Increased stability in the continuation for the Helmholtz equation with variable coefficient

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Abstract

In this paper we give analytical evidence of increasing stability in the Cauchy Problem for the Helmholtz equation when frequency is growing. This effect depends on convexity properties of the surface where the Cauchy Data are given and on some monotonicity properties of the variable coefficient of the Helmholtz equation. Proofs use Carleman estimates and the theory of elliptic and hyperbolic boundary value problems in Sobolev spaces.

1 Introduction

The study of the uniqueness in the Cauchy problem for partial differential equations has a long history. For theory and applications a stability issue is of fundamental importance. Needs of prospecting by acoustical and electromagnetic waves (in particular, of the inverse scattering theory) stimulate study of this problem for the Helmholtz equation

\[(\Delta + a_0^2 k^2)u = f \text{ in } \Omega \quad (1.1)\]

with the Cauchy data

\[u = u_0, \quad \partial_n u = u_1 \text{ on } \Gamma \quad (1.2)\]

where $\Omega$ be a domain in $\mathbb{R}^n$ and $\Gamma \in C^1$ be an (open) part of $\partial \Omega$. Due to the celebrated results of Fritz John [7] in general case one can expect only quite weak logarithmic stability. However in several important practical examples (e.g. in computations for nearfield acoustical holography [1]) it was observed that stability (and, as a consequence, resolution) in the Cauchy Problem (and hence in related inverse problems) is increasing with $k$. In the paper [2] the authors found new stability estimates explaining this phenomenon for constant $a_0$ and illustrated it by numerical solution of some important applied problems. The goal of this paper is to consider variable $a_0$. This is achieved by combining some results from [2] with use of Carleman estimates for the time dependent wave equation and by ”freezing” $a_0$ at certain points.
This paper is organized as follows. In Section 1 we describe the current state of the problem and our main results. In Section 2 we give energy estimates in low frequency zone for constant and variable coefficients. A key ingredient in the proof of the stability and uniqueness for the Cauchy problem is a Carleman type estimate for (1.1) which does not depend on \( k \). In Section 3 we will derive this estimate from a known estimate for hyperbolic equations. In Section 4 we give proofs of main results.

We write \( x = (x', x_n) \in \mathbb{R}^n \). Let \( \Omega \) be an open subset of the cylinder \( \{0 < x_n < h, |x'| < r \} \) with the Lipschitz boundary \( \partial \Omega \), \( \bar{\Omega} \subset \{x_n < h\} \) and \( \Gamma \) be the part of \( \partial \Omega \) contained in the layer \( \{0 < x_n < h\} \). Let \( \Omega(d) = \Omega \cap \{d < x_n\} \) and \( \Omega^*(d) = \mathbb{R}^{n-1} \times (d, h) \). Let \( e_n = (0, \ldots, 0, 1) \). By \( C, \kappa \) we denote constants which depend only on \( \Omega, S, a_0, d \). Any other dependence is indicated. \( \|u\|_{(0)}(\Omega) \) is the standard norm in the Sobolev space \( H_{(0)}(\Omega) \) and \( \|u\|_{(\Omega)} = \|u\|_{(0)}(\Omega) \). We let \( M_1 = \|u\|_{(1)}(\Omega), F = \|f\|(\Omega) + \|u\|(\Gamma) + \|\nabla u\|(\Gamma) \) and \( F(k) = F + k\|u\|(\Gamma) \). \( V(\xi, x_n) \) denotes the (partial) Fourier transformation \( \mathcal{F} v(\xi, x_n) \) of a function \( v(x) \) with respect to \( x' \).

Since we are interested in increasing wave numbers \( k \), for simplicity of the exposition we will assume that

\[
1 \leq k. \tag{1.3}
\]

**Theorem 1.1** Let \( a_0 \in C^1(\overline{\Omega}), 0 < a_0 \) on \( \overline{\Omega} \) and

\[
0 < a_0 + \nabla a_0 \cdot x + \beta_n \partial_n a_0, \ 0 \leq \partial_n a_0 \text{on} \ \overline{\Omega} \tag{1.4}
\]

for some positive \( \beta_n \).

Then for any \( \varepsilon \) there are \( C, C(\varepsilon), \kappa \in (0, 1) \) such that

\[
\|u\|_{(0)}(\Omega(d)) \leq C(F + \varepsilon M_1 + C(\varepsilon) \frac{M_1^{1-\kappa} F(k)^\kappa + F}{k}) \tag{1.5}
\]

for all \( u \) solving (1.1), (1.2).

Theorem 1.1 allows to consider more general domains \( \Omega \). Let \( S \) be a compact subset of \( \Omega \). By \( P(\nu; d) \) we denote the half-space of \( \mathbb{R}^n \) with the exterior normal \( \nu \) which is at the distance \( d \) from \( S \). By \( \gamma \) we denote all \( \nu \) such that \( P(\nu; d) \cap \partial \Omega \) is contained in \( \Gamma \). Let \( \Omega(\nu; \Gamma, d) \) be \( P(\nu; d) \cap \Omega \) and, finally, let \( \Omega(\Gamma, d) \) be the union of all such \( \Omega(\nu; \Gamma, d) \) over \( \nu \in \gamma \). If \( \Gamma = \partial \Omega \) then \( \Omega(\Gamma, 0) \) is the difference of \( \Omega \) and of the convex hull of \( \Gamma \) and \( \Omega(\Gamma, d) \) is the collection of points of \( \Omega(\Gamma, 0) \) which are at distance \( d \) from \( S \). As in [2], applying Theorem 1.1 to any \( \Omega = \Omega(\nu; \Gamma, d), \nu \in \gamma \) and using appropriate partition of the unity we obtain

**Corollary 1.1** Let the condition (1.4) be satisfied in any \( \Omega(\nu; \Gamma, d), \nu \in \gamma \) with \( x_n \) direction replaced by \( \nu \).

Then the bound (1.5) with \( \Omega(\Gamma, d) \) instead of \( \Omega(d) \) is valid.
There is an important particular case when the norm of the data does not explicitly depend on \( k \). Let us keep the notation of Corollary 1.1. Let \( \omega \) be an open subset of \( \Omega \) with \( \Gamma \subset \partial \omega \) (boundary layer) such that \( \Gamma \) is at the distance \( d_0 \) from \( \partial \omega \cap \Omega \).

Let \( F_\omega = \| f \|_1(\Omega) + \| u \|_1(\omega) \).

**Corollary 1.2** Under the conditions of Corollary 1.1 there are constants \( C, C(d_0), C(\varepsilon) \) such that for any solution \( u \) to the Cauchy problem \((1.1),(1.2)\)

\[
\| u \|_1(\Omega(\Gamma, d)) \leq C(d_0)(F_\omega + \varepsilon\| u \|_1(\Omega) + C(\varepsilon)\frac{M^{1-\kappa}F_\kappa + F_\omega}{k}).
\]  

(1.6)

To derive this Corollary from Corollary 1.1 we let \( \chi \) to be a cut off function which is 1 on \( \Omega \setminus \omega \) and 0 near \( \Gamma \). Applying Corollary 1.1 to \( \chi u \) instead of \( u \) and using that \((\Delta + k^2)(\chi u) = \chi((\Delta + k^2)u) + 2\nabla \chi \cdot \nabla u + \Delta \chi u = \chi f + 2\nabla \chi \cdot \nabla u + \Delta \chi u \) and that the function \( \chi u \) has zero Cauchy data on \( \Gamma \), (1.6) follows from Corollary 1.1.

Theorem 1.1 and its corollaries show an improved stability in the Cauchy Problem (1.1), (1.2) when one continues the solution to the differential equation inside the convex hull of \( \Gamma \). Due to the results of John [7] this is impossible when one continues to the outside of a convex \( \Gamma \).

Our proof of Theorem 1.1 is based on the following

**Theorem 1.2** Let the condition \((1.4)\) be satisfied.

Then there are constants \( C, \kappa \in (0,1) \) such that for any solution \( u \) to the Cauchy problem \((1.1), (1.2)\)

\[
\| u \|_1(\Omega(d)) \leq C(F(k)^\kappa(M_1)^{1-\kappa} + F)
\]

(1.7)

which is of its own interest due to independence of \( C \) on \( k \).

**2 Proofs of auxiliary energy type estimates**

We remind a simple result imitating the standard energy estimate for hyperbolic initial value problems.

**Lemma 2.1** Let \( a_n \in C^1([0, h]) \) and depend only on \( x_n \). Let \( v \in C^2(\Omega^*(h)) \) solve the initial value problem

\[
(\Delta + a_n^2k^2)v = \partial_1 f_1 + \ldots + \partial_n f_n + kf_{n+1} + k^2 f_0 \text{ in } \Omega^*(h),
\]

\[
v = 0 \text{ on } \Omega^*(h_1)
\]

for some \( h_1 < h \), \( f_j \in C^\infty(\Omega^*(d)) \), \( f_j = 0 \text{ on } \Omega(h_1) \), and

\[
V(\xi, x_n) = 0 \text{ when } \frac{a_n^2(x_n)}{2}k^2 < |\xi|^2
\]

(2.2)
Then there is constant $C$ depending only on $h$, $\sup|\partial_0 a_n|$, $\sup a_n^{-1}$ over $(0, h)$ such that

$$
\|v\|\Omega^*(d)) \leq C(\|f_1\|\Omega^*(d)) + \cdots + \|f_n+1\|\Omega^*(d)) + \|f_0\|\Omega^*(d)) + \|\partial_0 f_0\|\Omega^*(d))
$$

(2.3)

**Proof.** Due to Parseval’s identity it suffices to show that solutions to the initial value problems

$$
\partial^2_n V_j + (a^2_n k^2 - |\xi|^2)V_j = -i\xi V_j \text{ on } (d, h), j = 1, \ldots, n-1,
$$

(2.4)

$$
\partial^2_n V_n + (a^2_n k^2 - |\xi|^2)V_n = \partial_n F_n \text{ on } (d, h),
$$

(2.5)

$$
\partial^2_n V_{n+1} + (a^2_n k^2 - |\xi|^2)V_{n+1} = k F_{n+1} \text{ on } (d, h),
$$

(2.6)

$$
\partial^2_n V_0 + (a^2_n k^2 - |\xi|^2)V_0 = k^2 F_0 \text{ on } (d, h)
$$

(2.7)

with the zero final conditions

$$
V_j = 0, F_j = 0 \text{ on } (h_1, h),
$$

(2.8)

satisfy the bounds

$$
\int_{d}^{h} |V_j|^2(\xi, s)ds \leq C \int_{d}^{h} |F_j|^2(\xi, s)ds, j = 1, \ldots, n+1,
$$

(2.9)

$$
\int_{d}^{h} |V_0|^2(\xi, s)ds \leq C \int_{d}^{h} (|\partial_n F_0|^2(\xi, s) + |F_0|^2(\xi, s))ds.
$$

(2.10)

Indeed, solving the Cauchy problems for ordinary differential equations (2.4)-(2.7), letting $V = V_1 + \cdots + V_{n+1} + V_0$ and using the inverse Fourier transformation in $x_1, \ldots, x_{n-1}$ we obtain (2.3).

Obviously, if suffices to obtain bounds (2.9),(2.10) for real and imaginary parts of solutions via real and imaginary parts of the right sides. In other words, we can assume that $V_j, iF_j, V_n, F_n, V_{n+1}, F_{n+1}, V_0, F_0$ are real-valued.

Let us first consider $j = 1, \ldots, n-1$. Multiplying the both sides of (2.4) by $-\partial_n V_j e^{\tau x_n}$, observing that

$$(\partial^2_n V_j)\partial_n V_j = \frac{1}{2}\partial_n (\partial_n V_j)^2, (\partial_n V_j)V_j = \frac{1}{2}\partial_n (V_j^2)$$

and integrating by parts over the interval $(x_n, h)$ with use of (2.8) we obtain

$$
\frac{1}{2}(\partial_n V_j)^2(x_n) e^{\tau x_n} + \frac{\tau}{2} \int_{x_n}^{h} (\partial_n V_j)^2(s) e^{\tau s}ds + \frac{1}{2}(a^2_n (x_n) k^2 - |\xi|^2)V_j^2(x_n) e^{\tau x_n} + \frac{1}{2} \int_{x_n}^{h} (\tau(a^2_n (s) k^2 - |\xi|^2) + 2a_n (s) \partial_n a_n (s))V_j^2(s) e^{\tau s}ds =
$$
Multiplying by $\xi_j F_j \partial_n V_j(\cdot, s)e^{\tau s} ds$.

Using the elementary inequality $|\xi_j F_j \partial_n V_j| \leq \frac{1}{2}(|\xi|^2|F_j|^2 + |\partial_n V_j|^2)$, dropping the first and the third term in the left side in the previous equality and cancelling the second term in the left side and the second term in the right side we yield

$$\frac{1}{2} \int_{x_n}^{h} ((\tau a_n^2(s)k^2 - |\xi|^2) + 2a_n(s)\partial_n a_n(s))V_j^2(\cdot, s)e^{\tau s} ds \leq \frac{1}{2} \int_{x_n}^{h} |\xi|^2|F_j|^2(\cdot, s)e^{\tau s} ds.$$  

Let us choose

$$\tau = \max(1, \sup(1 - \frac{4\partial_n a_n}{a_n}(x_n)), x_n \in (0, h)).$$  

Then

$$\tau (a_n^2k^2 - |\xi|^2) + 2a_n\partial_n a_n \geq \frac{a_n^2k^2}{2}.$$  

Replacing $e^{\tau s}$ by its minimum 1 in the left side and by its maximum $e^{\tau h}$ in the right side and using that $|\xi|^2 < \frac{k^2a_n^2}{2}$ we obtain (2.9) with $C = e^{\tau h}$ where $\tau$ is given by (2.12).

Integrating the equation (2.5) over $(x_n, h)$ and using the final conditions (2.8) we obtain

$$-\partial_n V_n(\cdot, x_n) + \int_{x_n}^{h} (a_n^2(s)k^2 - |\xi|^2)V_n(\cdot, s) ds = -F_n(\cdot, x_n).$$

Multiplying this equation by $V_n(\cdot, x_n)$ we will have

$$-\frac{1}{2} \partial_n(V_n^2)(\cdot, x_n) - (k^2 - |\xi|^2)\frac{1}{2} \partial_n(\int_{x_n}^{h} V_n(\cdot, s) ds)^2 = -F_n V_n(\cdot, x_n).$$

Multiplying by $e^{\tau x_n}$ and integrating by parts over $(x_n, h)$ we arrive at

$$\frac{1}{2} V_n^2(\cdot, x_n)e^{\tau x_n} + \frac{\tau}{2} \int_{x_n}^{h} V_n^2(\cdot, s)e^{\tau s} ds + \frac{\tau}{2} \int_{x_n}^{h} \frac{e^{\tau x_n}}{2(a_n^2(s)k^2 - |\xi|^2)}(\int_{x_n}^{h} (a_n^2(s)k^2 - |\xi|^2)V_n(\cdot, s) ds)^2 ds +$$

$$\frac{1}{2} \int_{x_n}^{h} \frac{\tau a_n^2(s_1)k^2 - |\xi|^2 - 2a_n(s_1)\partial_n a_n(s_1)}{(a_n^2(s_1)k^2 - |\xi|^2)}e^{\tau s_1}(\int_{s_1}^{h} (a_n^2(s)k^2 - |\xi|^2)V_n(\cdot, s) ds)^2 ds_1 =$$

$$- \int_{x_n}^{h} F_n V_n(\cdot, s)e^{\tau s} ds \leq \frac{1}{\tau} \int_{x_n}^{h} F_n^2(\cdot, s)e^{\tau s} ds + \frac{\tau}{4} \int_{x_n}^{h} V_n^2(\cdot, s)e^{\tau s} ds$$

by the elementary inequality $ab \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}$. Now we will choose

$$\tau = \max(1, \sup(\frac{4\partial_n a_n}{a_n}(x_n)), x_n \in (0, h)).$$

(2.14)
Due to this choice of $\tau$, the assumptions (1.3) and (2.2) last integral in the left side of (2.13) is nonnegative. Using also (2.2) and dropping the first, third, and the fourth term in the left side of (2.13), absorbing the alst term in the right side by the left side and replacing $e^{r^s}$ by its minimum $1$ in the left side and by its maximum $e^{r^h}$ in the right side we arrive at (2.9) with $C = \frac{4e^{r^h}}{h}$ and $\tau$ defined by (2.14).

A proof of the bound (2.9) for $V_{n+1}$ (with $C = 2e^{2r} \sup a_n^{-2}(s), s \in (0, h)$) is similar to the previous bounds.

Our proof of the bound (2.9) for $V_0$ needs some modifications of the previous argument. Indeed, as in the bounds for $V_j$,

$$\frac{1}{2}(\partial_n V_0)^2(x_n)e^{\tau x_n} + \frac{\tau}{2} \int_{x_n}^{h} (\partial_n V_0)^2(s)e^{\tau s} ds +$$

$$\frac{1}{2}(a_n^2(x_n)k^2 - |\xi|^2)V_0^2(x_n)e^{\tau x_n} + \frac{\tau}{2} \int_{x_n}^{h} ((a_n^2(s)k^2 - |\xi|^2) + 2a_n(s)\partial_n a_n(s))V_0^2(s)e^{\tau s} ds =$$

$$- \int_{x_n}^{h} (k^2 F_0)\partial_n V_0(s)e^{\tau s} ds =$$

$$k^2 F_0 V_0(x_n)e^{\tau x_n} + k^2 \int_{x_n}^{h} (\partial_n F_0 + \tau F_0) V_0(s)e^{\tau s} ds.$$

Dropping the first two terms in the left side, using the elementary inequalities

$$F_0 V_0 \leq (\frac{2}{a_n^2} F_0^2 + \frac{a_n^2}{8} V_0^2), \quad (\partial_n F_0 + F_0) V_0 \leq \frac{4}{a_n^2} ((\partial_n F_0)^2 + \tau^2 F_0^2) + \frac{a_n^2}{8} V_0^2$$

and the assumption (2.2) and dividing by $k^2$ we yield

$$\frac{a_n^2(x_n)}{4} V_0^2(x_n)e^{\tau x_n} + \int_{x_n}^{h} \frac{a_n^2(s)}{4} V_0^2(s)e^{\tau s} ds \leq$$

$$\frac{a_n^2(x_n)}{8} V_0^2(x_n)e^{\tau x_n} + \frac{2}{a_n^2(x_n)} F_0^2(x_n)e^{\tau x_n} +$$

$$\int_{x_n}^{h} \frac{a_n^2(s)}{8} V_0^2(s)e^{\tau s} ds + \int_{x_n}^{h} \frac{4}{a_n^2(s)} ((\partial_n F_0)^2 + \tau^2 F_0^2)(s)e^{\tau s} ds.$$

which implies that

$$a_n^2(x_n) V_0^2(x_n)e^{\tau x_n} \leq \frac{16}{a_n^2(x_n)} F_0^2(x_n)e^{\tau x_n} + \int_{x_n}^{h} \frac{32}{a_n^2(s)} ((\partial_n F_0)^2 + F_0^2)(s)e^{\tau s} ds.$$

Let $a = \min a_n$ over $[0, h]$. Integrating with respect to $x_n$ over $(d, h)$ and replacing $a_n(x_n), e^{\tau x_n}$ on the left side by $a, 1$ and on the right side by $a, e^{\tau h}$ we arrive at (2.10) with $C = a^{-4}16(1 + 2h)e^{\tau h}$ where $\tau$ is defined by (2.12).

The proof is complete.
Now by using Lemma 2.2, freezing coefficient with respect to \( x' \) and partitioning of the unity, we will obtain energy type estimates for variable \( a_0 \).

Let \( \varepsilon > 0 \). By \( X(j) \) we denote points with integer coordinates. Let \( x(j), j = 1, \ldots, J \) be points \( \varepsilon X(j) \) which are contained in \( \Omega' \). It is clear that \( J \leq C\varepsilon^{-n} \). The spheres \( B'(x(j); \varepsilon) \) form an open covering of \( \overline{\Omega} \). We define \( \Omega_j = B'(x(j); \varepsilon) \times (0, h) \). Let \( \chi(x'; j) \) be partition of the unity subordinated to this covering. We can assume that

\[
0 \leq \chi(; j) \leq 1, \ |\nabla \chi(; j)| \leq C\varepsilon^{-1}, \ |\Delta \chi(; j)| \leq C\varepsilon^{-2} \tag{2.15}
\]

Now we will introduce a "low frequency" projector \( u_1 = Pu \) of a function \( v \). Let a function \( \chi \in C^\infty(\mathbb{R}) \), \( \chi = 1 \) on \((0, 1/2)\), \( \chi = 0 \) on \((3/4, \infty)\), \( 0 \leq \chi \leq 1 \). Let \( \chi_j(x_n; \xi) = \chi(k^{-1}a_0^{-1}(x(j), x_n)|\xi|) \). We define

\[
v(j) = \chi(j)v, \ P_j v(j) = F^{-1} \chi_j F v(j), \ v_1 = \sum_{j=1}^{J} P_j v(j). \tag{2.16}
\]

For brevity we let \( \|v\| = \|v\|_0(\Omega^*(d)) \).

**Lemma 2.2** Let \( v \in C^2(\overline{\Omega}^*) \) solve the initial value problem

\[
(\Delta + a_0^2k^2)v = \partial_1f_1 + \ldots + \partial_nf_n + kf_{n+1} + k^2f_0 \text{ in } \Omega^*(d),
\]

\[
v = 0 \text{ on } \Omega^*(h_1) \tag{2.17}
\]

for some \( h_1 < h \).

Then there is a constant \( C \) such that

\[
\|v\| \leq C((1 + \varepsilon^{-1-n/2}k^{-1})(\|f_1\| + \ldots + \|f_n\|) + \|f_{n+1}\| + \|f_0\| + \|\partial_n f_0\| + \varepsilon^{-1}k^{-1}\|v\|_1(\Omega^*) + \varepsilon(\|v\| + \|\partial_n v\|)). \tag{2.18}
\]

**Proof.** From (2.16) and from the Leibniz formula we have

\[
\Delta v(j) + k^2a_0^2(x)v(j) = \chi(j)(\partial_1f_1 + \ldots + \partial_nf_n + kf_{n+1} + k^2f_0) - 2\nabla \chi(j) \cdot \nabla v - (\Delta \chi(j))v,
\]

so

\[
\Delta v(j) + k^2a_0^2(x')v(j) = \partial_1(\chi(j)f_1) + \ldots + \partial_n(\chi(j)f_n) - \partial_1 \chi(j)f_1 - \ldots - \partial_{n-1} \chi(j)f_{n-1} + k\chi(j)f_{n+1} + k^2\chi(j)f_0 - 2\nabla \chi(j) \cdot \nabla v - (\Delta \chi(j))v + k^2((a_0^2(x') - a_0^2(x))v(j)).
\]

Applying the low frequency projector \( P_j \) to the both parts we yield

\[
\Delta P_j v(j) + k^2a_n(j)P_j v(j) = \partial_1 P_j(\chi(j)f_1) + \ldots + \partial_n P_j(\chi(j)f_n) - P_j, \chi(j)f_n) -
\]
where \( P_{j}(\partial \chi(j))f_{1} - ... - P_{j}(\partial \chi(j))f_{n-1} + P_{j}(k\chi(j)f_{n+1}) + P_{j}(k^{2}\chi(j)f_{0} + P_{j}(2\nabla^{'}\chi(j) \cdot \nabla v) + P_{j}(\Delta \chi(j)v) + + k^{2}P_{j}((a_{n}(j) - a_{0}^{2})v(j)) \),

Observing that \( |(a_{n}(j) - a_{0}^{2})| \leq C\varepsilon \) on support of \( v(j) \), that \( \|P_{j}f\| \leq \|f\| \), using (2.15), and applying Lemma 2.2 we obtain

\[
\|P_{j}v(j)\|^2 \leq C(\|\chi(j)f_{1}\|^2 + ... + \|\chi(j)f_{n}\|^2 + \varepsilon^{-2}k^{-2}(\|f_{1}\|^2 + ... + \|f_{n}\|^2) + \|\chi(j)f_{n+1}\|^2 + \|\chi(j)f_{0}\|^2 + \|\chi(j)\partial_{n}f_{0}\|^2 + \varepsilon^{-2}k^{-2}\|\nabla^{'}v\|^2(\Omega_{j}) + \varepsilon^{-4}k^{-4}\|v\|^2(\Omega_{j}) + \varepsilon^2(\|v\|^2(\Omega_{j}) + \|\partial_{n}v\|^2(\Omega_{j})).
\]

(2.19)

Now summing local estimates (2.19) we will obtain a bound for \( v_{1} \) given by (2.16). Support of \( v(j) \) intersects at most \( 2^{n} \) supports of other \( v(k) \), but this is not true for \( P_{j}v(j) \). To make certain constants be \( \varepsilon \) independent (as in (1.5), we will use that \( (I - P_{j})v(j) \) is a high frequency component of \( v(j) \) as defined by (2.16), hence

\[
\|(I - P_{j})v(j)\|^2 \leq Ck^{-2}\|v(j)\|^2_{(1)}
\]

and

\[
\|v(j)\|^2 = \|P_{j}v(j)\|^2 + \|(I - P_{j})v(j)\|^2 \leq \|P_{j}v(j)\|^2 + Ck^{-2}\|v(j)\|^2_{(1)}.
\]

Using that multiplicity of covering \( \Omega_{j} \) is at most \( 2^{n} \) and summing (2.19) over \( j = 1, ..., J \) we yield

\[
\|v\|^2 \leq C(\sum_{j=1}^{J} \|\chi(j)f_{0}\|^2 + ... + \sum_{j=1}^{J} \|\chi(j)f_{n+1}\|^2 + \sum_{j=1}^{J} \|\chi(j)\partial_{n}f_{0}\|^2 + \varepsilon^{-n}2k^{-2}(\|f_{1}\|^2 + ... + \|f_{n}\|^2) + \varepsilon^{-4}k^{-2}\|v\|^2_{(1)}(\Omega^{*}) + \varepsilon^2(\|v\|^2 + \|\partial_{n}v\|^2)).
\]

Using that \( \chi^{2}(1) + ... + \chi^{2}(J) \leq 1 \) we obtain (2.18) and complete the proof of Lemma 2.2.

### 3 Some Carleman estimates

Let

\[
w(x; \tau) = \int_{-T}^{T} \exp(2\tau e^{\alpha(|x - \beta t^{2} - \theta t^{2})}) dt
\]

(3.1)

where \( \beta = (0, ..., 0, \beta_{n}) \) is a vector to be chosen later and \( T \) is any positive number, e.g. \( T = 1 \).
Lemma 3.1 Let the condition (1.4) be satisfied. Then there is constant $C$ such that

$$
\int_{\Omega} \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) w(\cdot, \tau) \leq C \left( \int_{\partial\Omega} \left( \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) w(\cdot, \tau) \right) \right)
$$

(3.2)

for all functions $u \in H^2(\Omega_1)$ and all $\tau > C$.

Proof.

It is known [4], [10] that under the condition (1.4) there are positive $\sigma, \theta$ depending on $\Omega, a_0, \beta$ such that with

$$
\varphi(x, t) = e^{\sigma |x - \beta|^2 - \theta^2 t^2}
$$

we have the following Carleman estimate for the wave operator

$$
\int_{\Omega \times (-T, T)} \left( \tau^3|U|^2 + \tau|\nabla U|^2 + \tau|\partial_t U|^2 \right) e^{2\tau \varphi} \leq C \left( \int_{\Omega \times (-T, T)} \left( (\Delta - a_0^2 \partial_t^2)|U|^2 \right) e^{2\tau \varphi} \right)
$$

(3.3)

We will apply (3.3) to the function

$$
U(x, t) = u(x)e^{ikt},
$$

(3.4)

choose large $\tau$ to absorb the integral over $\Omega \times \{-T, T\}$ by the left side of (3.3) and integrate with respect to $t$ to obtain the weight function $w$.

From the definition (3.4),

$$
\nabla U(x, t) = \nabla u(x)e^{ikt}, \partial_t U(x, t) = ik u(x)e^{ikt},
$$

and

$$
(\Delta - a_0^2 (x) \partial_t^2)U(x, t) = (\Delta u(x) + a_0^2(x)u(x))e^{ikt}.
$$

Hence the Carleman estimate (3.3) implies that

$$
\int_{\Omega} \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) \left( \int_{-T}^{T} e^{2\tau \varphi(x, t)} dt \right) dx \leq C \left( \int_{\Omega} \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) \left( \int_{-T}^{T} e^{2\tau \varphi(x, t)} dt \right) dx + \int_{\partial\Omega} \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) \left( \int_{-T}^{T} e^{2\tau \varphi(x, t)} dt \right) dx + \int_{\partial\Omega} \left( (\tau^3 + \tau k^2)|u|^2 + \tau|\nabla u|^2 \right) \left( \int_{-T}^{T} e^{2\tau \varphi(x, t)} dt \right) dx
$$

9
\[
\int_{\Omega} (\tau^3 + \tau k^2) |u|^2 + \tau |\nabla u(x)|^2 e^{2\tau \varphi(x,T)} dx.
\] (3.5)

Now by choosing \(\tau\) large and using different growth rate in \(\tau\) of the weight function on the left side of (3.5) we will eliminate the last term in the right side. Indeed, let \(E > 0\). Due to its definition
\[
\varphi(x, t) - \varphi(x, T) = e^{\sigma|x-\beta|^2} (e^{-\theta t^2} - e^{-\theta T^2}) > \varepsilon_1(\theta)
\]
when \(|t| < T/2, x \in \Omega\). Hence
\[
E < \int_{-T/2}^{T/2} e^{2\tau(\varphi(x,t) - \varphi(x,T))} dt < \int_{-T}^{T} e^{2\tau(\varphi(x,t) - \varphi(x,T))} dt
\]
when \(C(E) < \tau\). Then
\[
E e^{2\tau \varphi(x, T)} < \int_{-T}^{T} e^{2\tau \varphi(x, t)} dt
\]
provided \(C(E) < \tau\). Letting \(E = 2C\) we can absorb the last term on the right side of (3.5) by the left side.

The proof is complete.

4 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. We will choose \(\beta_n = -(2r^2 - 3/8 \, d), \beta = (0, ..., 0, \beta_n)\) and we introduce the notation \(\Omega(d) = \Omega \cap \{(d - \beta_n)^2 < |x - \beta|^2\}\). Using our choice of \(\beta\) and considering the intersection of level surface \(|x - \beta|^2 = (1/2 \, d - \beta_n)^2\) with the lateral wall \(|x'| = r\) of the cylindrical domain one can be convinced that the boundary layer \(\{x_n < 1/2 \, d\} \cap \Omega\) does not intersect \(\Omega(\frac{d}{2})\). Hence there is a cut-off function \(\chi\) which is 1 on \(\Omega(\frac{d}{2})\), zero near \(\partial \Omega \cap \{x_n = 0\}\) and which satisfy the bounds \(|\nabla \chi| \leq C d^{-1}, |\Delta \chi| \leq C d^{-2}\).

Applying Lemma 3.1 to \(\chi u\) instead of \(u\) and shrinking the domain in the norms on the left side of (3.2) we get
\[
\int_{\Omega(d)} ((\tau^3 + \tau k^2) |u|^2 + \tau |\nabla u|^2)w(\tau) \leq
\]
\[
C(\int_{\Omega} |f|^2 w(\tau) + \int_{\Omega \cap \Omega(\frac{d}{2})} |\nabla \chi \cdot \nabla u + (\Delta \chi) u|^2 w(\tau) + \int_{\Gamma} ((\tau^3 + \tau k^2) |u|^2 + \tau |\nabla u|^2 + \tau |\nabla \chi u|^2)w(\tau)),
\] (4.1)
where we used that \(\chi = 1\) on \(\Omega(\frac{d}{2})\) and utilized the triangle inequality.

Let
\[
b = e^{\sigma \chi^2}, \, b_1 = e^{\sigma |d - \beta_n|^2}, \, b_2 = e^{\sigma |\frac{d}{2} - \beta_n|^2},
\]
where \( X = \sup |x + \beta| \) over \( x \in \Omega \),

\[
W(\tau) = \int_{-T}^{T} e^{2\tau b_2 e^{-\alpha t^2}} dt, \quad w_1(\tau) = \int_{-T}^{T} e^{2\tau b_1 e^{-\alpha t^2}} dt, \quad w_2(\tau) = \int_{-T}^{T} e^{2\tau b_2 e^{-\alpha t^2}} dt.
\]

Observing that \( w_1 \leq w \) on \( \Omega(d) \), \( w \leq W \) on \( \Omega \), and \( w \leq w_2 \) on \( \Omega \setminus \Omega(d) \) and replacing \( w \) by its minimal value in the left side and by maximal values on the right side of (4.1) we yield

\[
\tau^3 w_1(\tau) \| u \|^2(\Omega(d)) + \tau w_1(\tau) \| \nabla u \|^2(\Omega(d)) \leq C(W(\tau)(\| f \|^2(\Omega) + (\tau^3 + \tau(k^2 + d^{-2})\| u \|^2(\Gamma) + \tau \| \nabla u \|^2(\Gamma)) + d^{-4} w_2(\tau)(\| \nabla u \|^2(\Omega) + \| u \|^2(\Omega))).
\]

Dividing the both parts of this inequality by \( w_1 \) we obtain

\[
\tau^3 \| u \|^2(\Omega(d)) + \tau \| \nabla u \|^2(\Omega(d)) \leq C(W(\tau)w_1^{-1}(\tau)(\| f \|^2(\Omega) + (\tau^3 + \tau(k^2 + d^{-2})\| u \|^2(\Gamma) + \tau \| \nabla u \|^2(\Gamma)) + d^{-4} w_2(\tau)w_1^{-1}(\tau)(\| \nabla u \|^2(\Omega) + \| u \|^2(\Omega))).
\]

(4.2)

Obviously,

\[
W(\tau)w_2^{-1}(\tau) \leq Ce^{C/d^2}.\]

A crucial observation is that

\[
w_2(\tau)w_1^{-1}(\tau) \leq Ce^{-\tau/2}.
\]

Indeed, from the definition of \( b_j \) and \( \beta \) by elementary calculations

\[
b_1 - b_2 = e^{\sigma(2r^2 - d^2/(2r) + (2r^2 - d^2)/8)}(e^{\sigma(3d^2/8 + 2r^2 - 1)} - 1) \geq C^{-1},
\]

and therefore

\[
w_1(\tau) \geq \int_{-T}^{T} e^{2\tau b_2 e^{-\alpha t^2}} e^{2\tau b_1 - b_2} e^{-\alpha t^2} dt \geq w_2(\tau) e^{2\tau/C}.
\]

Hence from (4.2) we have

\[
\| u \|^2(\Omega(d)) + \| \nabla u \|^2(\Omega(d)) \leq C(e^{C/d^2} e^{C/d^2} F^2(k) + e^{-C/d^4} M^2).
\]

(4.3)

By increasing \( C \) we can eliminate \( \tau^3 \) in the right side.

Now we will try to minimize this expression by equalizing the two terms, which leads to the choice

\[
\tau = \frac{C}{Ce^{C/d^2} + 1} \frac{M}{d^2 F(k)}.
\]
Due to remark after (4.3) the right side in this inequality is getting
\[ Cd^{-4}F(k)^2 \lambda^2 M^2(1-\lambda), \quad \lambda = \frac{1}{Ce^{C/d^2} + 1} \]
and we obtain (1.7).

The proof is complete.

**Proof of Theorem 1.1.**

Since \( \Gamma \) is Lipschitz, by known extension theorems there is a function \( u^* \) such that \( u = u^* \), \( \nabla u = \nabla u^* \) on \( \Gamma \) and
\[
\| u^* \|_{(1)}(\Omega^*(0)) \leq C(\| u \|_{(\Gamma)} + \| \nabla u \|_{(\Gamma)}) \leq CF, \tag{4.4}
\]
where we used the definition of \( F \). Let \( v = u - u^* \) on \( \Omega \) and \( v = 0 \) on \( \Omega^*(0) \setminus \Omega \). It suffices to obtain the bound (1.5) for \( v \) instead of \( u \). Observe that
\[
\Delta v + a_0^2 k^2 v = f + f^* - k^2 u^* \text{ in } \Omega^*(0). \tag{4.5}
\]
where
\[
f^* = -\text{div}(\nabla u^*).\]

Since \( v \) is zero outside some cylinder by using known results about \( H^1 \)-approximation of energy solutions by \( H^2 \)-solutions we can assume that \( v \in H^2(\mathbb{R}^{n-1} \times (0, h)) \) and hence \( f^* = \partial_1 f_1 + \ldots + \partial_n f_n + f_{n+1} \) with \( \| f_j \| \leq CF \). By (4.5) and Lemma 2.2
\[
\| v \|((\mathbb{R}^{n-1} \times (d, h)) \leq C((1 + \varepsilon^{-n/2-1}k^{-1})F + F + \varepsilon^{-2}k^{-1}(\| u \|_{(1)}(\Omega(d) \cap \Omega) + F) + \varepsilon(\| u \|_{(1)}(\Omega) + F))
\]

From this bound and from (1.7) we obtain the needed bound (1.5) for \( v \) and complete the proof.

## 5 Conclusion

At our opinion, difficulties in theory and applications of many important inverse problems are due to their notorious (exponential) instability. So any way to increase stability is indeed valuable. While increasing stability with the wave number is observed experimentally in several basic inverse problems, before there was no theoretical explanation.

One of next natural questions is to trace the dependence of constants on the distance \( d \) and to study stability in the whole domain \( \Omega \). For example, we expect that \( C \leq C_0 k^{-4} \). To show it we need more detailed Carleman estimates. We expect that this increased stability is more dramatic in the three-dimensional case, when the data are given at a larger distance from \( \Gamma_0 \), when singularities of the solution are distributed over \( \Gamma_0 \) and certainly for large frequencies. Accordingly, the most stable solution (for the same space geometry as in section 1) is anticipated in the time

12
domain (i.e. when the Helmholtz equation is replaced by the wave equation) provided the initial data are zero. In near future we plan to study this issue theoretically and to link it to the increased stability for the Helmholtz equation and to the (largely open) problem of the exact controllability in a subdomain. Observe that the exact controllability in the whole domain is relatively well understood [5], [8].

The author already showed increased stability of recovery a potential in the Schroedinger equation $(-\Delta - k^2 + c(x))u = 0$ from its Dirichlet-to-Neumann map. The results were presented at the international conferences Applied Inverse Problems 2005 in Cirencester, England. The paper with complete proofs using complex geometrical optics technique and some sharp estimates of regular fundamental solutions of operators with constant coefficients [4] is in preparation. Probably, it is harder to show increased stability for the coefficient $a_0$ in the equation $(-\Delta - k^2 a_0^2(x))u = 0$. At present, methods of (complex) geometrical optics do not look promising, and there is no good alternative. The next natural step is to obtain similar estimates for the inverse scattering problems by obstacles and by the medium. It is clear that one has to impose some (pseudo)convexity condition on unknown coefficients or obstacles.

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References


