Increased stability for the Schrödinger potential from the Dirichlet-to-Neumann map

Victor Isakov

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Abstract

We derive some bounds which can be viewed as an evidence of increasing stability in the problem of recovery of the potential coefficient in the Schrödinger equation from the Dirichlet-to-Neumann map, when frequency (energy level) is growing. These bounds hold under certain a-priori bounds on the unknown coefficient. Proofs use complex- and real-valued geometrical optics solutions. We outline open problems and possible future developments.

1 Introduction

We consider the problem of recovery of the potential in the Schrödinger equation from many boundary measurements, i.e. from the Neumann data given for all Dirichlet data (the Dirichlet-to-Neumann map). Calderon [8] and Faddeev [10] proposed the idea of using complex exponential solutions to demonstrate uniqueness in the closely related linearized inverse conductivity problem and in the inverse potential scattering problem for the Schrödinger equation. Sylvester and Uhlmann in their fundamental paper [23] attracted ideas from geometrical optics, constructed almost complex exponential solutions, and proved global uniqueness of \( c \) (potential in the Schrödinger equation) in the three-dimensional case. In the two-dimensional case this problem is less overdetermined, and the methods of [23] are not applicable, but one enjoys advantages of the methods of inverse scattering and of theory of complex variables. Using these methods Nachman [19] and Astala and Päivärinta [4] showed uniqueness of the conductivity coefficient, and using the results of [19], Isakov and Nachman [16] proved uniqueness of \( c \) resulting from a transformation of the conductivity equation with a real-valued coefficient. Recently, Bukhgeim [6] introduced new ideas combining methods of theory of complex variables, complex geometrical optics, and the stationary phase to demonstrate uniqueness of the (complex-valued) potential \( c \) in the plane case.

A logarithmic stability estimate for \( c \) from the Dirichlet-to-Neumann map was obtained by Alessandrini [1]. Then Mandache [18] showed that this estimate is optimal (at zero energy). The logarithmic stability is quite discouraging for applications,
since small errors in the data of the inverse problem result in large errors in numerical reconstruction of physical properties of the medium. In particular, it severely restricts resolution in the electrical impedance tomography.

In this paper we show that the stability is increasing when the energy level/frequency \( k \) is growing. Let \( \varepsilon \) be the operator norm of the difference of two Dirichlet-to-Neumann maps corresponding to different potentials. We give conditional estimates for difference of potentials by a function of \( \varepsilon \) which goes to zero as \( \varepsilon \) goes to zero. This function is the sum of the terms containing powers of \( \varepsilon \) and of \( -\log \varepsilon \), moreover the terms containing \( \log \varepsilon \) tend to zero (as powers of \( k \)) when \( k \to \infty \). In the proofs we use complex and real valued geometrical optics solutions and explicit sharp bounds on fundamental solutions of elliptic operators with parameter \( k \).

2 Main results

Let \( \Omega \) be a domain in \( \mathbb{R}^3 \) with Lipschitz boundary. We consider the Schrödinger equation

\[
-\Delta u - k^2 u + cu = 0 \text{ in } \Omega
\]

with the Dirichlet boundary data

\[
u = g \text{ on } \partial \Omega.
\]

We will assume that the (complex valued) potential \( c \in L_\infty(\Omega) \). We define the Dirichlet-to-Neumann map

\[
\Lambda_c g = \partial_\nu u \text{ on } \partial \Omega, \quad g \in H^{\frac{1}{2}}(\partial \Omega).
\]

It is well-known that \( \Lambda_c \) is a continuous linear operator from \( H^{\frac{1}{2}}(\Gamma) \) into \( H^{-\frac{1}{2}}(\Gamma) \). We denote its operator norm by \( \|\Lambda_c\| \).

Let \( B \) be the unit ball in \( \mathbb{R}^3 \). We assume that \( \Omega \subset B \) and that \( c \) is zero near \( \partial \Omega \). We denote by \( C_0 \) generic constants. These constants do not depend on \( c \), \( k \), or \( \Omega \). They are only determined by our proofs. Generic \( C_\Omega \) might in addition depend on \( \Omega \).

Theorem 2.1 Let

\[
\|c_j\|_\infty(\Omega) \leq M, \quad \|c_j\|_{1,\infty}(\Omega) \leq M_1, \quad j = 1, 2.
\]

and \( \varepsilon = \|\Lambda_{c_2} - \Lambda_{c_1}\|, E = -\log \varepsilon \). Let us assume that \( 2 \leq E \).

There are constants \( C_0, C_\Omega \) such that if

\[
k \leq \frac{E^2}{2} - \frac{E}{4} - C_0^2 M < \sqrt{\frac{E^2}{2} - \frac{E}{4} - k + 2k^2 + 4},
\]

then

\[
\|c_2 - c_1\|_2(\Omega) \leq C_0 M^3 (E + k)^{-\frac{1}{4}} + \frac{M_1}{\sqrt{E + k}} + C_\Omega E^2 (E^2 + M^2) \varepsilon^{1 - \frac{1}{\sqrt{2}}}.
\]
There are constants $C_0, C_\Omega$ such that if

\[ E \leq k, \quad C_0^2 M^2 < k^2 + 2, \quad (2.7) \]

then

\[ ||c_2 - c_1||_2(\Omega) \leq \frac{C_0 + M_1}{\sqrt{k + E + 1}} + C_\Omega (k + M^2) k \varepsilon. \quad (2.8) \]

There are constants $C_0, C_\Omega$ such that if

\[ E \leq k < \varepsilon^{-\frac{9}{10}}, \quad C_0^2 M^2 < k^2 + 2, \quad (2.9) \]

then

\[ ||c_2 - c_1||_2(\Omega) \leq C_0 M^3 \left( \frac{E^{\frac{3}{2}}}{k} + (E + k)^{-\frac{9}{10}} \right) + \frac{2M_1}{E + k^{\frac{9}{10}}} + C_\Omega (\varepsilon^{\frac{1}{3}} + M^2 \varepsilon^{\frac{2}{10}} + E^{\frac{3}{2}} \varepsilon^{\frac{1}{10}}). \quad (2.10) \]

The bounds (2.6), (2.8), (2.10) correspond to low, high, and medium frequency ranges of $k$. In all of them the logarithmically unstable component goes to zero as $k$ grows. So these bounds can be viewed as a proof of increasing stability of recovery of $c$ for larger energies/frequencies $k$. The factor $k$ in the second term of the bound (2.8) most likely is necessary. Indeed, one needs time derivatives in the closely related inverse problems for the wave equation. At present it is not clear however if one can weaken dependence on $k$. In any event this second term contributes only to (the best possible) Lipschitz stability, while logarithmic terms are decaying for larger $k$. To remove this term we consider the midfrequencies range of $k$ given by (2.9). We tried to keep the most explicit track of dependencies on $k$ and $c$, as well as to give explicit numerical values of all powers of $\varepsilon$ and $E$. While constants $C_0, C_\Omega$ are hard to evaluate for general $\Omega$, it is very likely that for $\Omega = B$ one can obtain relatively simple explicit bounds on these constants.

We observe that for real-valued $c$ the Dirichlet problem (2.1), (2.2) might have eigenvalues $k$ when its solution fail to exist and be unique, so that the Dirichlet-to-Neumann map is not well defined. Then one can consider instead the Neumann-to-Dirichlet map, or replace these maps by the Cauchy set with naturally defined norm.

3 Almost exponential solutions

We start with

Lemma 3.1 Let

\[ k^2 \leq \tau^2 + \frac{|\xi|^2}{4}, \quad C_0^2 M^2 < |\xi|^2 + 4\tau^2 - 2k^2 + 4. \quad (3.1) \]
Then there are solutions

\[ u(x; j) = e^{ix\zeta(j)}(1 + v(x; j)) \]  

(3.2)
to the equations

\[-\Delta u_j - k^2 u_j + c_j u_j = 0 \text{ in } B,\]  

(3.3)

with

\[ \zeta(1) + \zeta(2) = \xi, \ |\zeta(j)| = \sqrt{\frac{|\xi|^2}{2} + 2\tau^2 - k^2}, \ |3\zeta(j)| = \sqrt{\frac{|\xi|^2}{4} + \tau^2 - k^2}, \]  

(3.4)

\[ \|v(; j)\|_2(B) \leq \frac{C_0 M}{\sqrt{|\xi|^2 + 4\tau^2 - 2k^2 + 4}} \]  

(3.5)

and

\[ \|v(; j)\|_1(B) \leq C_0 M. \]  

(3.6)

Proof
Let \( \xi \in \mathbb{R}^3, \xi \neq 0 \) and

\[ \xi = |\xi|e_1. \]

We introduce

\[ \zeta(1) = \frac{|\xi|}{2} e_1 + i\left(\frac{|\xi|^2}{4} + \tau^2 - k^2\right)^{\frac{1}{2}} e_2 + \tau e_3, \]

\[ \zeta(2) = \frac{|\xi|}{2} e_1 - i\left(\frac{|\xi|^2}{4} + \tau^2 - k^2\right)^{\frac{1}{2}} e_2 - \tau e_3, \]  

(3.7)

where \( e_1, e_2, e_3 \) is an orthonormal basis in \( \mathbb{R}^3 \). Then (3.3) holds if and only if

\[-\Delta v(; j) - 2i\zeta(j) \cdot \nabla v(; j) = c_j (1 + v(; j)) \text{ in } B. \]  

(3.8)

Let \( P(\zeta; j) = \zeta \cdot \zeta + 2\zeta(j) \cdot \zeta \). By known results [12], [14], there is a regular fundamental solution \( E(j) \) of \( P(; j) \) such that for any linear partial differential operator \( Q \) with constant coefficients

\[ \|Q E(j)f\|_2(B) \leq C_0 \sup_{\tilde{\xi} P(\xi^*)} \tilde{Q}(\xi^*) \|f\|_2(B) \sup \text{ over } \xi^* \in \mathbb{R}^3 \]  

(3.9)

for any \( f \in L^2(B) \), where

\[ \tilde{P}(\xi) = \left( \sum_{|\alpha| \leq 2} |\partial_\xi^\alpha P(\xi)|^2 \right)^{\frac{1}{2}}. \]

In our particular case, by letting \( \zeta(j) = \xi(j) + i\eta(j), \xi(j), \eta(j) \in \mathbb{R}^3 \), for any \( \xi^* \in \mathbb{R}^3 \) we yield

\[ \tilde{P}^2(\xi^*; j) = (|\xi^*|^2 + 2\xi(j) \cdot \xi^*)^2 + 4(\eta(j) \cdot \xi^*)^2 + 4(|\xi^* + \xi(j)|^2 + |\eta(j)|^2) + 12 = \]
due to the choice of $\zeta$ in (3.7).

Similarly,

$$\tilde{P}^2(\xi^*; j) \geq (|\xi^* + \xi(j)|^2 - |\xi(j)|^2)^2 + 4|\xi^* + \xi(j)|^2 + 12 \geq 11 \geq |\xi^*|^2 + 1,$$

(3.11)
due to the elementary inequalities $(a - b)^2 + 2(a - b) + 1 \geq 0$, or $(a - b)^2 + 4a + 2 \geq 2(a + b) + 1$, with $a = |\xi^* + \xi(j)|^2$, $b = |\xi(j)|^2$.

Since $E(j)$ is a fundamental solution, any solution $v(\cdot; j)$ to the equation

$$v(\cdot; j) = E(j)(c_j(1 + v(\cdot; j))) \text{ on } B$$

solves (3.8).

From (3.9) with $Q = 1$ and (3.10) it follows that

$$||E(j)f||_2(B) \leq C_0(|\xi|^2 + 4\tau^2 - 2k^2 + 4)^{-\frac{1}{2}}||f||_2(B),$$

or

$$||E(j)f||_2(B) \leq \theta||f||_2(B),$$

(3.13)

where $|\xi|^2 + 4\tau^2 - 2k^2 + 4 = \theta^{-2}C_0^2$. So the operator $F(v(\cdot; j))$ in the right side of (3.12) maps the ball $B(\rho) = \{v : ||v||_2(\Omega) \leq \rho\}$ into the ball $B(\theta||c_j||_\infty(\Omega)Vol^\frac{1}{2}(B) + \theta||c_j||_\infty(\Omega)\rho)$, and hence into $B(\rho)$ when

$$\theta M Vol^\frac{1}{2}(B) \leq (1 - \theta M)\rho$$

(3.14)

where we used the apriori bound (2.4). The second condition (3.1) and (3.13) imply that $\theta M < 1$ and hence (3.14) holds with

$$\rho = \frac{C_0 M Vol^\frac{1}{2}(B)}{\sqrt{|\xi|^2 + 4\tau^2 - 2k^2 + 4}}.$$

(3.15)

Similarly, using (3.11) and (3.9) with $Q(\xi^*) = \xi_k$, $k = 1, 2, 3$, from (3.12) we will have

$$||v(\cdot; j)||_{(1)}(B) \leq C_0||c_j(1 + v(\cdot; j))||_2(B) \leq C_0 M(1 + ||v(\cdot; j)||_2(\Omega)) \leq C_0 M\left(1 + \frac{M}{\sqrt{|\xi|^2 + 4\tau^2 - 2k^2 + 4}}\right) \leq C_0 M,$$

(3.16)
due to the bound (3.5) provided $||v(\cdot; j)||_{(0)}(B) \leq \rho$, given by (3.15). Therefore the operator $F$ is continuous from $L^2(B)$ into $H^1(B)$ and therefore compact from $L^2(B)$ into itself. Now this operator maps convex closed set $B(\rho) \subset L^2(B)$ into itself and is compact, hence by Schauder-Tikhonov Theorem it has a fixed point $v(\cdot; j) \in B(\rho)$. Due to (3.15) we have the bound (3.5) and due to (3.16) we have the bound (3.6).

The proof is complete.
Lemma 3.2 Let
\[ |\xi| \leq 2k, \quad C_0^2 M^2 \leq 4k^2 + 8. \] (3.17)
Then there are solutions
\[ u(x; j) = e^{ix \cdot \zeta(j)}(1 + v(x; j)) \] (3.18)
to the equations
\[ -\Delta u_j - k^2 u_j + c_j u_j = 0 \] in \( B \),
with
\[ \zeta(1) + \zeta(2) = \xi, \quad |\zeta(j)| = k, \quad \Im \zeta(j) = 0, \] (3.19)
\[ \|v(; j)\|_2(B) \leq \frac{C_0 M}{2\sqrt{k^2 + 2}} \] (3.20)
and
\[ \|v(; j)\|_{(1)}(B) \leq C_0 M. \] (3.21)

Proof
Using the notation of Lemma 3.1 we introduce
\[ \zeta(1) = \frac{|\xi|}{2} e_1 + (k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}} e_2, \quad \zeta(2) = \frac{|\xi|}{2} e_1 - (k^2 - \frac{|\xi|^2}{4})^{\frac{1}{2}} e_2. \] (3.22)
Then, as in Lemma 3.1, (3.18) solves the Schrödinger equation (3.3) if and only if \( v(; j) \) satisfies (3.8). Again let \( P(\zeta; j) = \zeta \cdot \zeta + 2\zeta(j) \cdot \zeta \). As in the proof of Lemma 2.1, for \( P(\zeta; j) \) with our particular choice of \( \zeta(j) \) in (3.22) we obtain
\[ \tilde{P}^2(\xi^*; j) = (|\xi|^2 + 2\zeta(j) \cdot \xi^* + 2)^2 + 4|\zeta(j)|^2 + 8 \geq 4|\zeta(j)|^2 + 8 \geq 4k^2 + 8. \] (3.23)
and
\[ \tilde{P}^2(\xi^*; j) \geq |\xi^*|^2 + 1. \] (3.24)
As in the proof of Lemma 3.1 from (3.9) with \( Q = 1 \) and (3.23) it follows that
\[ \|E(j)f\|_2(B) \leq C_0(4k^2 + 8)^{-\frac{1}{2}}\|f\|_2(B). \]
or
\[ \|E(j)f\|_2(B) \leq \theta_1\|f\|_2(B), \] where \( 4k^2 + 8 = \theta_1^{-2} C_0^2 \). (3.25)
So using the apriori bound (2.4) we conclude that the operator \( F(v(; j)) \) in the right side of (3.12) maps the ball \( B(\rho) = \{v : \|v\|_2(B) \leq \rho\} \) into the ball \( B(\theta_1 MV\, ol^{\frac{1}{2}}(B) + \theta_1 M\rho) \), and hence into \( B(\rho) \) when
\[ \theta_1 MV\, ol^{\frac{1}{2}}(B) \leq (1 - \theta_1 M)\rho. \] (3.26)
The second condition (3.17) and (3.25) imply that \( \theta_1 M < 1 \) and hence (3.26) holds with

\[
\rho = \frac{C_0 M}{2\sqrt{k^2 + 2}}. \tag{3.27}
\]

Similarly, using (3.24) and (3.9) with \( Q(\xi_k) = \xi_k, k = 1, 2, 3 \) from (3.12) we will have

\[
\|v(j)\|_{(1)}(B) \leq C_0 M (Vol^{1/2}(B) + \|v(j)\|_2(B)) \leq C_0 M (1 + \frac{C_0 M}{\sqrt{4k^2 + 8}}) \leq C_0 M, \tag{3.28}
\]

provided (3.20) holds. Therefore the operator \( F \) is continuous from \( L^2(B) \) into \( H^1(B) \) and therefore compact from \( L^2(B) \) into itself. Now this operator maps convex closed set \( B(\rho) \) into itself and is compact, hence by Schauder-Tikhonov Theorem it has a fixed point \( v(j) \in B(\rho) \). Due to (3.27) we have the bound (3.20) and due to (3.28) we have the bound (3.21).

The proof is complete.

**Lemma 3.3** Let \( u(j) \) be the solutions (3.2) to the Schrödinger equations 

\[-\Delta u_j - k^2 u_j + c_j u_j = 0 \text{ in } \Omega \]

from Lemma 3.1. Then one has

\[
\|\nabla e(j)\|^2_{(1)}(\partial \Omega) \leq C_\Omega (\|e(j)\|^2_{(1)}(\partial \Omega) + 2|\zeta(j)|^2 e^{\frac{|\zeta(j)|}{2} + \frac{1}{2} + M e^{\frac{1}{2}}} (3.29)
\]

**Proof.** Let \( e(x; j) = e^{ix \xi(j)} \). By direct calculations we obtain

\[
\|e(j)\|_{(1)}(\partial \Omega) \leq C_{\Omega} (1 + |\zeta(j)|) e^{\frac{|\zeta(j)|}{2}}. 
\]

Hence from the interpolation inequality \( \|\nabla e(j)\|_{(\frac{1}{2})}(\partial \Omega) \leq C_{\Omega} \|\nabla e(j)\|_{(0)}(\partial \Omega) \) we yield

\[
\|e(j)\|_{(\frac{1}{2})}(\partial \Omega) \leq C_{\Omega} (1 + |\zeta(j)|)^{\frac{1}{2}} e^{\frac{|\zeta(j)|}{2}}. \tag{3.30}
\]

Obviously,

\[
\nabla (e(j) v(j)) = i\zeta(j) e(j) v(j) + e(j) \nabla v(j),
\]

so

\[
\|e(j) v(j)\|_{(1)}(\Omega) \leq C_{\Omega} M^2 e^{2|\zeta(j)|} \]

due to (3.4), (3.5), (3.6). Hence from trace theorems for Sobolev spaces

\[
\|e(j) v(j)\|_{(1)}(\partial \Omega) \leq C_{\Omega} M e^{\frac{|\zeta(j)|}{2}}. \tag{3.31}
\]

Recalling that \( u(j) = e(j) + e(j) v(j) \) and using (3.4) from (3.30), (3.31) we yield (3.29).

The proof is complete.
Lemma 3.4 Let \( u(j) \) be the solutions (3.18) to the Schrödinger equations \(-\Delta u_j - k^2u_j + c_ju_j = 0 \) in \( \Omega \) from Lemma 3.2. Then one has
\[
\|u(j)\|_{(\frac{1}{2})}(\partial\Omega) \leq C_{\Omega}(1 + \sqrt{k} + M)
\] (3.32)

Proof Let \( e(x; j) = e^{ix\cdot\zeta(j)} \). By direct calculations with use of (3.19) we obtain
\[
\|e(j)\|_{(1)}(\partial\Omega) \leq C_{\Omega}(1 + |\zeta(j)|)
\]
As in the proof of Lemma 3.3, using the interpolation theory we yield
\[
\|e(j)\|_{(\frac{1}{2})}(\partial\Omega) \leq C_{\Omega}(1 + |\zeta(j)|)^{\frac{1}{2}}.
\] (3.33)

Similarly to the proof of Lemma 3.3,
\[
\|e(j)v(j)\|_{(1)}(\Omega) \leq
\|e(j)v(j)\|_{0}(\Omega) + 2|\zeta(j)|^2\|e(j)v(j)\|_{0}(\Omega) + 2\|e(j)v(j)\|_{0}(\Omega) \leq
C_0M^2,
\]
due to (3.20), (3.21). Hence from trace theorems for Sobolev spaces
\[
\|e(j)v(j)\|_{(\frac{1}{2})}(\partial\Omega) \leq C_{\Omega}M.
\] (3.34)

Recalling that \( u(j) = e(j) + e(j)v(j) \) and using (3.19) from (3.33), (3.34) we yield (3.32).

4 Proofs of stability estimate

The following standard orthogonality result [1], [14] follows by simple application of the Green’s formula.

Lemma 4.1
\[
\int_{\Omega} (c_1 - c_2)u_1u_2 = \int_{\partial\Omega} ((\Lambda_{c_2} - \Lambda_{c_1})u_1)u_2
\] (4.1)
for all functions \( u_1, u_2 \in H^1(\Omega) \), solving the Schrödinger equations
\[-\Delta u_1 - k^2u_1 + c_1u_1 = 0 \text{ in } \Omega \]
and
\[-\Delta u_2 - k^2u_2 + c_2u_2 = 0 \text{ in } \Omega.\]
Proof of Theorem 2.1
First we handle the case (2.5).
Using the almost complex-exponential solutions (3.2) in the identity (4.1) and observing that
\[ u(x; 1)u(x; 2) = e^{ix\xi}(1 + v(x; 1) + v(x; 2) + v(x; 1)v(x; 2)) \]
due to (3.4) we obtain for the Fourier transform \((\hat{c}_2 - \hat{c}_1)(\xi)\) of \(c_2 - c_1\)
\[ |(\hat{c}_2 - \hat{c}_1)(\xi)| \leq \int_{\Omega} (|c_1| + |c_2|)(|v(1)| + |v(2)| + |v(1)||v(2)|) + \varepsilon\|u(1)\|_{(\frac{1}{2})(\partial\Omega)}\|u(2)\|_{(\frac{1}{2})(\partial\Omega)} \leq 2M(\|v(1)\|_{2(\Omega)} + \|v(2)\|_{2(\Omega)} + \|v(1)\|_{2(\Omega)}\|v(2)\|_{2(\Omega)}) + \varepsilon\|u(1)\|_{(\frac{1}{2})(\partial\Omega)}\|u(2)\|_{(\frac{1}{2})(\partial\Omega)} \]
by the Cauchy-Schwarz inequality and (2.4). Using (3.5) and (3.29) and letting \(T^2 = 4(\tau^2 - k^2), k \leq \tau\), we yield
\[ |(\hat{c}_2 - \hat{c}_1)(\xi)| \leq C_0 \frac{M^2}{\sqrt{|\xi|^2 + T^2 + 2k^2 + 4}} + C_\Omega\varepsilon(\|\xi\|^2 + T^2 + 2k^2 + 1)^{\frac{1}{2}} + M^2 e^{\sqrt{|\xi|^2 + T^2}} \leq C_0 \frac{M^3}{\sqrt{|\xi|^2 + T^2 + 2k^2 + 4}} + C_\Omega(\sqrt{|\xi|^2 + T^2 + k^2 + M^2} + 1)e^{\sqrt{|\xi|^2 + T^2}} \varepsilon. \quad (4.2) \]
From (2.4),
\[ \int_{\rho < |\xi|} |\hat{c}_2 - \hat{c}_1|^2(\xi)d\xi \leq \frac{4M_1^2}{\rho^2 + 1}, \]
so from (4.2) by using the Parseval identity we conclude that
\[ \|c_2 - c_1\|^2_{2(\Omega)} \leq \int_{|\xi|<\rho} |\hat{c}_2 - \hat{c}_1|^2(\xi)d\xi + \int_{\rho < |\xi|} |\hat{c}_2 - \hat{c}_1|^2(\xi)d\xi \leq \]
\[ C_0M^6 \int_0^\rho \frac{\rho^2}{r^2 + T^2 + 2k^2 + 4}dr + C_\Omega \int_0^\rho ((\rho^2 + T^2 + k^2 + M^4 + 1)e^{2\sqrt{\rho^2 + T^2}}\rho^2 dr \varepsilon^2 + \frac{M_1^2}{\rho^2 + 1} \leq \]
\[ C_0M^6 \rho^3 \frac{\rho^3}{\rho^2 + T^2 + 2k^2 + 4} + C_\Omega \rho^3 (\rho^2 + T^2 + k^2 + 1 + M^4)e^{2\sqrt{\rho^2 + T^2}}\varepsilon^2 + \frac{M_1^2}{\rho^2 + 1}. \quad (4.3) \]
provided the assumption (3.1) holds.
Let
\[ T^2 = \frac{E^2}{2} - \frac{E}{4} - k, \rho^2 = \frac{1}{4}E + k. \]
This choice is possible due to (2.5), moreover (3.1) is satisfied. Hence we obtain from (4.3)
\[ \|c_2 - c_1\|_2^2(\Omega) \leq C_0M^6(E + k)^{-\frac{1}{2}} + C_\Omega\frac{E^2}{2} + k^2 + 1 + M^4)\left(\frac{E}{4} + k\right)^{\frac{3}{2}}\varepsilon^{2 - \frac{2}{\sqrt{2}}} + \frac{4M_1^2}{E + 4k + 4} \leq C_0M^6((E + k)^{-\frac{1}{2}} + C_\Omega E^3(E^4 + M^4)\varepsilon^{2 - \frac{2}{\sqrt{2}}} + \frac{4M_1^2}{E + k + 1}, \]
because
\[ \frac{\rho^3}{\rho^2 + T^2 + 2k^2 + 4} = \frac{(E + 4k)^{\frac{3}{2}}}{4(E^2 + 4k^2 + 8)} \leq C_0\frac{(E^2 + 4k^2)^{\frac{3}{2}}}{E^2 + 4k^2 + 8} \leq C_0(E^2 + k^2)^{-\frac{1}{4}} = C_0(E + k)^{-\frac{1}{2}} \]
and \( k < E^2 \) due to (2.7).
This completes the proof in the case (2.5).
Now we handle the case (2.7).
As above, using the almost exponential solutions (3.18) and the bounds (3.20), (3.32) in the identity (4.1) we obtain
\[ |(\hat{c}_2 - \hat{c}_1)(\xi)| \leq C_0M\frac{M}{\sqrt{k^2 + 2}} + \frac{M^2}{k^2 + 2} + C_\Omega(k + M^2)\varepsilon \leq C_0\frac{M^3}{\sqrt{k^2 + 2}} + C_\Omega(k + M^2)\varepsilon, \quad |\xi| \leq 2k. \]
Hence as above,
\[ \|c_2 - c_1\|_2^2(\Omega) \leq C_0M^6\frac{\rho^3}{k^2 + 2} + C_\Omega(k^2 + M^4)\rho^3\varepsilon^2 + \frac{M_1^2}{\rho^2 + 1}, \quad \rho \leq 2k. \]  
(4.4)
Let us choose \( \rho = k^{\frac{3}{2}}. \)
Then from (4.4) we have
\[ \|c_2 - c_1\|_2^2(\Omega) \leq C_0M^6\frac{k^{\frac{3}{2}}}{k^2 + 2} + C_\Omega(k^2 + M^4)k^{\frac{3}{2}}\varepsilon^2 + \frac{M_1^2}{k^{\frac{3}{2}} + 1} \leq \]
\[ \frac{C_0M^6 + M_1^2}{k^{\frac{3}{2}}} + C_\Omega(k^2 + M^4)k^{\frac{3}{2}}\varepsilon^2. \]
Taking square roots completes the proof in the case (2.7).
Finally we will handle the case (2.9). Using the assumptions (2.9) and letting
\[ \rho = E + k^{\frac{1}{30}}, \]
we obtain from (4.4)
\[ \| c_2 - c_1 \|_2^2(\Omega) \leq \]
\[ C_0 M^6 \left( \frac{E + k^{\frac{1}{30}}}{k^2 + 2} \right)^3 + C_\Omega (k^2 + M^4)(E + k^{\frac{1}{30}})^3 \epsilon^2 + \frac{M^2_1}{(E + k^{\frac{1}{30}})^2} \leq \]
\[ C_0 M^6 \left( \frac{E^3}{k^2} + k^{-\frac{10}{15}} \right) + C_\Omega (k^2 + M^4)(E^3 + k^{\frac{1}{30}}) \epsilon^2 + \frac{M^2_1}{(E + k^{\frac{1}{30}})^2} \leq \]
\[ C_0 M^6 \left( \frac{E^3}{k^2} + k^{-\frac{10}{15}} \right) + C_\Omega (\epsilon^{-\frac{10}{15}} + M^4)(E^3 + \epsilon^{-\frac{10}{15}}) \epsilon^2 + \frac{M^2_1}{(E + k^{\frac{1}{30}})^2}, \]
where we again used the assumption (2.9). Taking square roots, doing elementary calculations, taking square roots, using that \( E^3 < C_0 \epsilon^{\frac{11}{10}} \), and adjusting constants, we obtain (2.10).

The proof is complete.

5 Conclusion

It is clear that difficulties in theory and applications of many important inverse problems are due to their notorious (exponential) instability (or logarithmic stability) with respect to data changes. In practical situations, logarithmic stability permits as a rule to find only 10-20 Fourier coefficients of unknown coefficients or source terms. This results in poor resolution of various numerical methods and consequently in a disappointment of engineers or scientists expecting effective mathematical processing of experimental data. An acquisition of such data is often very laborous and expensive. So any way to increase stability and resolution is indeed valuable. While increasing stability with the wave number is observed in numerical experiments for several important inverse problems [5], [20], before there was no theoretical explanation. Moreover, there is a belief that stability always grows with frequency. As shown in [17], [13], [15], at least for the continuation of solutions of elliptic equations it is true only under some (convexity type) conditions. Otherwise, stability might deteriorate. This paper contains the first proof of increasing stability for potential from the Dirichlet-to-Neumann map.

Now we outline possible future developments.

We considered compactly supported potentials \( c \). Since the boundary reconstruction (at least when \( k = 0 \) is stable [24]) by combining the methods of this paper with the proofs in [1], [14], section 5.1, one can most likely consider general potentials. Constants \( C_0 \) in the stability estimates may be evaluated more explicitly by using
periodic Faddeev type solutions [11], instead of the Fourier transform and regular fundamental solutions. Constants in our estimates are not explicit, but by using the bounds in [11] we expect to obtain realistic bounds on them at least when $\Omega$ is the unit ball $B$. There is a need in a numerical evidence of increasing stability in the important inverse problem we considered in this paper, as well as in similar inverse medium problems studied in particular in [5], [19].

Probably, it will be more difficult to show increasing stability for the coefficient $a_0$ in the equation $(-\Delta - k^2 a_0^2(x))u + cu = 0$. At present, there are only some preliminary results (in low frequency zone) [22], methods of (complex) geometrical optics do not look promising, and we do not know a good alternative. We observe that the conductivity equation $\text{div}(a\nabla u) + k^2 u = 0$ can be transformed into this Helmholtz type equation with $a_0 = a^{-\frac{1}{2}}, c = a^{-\frac{1}{2}} \Delta a^\frac{1}{2}$ [14], p.181, so increasing stability for the conductivity equation remains a challenge.

The next challenge is to obtain similar estimates for the inverse scattering problems by obstacles and by the medium. In particular, it is still an open problem whether stability of recovery of near field from far field pattern is improving with growing frequency. Near logarithmic stability estimates with constants depending on $k$ were obtained in [7], [14], section 6.1, [25], p. 222. Preliminary stability results for the inverse scattering by potential have been given in [21]. It is realistic to demonstrate the increasing stability for hard or transparent obstacles from the Dirichlet-to-Neumann map by combining the results of this paper with the methods of [2]. A better stability for larger $k$ was observed in numerical solution of the inverse obstacles scattering problem in [9]. It is clear that one has to impose some (pseudo)convexity condition on unknown coefficients or obstacles. It is also important to collect numerical evidence of decreasing stability when these convexity conditions are not satisfied.

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References


