} \varphi(x))v|^2
\]

for \( v \in C^0_0(\Omega_\varepsilon \cap B(x_0; \delta)) \). To complete the proof of Theorem 1.4 we use the partition of the unity argument in the estimate (27). Since our choice of \( \delta_0 \) depends on \( \gamma \) we give this argument in some detail.

The balls \( B(x_0; \delta_0) \) form an open covering of the compact set \( \bar{\Omega} \). Hence we can find a finite subcovering \( B(x_{0j}; \delta_0) \) and a special partition of the unity \( \chi_j(\cdot; \gamma) \) subordinated to this subcovering. In particular, \( \chi_j \in C^0_0(B(x_{0j}; \delta_0)) \), \( 0 \leq \chi_j \leq 1 \), and \( \sum_{j=0}^n \chi_j^2 = 1 \) on \( \bar{\Omega} \). By Leibniz’ formula

\[
\partial_\alpha (\chi_j v) = \chi_j \partial_\alpha v + (\partial_\alpha \chi_j)v,
\]

\[
P(\cdot, D + i\tau \nabla_{x,t} \varphi)(\chi_j v) = \chi_j P(\cdot, D + i\tau \nabla_{x,t} \varphi) v + \sum_{|\beta|\leq 1} a^\beta \tau^{-1-|\beta|} \partial_\beta v
\]

with \( |a^\beta| \leq C(\gamma) \). Hence applying the Carleman estimate (27) to \( \chi_j v \) and using the elementary inequality \( |a + b|^2 \geq \frac{1}{2} a^2 - b^2 \) we obtain

\[
\frac{1}{2} \gamma \left( \int \sigma^{-1} |\chi_j \partial_t v|^2 + \sum_{\alpha_0=0} \int \sigma^{3-2|\alpha|} |\chi_j \partial_\alpha v|^2 \right) \leq C \int |\chi_j P(x, D + i\tau \nabla_{x,t} \varphi(x))v|^2 \]

Summing up over \( j = 1, \ldots, J \) and using that \( \sum_{j=0}^n \chi_j^2 = 1 \) we yield

\[
\frac{1}{2} \gamma \left( \int \sigma^{-1} |\partial_t v|^2 + \sum_{\alpha_0=0} \int \sigma^{3-2|\alpha|} |\partial_\alpha v|^2 \right) \leq C \int |P(x, D + i\tau \nabla_{x,t} \varphi(x))v|^2 + C(\gamma) \sum_{|\beta|\leq 1} \int \tau^{2-2|\beta|} |\partial_\beta v|^2.
\]

Since the highest powers of \( \tau \) are in the first term of the left side, choosing \( C(\gamma) < \tau \) we absorb both the second terms on the left and the right into the first term on the left.

Now by setting \( u = e^{-\tau \varphi} v \) we have \( P(x, D + i\tau \nabla_{x,t} \varphi(x))v = e^{\tau \varphi} P(x, D)u \). Hence the estimate (27) yields

\[
\gamma \left( \int \sigma^{-1} e^{2\tau \varphi} |\partial_t u|^2 + \sum_{\alpha_0=0} \int \sigma^{3-2|\alpha|} e^{2\tau \varphi} |\partial_\alpha u|^2 \right) \leq C \int e^{2\tau \varphi} |P(x, D)u|^2.
\]

The proof is complete.
Lemma 3.1. Let $|\nabla \psi| > 0$ on $\Omega$. Then, for a second order elliptic operator $A$, there are constants $C, C_0(\gamma)$ such that
\[
\gamma \int_{\Omega} \sigma^{4-2|\alpha|} e^{2\gamma \varphi} |\partial^\alpha v|^2 \leq C \int_{\Omega} \sigma e^{2\gamma \varphi} |Av|^2
\]
for all $v \in C_0^2(\Omega), |\alpha| \leq 2, C < \gamma$, and $C_0(\gamma) < \tau$.

Lemma 3.2. Under the conditions of Theorem 1.4, for a second order parabolic operator $P$ there are constants $C(\varepsilon_0), C_0(\varepsilon_0, \gamma)$ such that
\[
\gamma^{\frac{1}{2}} \left( \int_{\Omega} e^{2\gamma \varphi} |\partial_t v|^2 + \int_{\Omega} \sigma^{4-2|\alpha|} e^{2\gamma \varphi} |\partial^\alpha v|^2 \right) \leq C \int_{\Omega} \gamma^{-\frac{1}{2}} \sigma e^{2\gamma \varphi} |Pv|^2
\]
for all $v \in C_0^2(\Omega), |\alpha| \leq 2$ with $\alpha_0 = 0, C < \gamma$, and $C_0 < \tau$.

Proof. We apply Carleman estimate with a second large parameter $\gamma$ for a parabolic operator $P$ in Theorem 1.4,
\[
\int_{\Omega} \left( \frac{1}{\tau \varphi} (|\partial_t u|^2 + \sum_{j,k=1}^n |\partial_j \partial_k u|^2) + \gamma \sigma |\nabla u|^2 + \gamma \sigma^3 |u|^2 \right) e^{2\gamma \varphi} \leq C \int_{\Omega} |Pu|^2 e^{2\gamma \varphi},
\]
with $u = \gamma^{-\frac{1}{2}} \sigma \frac{1}{2} v$. By the Leibniz formula
\[
\partial^\alpha \left( \gamma^{-\frac{1}{2}} \sigma \frac{1}{2} v \right) = \gamma^{-\frac{1}{2}} \sigma \frac{1}{2} \partial^\alpha v + \sigma \frac{1}{2} A_{|\alpha|, -1}(x, D)v, |\alpha| = 1, 2
\]
and
\[
P(x, D)(\gamma^{-\frac{1}{2}} \sigma \frac{1}{2} v) = \gamma^{-\frac{1}{2}} \sigma \frac{1}{2} P(x, D)v + \sigma \frac{1}{2} A_1(x, D)v
\]
where $A_m$ is a linear partial differential operator of order $m$ with coefficients bounded by $C(\gamma)$. By using these relations and the triangle inequality from (30) we get
\[
\gamma^{\frac{1}{2}} \int_{\Omega} \left( |\partial_t v|^2 + \sum_{j,k=1}^n |\partial_j \partial_k v|^2 + \sigma^2 |\nabla v|^2 + \sigma^4 |v|^2 \right) e^{2\gamma \varphi} - C(\gamma) \int_{\Omega} (|\nabla v|^2 + \sigma^2 |v|^2) e^{2\gamma \varphi}
\]
\[
\leq C \int_{\Omega} \gamma^{\frac{1}{2}} \sigma e^{2\gamma \varphi} |Pv|^2 + C(\gamma) \int_{\Omega} (\sigma |\nabla v|^2 + \sigma |v|^2) e^{2\gamma \varphi}.
\]
Since $\sigma = \gamma \tau \varphi$, by choosing a large $\tau > C(\gamma)$ the second integrals on both sides are absorbed by the first integral on the left side. We proved Lemma 3.2.

By using two auxiliary functions $v = div u$ and $w = curl u$, the operator (1)-(2) implies an extended principally triangular matrix differential operator of first seven scalar equations with wave operators in the diagonal ([10], Section 2) and last one scalar equation with parabolic operator. This extended operator becomes the coupled hyperbolic-parabolic operator:
\[
\Box(\mu; R)u = \frac{A_T(u; \theta)}{\rho} + A_{1,1}(u, v, \theta),
\]
\[
\Box(\lambda + 2\mu; R)v = div \frac{A_T(u; \theta)}{\rho} + \sum_{j,k=1}^3 \nabla \left( \frac{T_{jk}}{\rho} \right) \cdot \partial_j \partial_k u + A_{2,1}(u, v, w, \theta) + A_{2,2}(u, \theta),
\]
$\Box(\mu; R)w = \text{curl} \frac{A_T(u; \theta)}{\rho} + \sum_{j,k=1}^{3} \nabla \left( \frac{r_{jk}}{\rho} \right) \times \partial_j \partial_k u + A_{3;1}(u, v, w, \theta) + A_{3;2}(u, \theta),$  

(31)

$\partial_\gamma \theta - \sum_{j,k=1}^{3} a_{jk} \partial_j \partial_k \theta = P_T(u; \theta) - \sum_{j=1}^{3} a_j \partial_j \theta - a \theta + A_{4;1}(u, v, w)$

where $A_{i;1}(A_{i;1}), A_{i;2}(A_{i;2}), i = 1, 2, 3$ are first order, second order matrix (scalar) linear partial differential operators which do not involve $\partial_\theta \theta$ and $\partial_\theta^2 \theta$, and $A_{4;1}$ is a first order scalar linear partial differential operator, with the bounded coefficients with $C^1(\Omega)$-norms bounded by $C$.

Applying Theorem 1.2 to each of first seven scalar differential operators forming the extended operator (31) and summing up seven Carleman estimates, we get

$$\int_\Omega (\sigma(|\nabla_{x,t}u|^2 + |\nabla_{x,t}v|^2 + |\nabla_{x,t}w|^2) + \sigma^3(|u|^2 + |v|^2 + |w|^2))e^{2\tau \phi}$$

$$\leq C \int_\Omega (|A_T(u; \theta)|^2 + |\nabla A_T(u; \theta)|^2)e^{2\tau \phi} + C \int_\Omega \sum_{j,k=1}^{3} |\partial_j \partial_k u|^2 e^{2\tau \phi}$$

$$+ C \int_\Omega (|\nabla_{x,t}u|^2 + |\nabla_{x,t}v|^2 + |\nabla_{x,t}w|^2 + |u|^2 + |v|^2 + |w|^2)e^{2\tau \phi}$$

$$+ C \int_\Omega \left( \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + |\nabla \theta|^2 + |\theta|^2 \right)e^{2\tau \phi}. \tag{32}$$

By choosing $\tau > 2C$, we absorb the third integral in the right side by the left side, arriving at the inequality

$$\int_\Omega (\sigma(|\nabla_{x,t}u|^2 + |\nabla_{x,t}v|^2 + |\nabla_{x,t}w|^2) + \sigma^3(|u|^2 + |v|^2 + |w|^2))e^{2\tau \phi}$$

$$\leq C \int_\Omega (|A_T(u; \theta)|^2 + |\nabla A_T(u; \theta)|^2)e^{2\tau \phi} + C \int_\Omega \sum_{j,k=1}^{3} |\partial_j \partial_k u|^2 e^{2\tau \phi}$$

$$+ C \int_\Omega \left( \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + |\nabla \theta|^2 + |\theta|^2 \right)e^{2\tau \phi}. \tag{32}$$

To eliminate the second integral in the right side, we need a second large parameter $\gamma$. By Lemma 3.1

$$\gamma \int_\Omega \sum_{j,k=1}^{3} |\partial_j \partial_k u|^2 e^{2\tau \phi} \leq C \int_\Omega |\Delta u|^2 e^{2\tau \phi} \leq C \int_\Omega \sigma(|\nabla u|^2 + |\nabla w|^2)e^{2\tau \phi}$$

$$\leq C \int_\Omega (|A_T(u; \theta)|^2 + |\nabla A_T(u; \theta)|^2)e^{2\tau \phi} + C \int_\Omega \sum_{j,k=1}^{3} |\partial_j \partial_k u|^2 e^{2\tau \phi}$$

$$+ C \int_\Omega \left( \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + |\nabla \theta|^2 + |\theta|^2 \right)e^{2\tau \phi}.$$
where we used the known identity $\Delta u = \nabla v - curl w$ and (32). Choosing $\gamma > 2C$, we can see that the second integral on the last inequality is absorbed by the left side. This yields

$$
\gamma \int_{\Omega} \sum_{j,k=1}^{3} |\partial_j \partial_k u|^2 e^{2\tau \phi} 
$$

$$
\leq C \int_{\Omega} (|A_T(u;\theta)|^2 + |\nabla A_T(u;\theta)|^2) e^{2\tau \phi} + C \int_{\Omega} (\sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + |\nabla \theta|^2 + |\theta|^2) e^{2\tau \phi}.
$$

Using again (32) we get

$$
\int_{\Omega} (\sigma(|\nabla_x \theta|^2 + |\nabla_x \varepsilon|^2 + |\nabla_x \gamma|^2)) + \sigma^3(|u|^2 + |v|^2) e^{2\tau \phi}
$$

$$
\leq C \int_{\Omega} (|A_T(u;\theta)|^2 + |\nabla A_T(u;\theta)|^2) e^{2\tau \phi} + C \int_{\Omega} (\sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + |\nabla \theta|^2 + |\theta|^2) e^{2\tau \phi}.
$$

Now we apply Lemma 3.2 to the last scalar operator of the extended operator (31). Then

$$
\gamma \frac{1}{2} \int_{\Omega} (|\partial \theta|^2 + \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + \sigma^2 |\nabla \theta|^2 + \sigma^4 |\theta|^2) e^{2\tau \phi}
$$

$$
\leq C \int_{\Omega} \gamma^{-\frac{1}{2}} \sigma (|P_T(u;\theta)|^2 + |\nabla_x \theta|^2 + |\nabla_x \varepsilon|^2 + |\nabla_x \gamma|^2 + |\nabla \theta|^2)
$$

$$
+ |u|^2 + |v|^2 + |w|^2 + |\theta|^2) e^{2\tau \phi}.
$$

Adding up two estimates (33) and (34),

$$
\int_{\Omega} (\sigma(|\nabla_x \theta|^2 + |\nabla_x \varepsilon|^2 + |\nabla_x \gamma|^2 + |\nabla \theta|^2)
$$

$$
+ \sigma^3(|u|^2 + |\nabla u|^2 + |\nabla \gamma|^2 + |\theta|^2)) e^{2\tau \phi} + \gamma \frac{1}{2} \int_{\Omega} (|\partial \theta|^2 + \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2) e^{2\tau \phi}
$$

$$
\leq C \int_{\Omega} (|A_T(u;\theta)|^2 + |\nabla A_T(u;\theta)|^2 + \gamma^{-\frac{1}{2}} \sigma (|P_T(u;\theta)|^2)) e^{2\tau \phi}
$$

$$
+ C \int_{\Omega} \left( \sum_{j,k=1}^{3} |\partial_j \partial_k \theta|^2 + \gamma^{-\frac{1}{2}} \sigma (|\nabla_x \theta|^2 + |\nabla_x \varepsilon|^2 + |\nabla_x \gamma|^2 + |\nabla \theta|^2)
$$

$$
+ |u|^2 + |v|^2 + |w|^2 + |\theta|^2) e^{2\tau \phi}.
$$

With $\gamma > 4C^2$, the last integral on the right side is absorbed by the integrals of the left. We proved Theorem 1.5.
4. Hölder type stability in the Cauchy problem. In this section we prove Theorem 1.6.

Since the surface $\Gamma \subset C^3$ is noncharacteristic for the linear thermoelasticity system we can uniquely solve (14) on $\Gamma$ for $\partial_\nu^2 u$ in terms of $f_1, f_2, g_0, g_1, h_0, h_1$ and their tangential derivatives. Moreover, $\|\partial_\nu^2 u\|_{(\frac{1}{2})}(\Gamma) \leq C(\|f_1\|_{(2)}(\Gamma) + \|f_2\|_{(2)}(\Gamma) + \|g_0\|_{(2)}(\Gamma) + \|g_1\|_{(2)}(\Gamma) + \|h_0\|_{(\frac{3}{2})}(\Gamma) + \|h_1\|_{(2)}(\Gamma))$.

Then extension theorems we can find $u^* \in H^2(\Omega)$ and $\theta^* \in H^2(\Omega)$ so that

$u^* = g_0, \partial_\nu u^* = g_1, \partial_\nu^2 u^* = \partial_\nu^2 u, \theta^* = h_0, \partial_\nu \theta^* = h_1$ on $\Gamma$

and

$$\|u^*\|_{(2)}(\Omega) + \|\theta^*\|_{(1)}(\Omega) \leq CF. \quad (36)$$

Let

$$v = u - u^*, \ s = \theta - \theta^*. \quad (37)$$

Then the functions $v$ and $s$ solve the Cauchy problem

$$A_T(v; s) = f_1 - A_T(u^*; \theta^*), \ P_T(v; s) = f_2 - P_T(u^*; \theta^*) \text{ in } \Omega,$$

$v = 0, \ \partial_\nu v = 0, \ s = 0, \ \partial_\nu s = 0$ on $\Gamma$. \quad (38)

Moreover, due to our construction of $u^*$ we have

$$\partial_\nu^2 v = 0 \text{ on } \Gamma. \quad (39)$$

To apply Carleman estimates from Theorem 1.5 we need zero Cauchy data on the whole boundary. To achieve them we introduce a cut-off function $\chi \in C^\infty(\bar{\Omega})$ so that $\chi = 1$ on $\Omega_{1/2}, \ \chi = 0$ on $\Omega \setminus \Omega_0$. By Leibniz formula

$$A_T(\chi v; \chi s) = \chi A_T(v; s) + A_1 v + B_0 s,$$

$$\nabla A_T(\chi v; \chi s) = \chi \nabla A_T(v; s) + A_2 v + B_1 s,$$

$$P_T(\chi v; \chi s) = \chi P_T(v; s) + A_3 v + B_2 s$$

where $A_1 (A_2)$, $A_2$ are matrix (scalar) linear partial differential operators with bounded coefficients of orders 1, 2 depending on $\chi$. And $B_1 (B_2)$ is a first order matrix (scalar) differential operator which does not involve $\partial_\nu s$. $B_0$ is a zero differential operator, with the bounded coefficients depending on $\chi$. Moreover $A_1 v = 0, A_2 v = 0, A_3 v = 0, B_1 s = 0, B_2 s = 0, B_3 s = 0$ on $\Omega_{1/2}$. Using the Cauchy data (38), (39) we conclude that $v \in H^2_0(\Omega)$, hence by the Carleman estimate of Theorem 1.5 we have

$$\int_{\Omega} (\sigma(\|\nabla_x t(\chi v)\|^2 + |\nabla_x t\text{div}(\chi v)|^2 + |\nabla_x t\text{curl}(\chi v)|^2 + |\nabla(\chi\theta)|^2) + \sigma^3(|\chi v|^2 + |\text{div}(\chi v)|^2 + |\text{curl}(\chi v)|^2 + |\chi\theta|^2)^2) e^{2r\varphi} \leq C \int_{\Omega} (|f_1|^2 + |\nabla f_1|^2 + \gamma^{-2}\sigma|f_2|^2 + |A_T(u^*; \theta^*)|^2 + |\nabla A_T(u^*; \theta^*)|^2$$

$$+ \gamma^{-2}\sigma|P_T(u^*; \theta^*)|^2) e^{2r\varphi} + C \int_{\Omega} (|A_2 v|^2 + |A_1 v|^2 + |A_3 v|^2 + |B_1 s|^2 + |B_2 s|^2 + |B_3 s|^2) e^{2r\varphi}$$

for $C < \gamma, C_0 < \tau$. Shrinking domain of integration on the left side to $\Omega_{1/4}$ (where $\chi = 1$) and splitting integration domain of the second integral of the right hand side into $\Omega_{1/8}$ and its complement we obtain

$$\int_{\Omega_{1/4}} (\sigma(\|\nabla_x t v\|^2 + |\nabla_x t\text{div} v|^2 + |\nabla_x t\text{curl} v|^2 + |\nabla v|^2)$$

$$+ \sigma^3(|v|^2 + |\text{div} v|^2 + |\text{curl} v|^2 + |\theta|^2)^2) e^{2r\varphi} \leq C \int_{\Omega_{1/4}} (|f_1|^2 + |\nabla f_1|^2 + \gamma^{-2}\sigma|f_2|^2 + |A_T(u^*; \theta^*)|^2 + |\nabla A_T(u^*; \theta^*)|^2$$

$$+ \gamma^{-2}\sigma|P_T(u^*; \theta^*)|^2) e^{2r\varphi} + C \int_{\Omega_{1/4}} (|A_2 v|^2 + |A_1 v|^2 + |A_3 v|^2 + |B_1 s|^2 + |B_2 s|^2 + |B_3 s|^2) e^{2r\varphi}$$

for $C < \gamma, C_0 < \tau$.
have

\[ + \sigma^3(|v|^2 + |\text{div}v|^2 + |\text{curl}v|^2 + |\theta|^2) e^{2\tau \phi} \]

\[ \leq C \int_{\Omega} (|f_1|^2 + |\nabla f_1|^2 + |f_2|^2 + |A_T(u^*; \theta^*)|^2 + |\nabla A_T(u^*; \theta^*)|^2 \]

\[ + \gamma^{-\frac{1}{2}} \sigma |P_T(u^*; \theta^*)|^2 \) e^{2\tau \phi} + C \int_{\Omega \setminus \Omega_\frac{3}{4}} (|A_2 v|^2 + |A_1 v|^2 + |A_1 v|^2 + |B_1 s|^2 + |B_0 s|^2 + |B_1 s|^2) e^{2\tau \phi} \]

where we used definition (15) of \( F \) and bound (36) and let \( \Phi = \sup \varphi \) over \( \Omega \) and \( \Phi_2 = \sup \varphi \) over \( \Omega \setminus \Omega_\frac{3}{4} \). Letting \( \Phi_1 = \inf \varphi \) over \( \Omega_\frac{3}{4} \) and replacing \( \varphi \) on the left side of the preceding inequality by \( \Phi_1 \) with fixed \( \gamma \) we yield

\[ \tau \left( ||v||^2_{(2)}(\Omega_\frac{3}{4}) + ||\text{div}v||^2_{(1)}(\Omega_\frac{3}{4}) + ||\text{curl}v||^2_{(1)}(\Omega_\frac{3}{4}) + ||s||^2_{(1)}(\Omega_\frac{3}{4}) \right) e^{2\tau \Phi_1} \]

\[ \leq CF^2 e^{2\tau \Phi} + C \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right) e^{2\tau \Phi_2}. \] (40)

Observe that \( \Phi_2 < \Phi_1 \).

We remind an interior Schauder type estimate for elliptic equations with zero Dirichlet data on \( \Gamma \):

\[ ||\omega||^2_{(2)}(\Omega_\delta) \leq \left( ||\omega||^2_{(0)}(\Omega_\frac{3}{4}) + ||\Delta \omega||^2_{(0)}(\Omega_\frac{3}{4}) \right) \]

and using in addition that \( \Delta = \nabla \text{div} - \text{curl} \text{curl} \) we obtain

\[ ||v||^2_{(2)}(\Omega_\delta) \leq \left( ||v||^2_{(0)}(\Omega_\frac{3}{4}) + ||\text{div}v||^2_{(1)}(\Omega_\frac{3}{4}) + ||\text{curl}v||^2_{(1)}(\Omega_\frac{3}{4}) \right), \]

and \( ||s||^2_{(1)}(\Omega_\delta) \leq ||s||^2_{(1)}(\Omega_\frac{3}{4}) \) since \( \Omega_\delta \) is smaller than \( \Omega_\frac{3}{4} \). Hence from (40) we have

\[ ||v||^2_{(2)}(\Omega_\delta) + ||s||^2_{(1)}(\Omega_\delta) \leq CF^2 e^{2\tau(\Phi_1 - \Phi_2)} + C \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right) e^{2\tau(\Phi_2 - \Phi_1)}. \] (41)

If \( \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right) F^{-2} < C \), then \( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \leq CF^2 \). Otherwise we let \( \tau = \frac{1}{2}(\Phi + \Phi_1 - \Phi_2)^{-1} \log \left( \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right) F^{-2} \right) \). Then the bound (41) implies that

\[ ||v||^2_{(2)}(\Omega_\delta) + ||s||^2_{(1)}(\Omega_\delta) \leq C \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right)^{1-\kappa} F^{2\kappa} \]

with \( \kappa = \frac{\Phi_1 - \Phi_2}{\Phi + \Phi_1 - \Phi_2} \). Combining both cases we yield

\[ ||v||^2_{(2)}(\Omega_\delta) + ||s||^2_{(1)}(\Omega_\delta) \leq C \left( F^2 + \left( ||v||^2_{(2)}(\Omega) + ||s||^2_{(1)}(\Omega) \right)^{1-\kappa} F^{2\kappa} \right). \]

Using that \( u = v + u^* \), \( \theta = s + \theta^* \), the triangle inequality, (36) and the elementary inequality \( (a + b)^\kappa \leq a^\kappa + b^\kappa \) we obtain (15) and complete the proof of Theorem 1.6.

5. Conclusion. We obtained a strong form of Carleman estimate with two large parameters for the linear thermoelasticity system with residual without any smallness assumption. The next goal is to obtain a form of Carleman estimate for the linear thermoelasticity system with residual stress without the \( \nabla_{x,t} \) in (13) by using Theorem 1.3. Then we will have better estimates of Hölder stability with reduced regularities in data. We believe one can apply this Carleman estimate to obtain the uniqueness of continuation, coefficients identification, and the identification of residual stress. As another goal we expect Carleman estimates and stability estimates to be obtained for transversely isotropic and more general anisotropic systems.

Acknowledgments. The author would like to thank Distinguished Professor Victor Isakov for his help and comments. This research is in part supported by the NSF grant DMS 10-08902.
REFERENCES

[12] V. Isakov and N. Kim, Carleman estimates with second large parameter for second order operators, Some application of Sobolev spaces to PDEs, International Math. Ser., Springer-Verlag, 10 (2009), 135-159.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: nanhee.kim@wichita.edu