Increased stability in the continuation of solutions to the Helmholtz equation

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Abstract

In this paper we give analytical and numerical evidence of increasing stability in the Cauchy Problem for the Helmholtz equation when frequency is growing. This effect depends on convexity properties of the surface where the Cauchy Data are given. Proofs use Carleman estimates and the theory of elliptic boundary value problems in Sobolev spaces. Our numerical testing is handling the nearfield acoustical holography and it is based on the single layer representation algorithm.

1 Introduction

The study of the uniqueness in the Cauchy problem for partial differential equations has a long history. For theory and applications a stability issue is of fundamental importance. Needs of prospecting by acoustical and electromagnetic waves (in particular, of the inverse scattering theory) stimulate study of this problem for the Helmholtz equation

$$(\Delta + k^2)u = f \text{ in } \Omega \tag{1.1}$$

with the Cauchy data

$$u = u_0,\ \partial_\nu u = u_1 \text{ on } \Gamma \tag{1.2}$$

where Ω be a domain in $\mathbf{R^n}$ and $\Gamma \in C^1$ be an (open) part of $\partial\Omega$. Due to the celebrated results of Fritz John [6] in general case one can expect only quite weak logarithmic stability. However in several important practical examples (e.g. in computations for nearfield acoustical holography [1]) it was observed that stability (and, as a consequence, resolution) in the Cauchy Problem (and hence in related inverse problems) is increasing with k. The goal of this paper is to give stability estimates explaining this phenomenon and to show that one can observe it in numerical solution of some important applied problems.

This paper is organized as follows. In Section 1, we describe the current state of the problem and our main results. Another key ingredient in the proof of the stability and uniqueness for the Cauchy problem is a Carleman type estimate for 1.1 which does not depend on k. In Section 2 we will derive this estimate and prove on its base Theorem 1.1 . In Section 3 we describe results of the numerical experiments.

Let Ω be an open subset of the cylinder $\{0 < x_n < h, |x'| < r\}$ with the Lipschitz boundary $\partial\Omega$, $\bar{\Omega} \subset \{x_n < h\}$ and Γ be the part of $\partial\Omega$ contained in the layer $\{0 < x_n < h\}$. Let $\Omega(d) = \Omega \cap \{d < x_n\}$ and $\Omega^*(d) = \mathbf{R}^{n-1} \times (d, h)$. Let $e_n = (0, ..., 0, 1)$. By C we denote constants which depend only on n, r, h. $\|u\|_{(l)}(\Omega)$ is the standard norm in the Sobolev space $H_{(l)}(\Omega)$ and $\|u\|(\Omega) = \|u\|_{(0)}(\Omega)$. We let $M = \|u\|_{(1)}(\Omega)$, $F = \|f\|(\Omega) + \|u\|(\Gamma) + \|\nabla u\|(\Gamma)$ and $F(k, d) = \|f\|(\Omega) + (k + d^{-1})\|u\|(\Gamma) + \|\nabla u\|(\Gamma)$. $V(\xi, x_n)$ denotes the (partial) Fourier transformation $\mathcal{F}\sqsubseteq(\xi, \S_\backslash)$ of a function $v(x)$.

Since we are interested in increasing wave numbers k, for simplicity of the exposition we will assume that

$$1 \leq k. \tag{1.3}$$

Theorem 1.1 *There is a constant C such that for any solution to the Cauchy problem (1.1), (1.2) one has*

$$\|u\|(\Omega(d)) \leq C(F + \frac{M^{1-\lambda} F(k, d)^\lambda}{d^{2-2\lambda} k}) \tag{1.4}$$

where

$$\lambda = \frac{2r^2 d + \frac{3}{8} d^3}{4r^2 h + h^2 d + \frac{5}{4} d^2 h + \frac{3}{8} d^3 + 3r^2 d}$$

Observe that

Theorem 1.1 permits to consider more general domains Ω. Let S be a compact subset of Ω. By $P(\nu; d)$ we denote the half-space of $\mathbf{R}^n$ with the exterior normal ν which has the distance d from S. By γ we denote all ν such that $P(\nu; d) \cap \partial\Omega$ is contained in Γ. Let $\Omega(\nu; \Gamma, d)$ be $P(\nu; d) \cap \Omega$ and, finally, let $\Omega(\Gamma, d)$ be the union of all such $\Omega(\nu; \Gamma, d)$ over $\nu \in \gamma$. If $\Gamma = \partial\Omega$ then $\Omega(\Gamma, 0)$ is the difference of Ω and of the convex hull of Γ and $\Omega(\Gamma, d)$ is the collection of points of $\Omega(\Gamma, 0)$ which are at distance d from S. Applying Theorem 1.1 to any $\Omega = \Omega(\nu; \Gamma, d), \nu \in \gamma$ we obtain

Corollary 1.1 *The bound (1.4) with $\Omega(\Gamma, d)$ instead of $\Omega(d)$ is valid.*

There is an important particular case when the norm of the data does not explicitly depend on k. Let us keep the notation of Corollary 1.1. Let ω be an open subset of Ω with $\Gamma \subset \partial\omega$ (boundary layer) such that Γ is at the distance d_0 from $\partial\omega \cap \Omega$. Let $F_\omega = \|f\|(\Omega) + \|u\|_{(1)}(\omega)$.

Corollary 1.2 *We have*

$$\|u\|(\Omega(\Gamma, d)) \leq C(d_0)(F_\omega + \frac{M^{1-\lambda} F_\omega^\lambda}{d^{2-2\lambda} k}). \tag{1.5}$$

To derive this Corollary from Corollary 1.1 we let χ to be a cut off function which is 1 on $\Omega \setminus \omega$ and 0 near Γ. Applying Corollary 1.1 to χu instead of u and using that $(\Delta + k^2)(\chi u) = \chi((\Delta + k^2)u) + 2\nabla\chi \cdot \nabla u + \Delta\chi u = \chi f + 2\nabla\chi \cdot \nabla u + \Delta\chi u$ and that the function χu has zero Cauchy data on Γ (1.5) follows from Corollary 1.1.

Theorem 1.1 and its corollaries show an improved stability in the Cauchy Problem (1.1), (1.2) when one continues the solution to the differential equation inside the convex hull of Γ. Due to the results of John [6] this is impossible when one continues to the outside of a convex Γ.

Our proof of Theorem 1.1 is based on the following

Theorem 1.2 *One has*

$$\|u\|_{(1)}(\Omega(d)) \leq CF(k,d)^{\lambda}(\frac{M}{d^2})^{1-\lambda} \tag{1.6}$$

which is of its own interest due to independence of C on k.

Theorems 1.1, 1.2 and their corollaries are due to Victor Isakov while the numerical example is handled by Tomasz Hrycak.

2 Proofs of auxiliary energy type estimates

We start with a simple result imitating the standard energy estimate for hyperbolic initial value problems.

Lemma 2.1 *Let $v_1 \in C^{\infty}(\bar{\Omega}^*$ solve the initial value problem*

$$(\Delta + k^2)v_1 = \partial_1 f_1 + ... + \partial_n f_n + f_{n+1} + k^2 f_0 \textit{ in } \Omega^*(d),$$

$$v = 0 \textit{ on } \Omega^*(h_1) \tag{2.1}$$

for some $h_1 < h$, $f_j \in C^{\infty}(\bar{\Omega}^*(d))$, $f_j = 0$ on $\Omega(h_1)$, and

$$V_1(\xi, x_n) = 0 \textit{ when } \frac{1}{2}k < |\xi| \tag{2.2}$$

Then

$$\|v_1\|(\Omega^*(d)) \leq$$
$$C(\|f_1\|(\Omega^*(d)) + ... + \|f_{n+1}\|(\Omega^*(d)) + \|f_0\|_{(1)}(\Omega^*(d)) \tag{2.3}$$

Proof. Due to Parseval's identity it suffices to show that solutions to the initial value problems

$$\partial_n^2 V_j + (k^2 - |\xi|^2)V_j = -i\xi_j F_j \text{ on } (d,h), j = 1, ..., n-1, \tag{2.4}$$

$$\partial_n^2 V_n + (k^2 - |\xi|^2)V_n = \partial_n F_n \text{ on } (d,h), \tag{2.5}$$

$$\partial_n^2 V_{n+1} + (k^2 - |\xi|^2)V_{n+1} = F_{n+1} \text{ on } (d,h) \tag{2.6}$$

$$\partial_n^2 V_0 + (k^2 - |\xi|^2)V_0 = k^2 F_0 \text{ on } (d,h) \tag{2.7}$$

with the zero final conditions

$$V_j = 0,\ F_j = 0 \text{ on } (h_1, h), \tag{2.8}$$

satisfy the bounds

$$\int_d^h |V_j|^2(\xi,s)ds \leq C \int_d^h |F_j|^2(\xi,s)ds,\ j = 1, ..., n+1. \tag{2.9}$$

$$\int_d^h |V_0|^2(\xi,s)ds \leq C \int_d^h (|\partial_n F_0|^2(\xi,s) + |F_0|^2(\xi,s)ds) \tag{2.10}$$

Obviously if suffices to obtain bounds (2.9),(2.10) for real and imaginary parts of solutions via real and imaginary parts of the right sides. In other words, we can assume that V_j, iF_j, F_n, F_0 are real-valued.

Let us first consider $j = 1, ..., n-1$. Multiplying the both sides of (2.4) by $\partial_n V_j e^{x_n}$, using that

$$(\partial_n^2 V_j)\partial_n V_j = \frac{1}{2}\partial_n(\partial_n V_j)^2,\ (\partial_n V_j)V_j = \frac{1}{2}\partial_n(V_j^2)$$

and integrating by parts over the interval (x_n, h) with use of (2.8) we obtain

$$-\frac{1}{2}(\partial_n V_j)^2(,x_n)e^{x_n} - \frac{1}{2}\int_{x_n}^h (\partial_n V_j)^2(,s)e^s ds -$$

$$\frac{1}{2}(k^2 - |\xi|^2)V_j^2(,x_n)e^{x_n} - \frac{1}{2}(k^2 - |\xi|^2)\int_{x_n}^h V_j^2(,s)e^s ds =$$

$$\int_{x_n}^h (-i\xi F_j)\partial_n V_j(,s)e^s ds$$

Multiplying the both sides by -1, using the elementary inequality $i\xi_j F_j \partial V_j \leq \frac{1}{2}(|\xi|^2|F_j|^2 + |\partial_n V_j|^2)$, dropping the first and the third term in the left side in the previous equality and cancelling the second term in the left side and the second term in the right side we yield

$$\frac{1}{2}\int_{x_n}^h (k^2 - |\xi|^2)V_j^2(,s)e^s ds \leq \int_{x_n}^h |\xi|^2|F_j|^2(,s)e^s ds.$$

Replacing e^s by its minimum 1 in the left side and by its maximum e^h in the right side and using that $|\xi| < \frac{k}{2}$ we obtain (2.9) with $C = e^h$.

Integrating the equation (2.5) over (x_n, h) and using the final conditions (2.8) we obtain

$$-\partial_n V_n(,x_n) + (k^2 - |\xi|^2)\int_{x_n}^h V_n(,s)ds = -F_n(,x_n),$$

Multiplying this equation by $V_n(,x_n)$ we will have

$$-\frac{1}{2}\partial_n(V_n^2)(,x_n) - (k^2-|\xi|^2)\frac{1}{2}\partial_n(\int_{x_n}^h V_n(,s)ds)^2 = -F_nV_n(,x_n)$$

and integrating again over (x_n,h) we arrive at

$$\frac{1}{2}V_n^2(,x_n) + \frac{1}{2}(k^2-|\xi|^2)(\int_{x_n}^h V_n(,s)ds)^2 =$$

$$-\int_{x_n}^h F_nV_n(,s)ds \leq \frac{1}{2(k^2-|\xi|^2)h}\int_{x_n}^h F_n^2(,s)ds + \frac{(k^2-|\xi|^2)h}{2}\int_{x_n}^h V_n^2(,s)ds.$$

Using the Cauchy-Schwarz inequality the second term in the left side is less than the second term on the right side. Dropping these terms and integrating over $(0,h)$ and using that due to (1.3) and (2.2) $\frac{1}{k^2-|\xi|^2} \leq \frac{4}{3k^2} \leq \frac{4}{3}$ we obtain (2.9) with $C = \frac{4}{3}$.

Thebound for (2.9) for V_{n+1} is similar to the previous bounds.

Our proof of the bound (2.9) for V_0 needs some modifications to the previous argument. Indeed, the repetition of the argument for V_j gives

$$-\frac{1}{2}(\partial_nV_0)^2(,x_n)e^{x_n} - \frac{1}{2}\int_{x_n}^h(\partial_nV_0)^2(,s)e^s ds -$$

$$\frac{1}{2}(k^2-|\xi|^2)V_0^2(,x_n)e^{x_n} - \frac{1}{2}(k^2-|\xi|^2)\int_{x_n}^h V_0^2(,s)e^s ds =$$

$$\int_{x_n}^h(k^2F_0)\partial_nV_0(,s)e^s ds =$$

$$-k^2F_0V_0(,x_n)e^{x_n} + k^2\int_{x_n}^h(\partial_nF_0+F_0)V_0(,s)e^s ds$$

Multiplying the both sides by -1, dropping the first two terms in the left side, and using the elementary inequalities

$$F_0V_0 \leq (F_0^2 + \frac{1}{4}V_0^2),\ (\partial_nF_0+F_0)V_0 \leq 2(\partial_nF_0)^2 + 2F_0^2 + \frac{1}{4}V_0^2$$

and the assumption that $|\xi| < \frac{1}{2}k$ we yield

$$\frac{3}{8}k^2V_0^2(,x_n)e^{x_n} + \frac{3}{8}k^2\int_{x_n}^h V_0^2(,s)e^s ds \leq$$

$$k^2F_0^2 + \frac{k^2}{4}V_0^2(,x_n) + k^2\int_{x_n}^h 2((\partial_nF_0)^2+F_0^2)(,s)e^s ds + \frac{1}{4}k^2\int_{x_n}^h V_0^2(,s)e^s ds.$$

which implies that

$$\frac{1}{8}V_0^2(,x_n)e^{x_n} \leq F_0^2(,x_n)e^{x_n} + 2\int_{x_n}^{h}((\partial_n F_0)^2 + F_0^2)e^s ds$$

Replacing e^s by e^h and increasing the integration interval we conclude that

$$V_0^2(,x_n) \leq 8F_0^2(,x_n) + 16e^h \int_0^h ((\partial_n F_0)^2 + F_0^2)(,s)ds$$

Integrating over $(0,h)$ we arrive at (2.10) (with $C = 8 + 16he^h$).

The proof is complete.

The following result is known for hyperbolic equations of second order [7] and for general partial differential operators [9]. Our contribution is to trace how constants depend on k and other parameters. We denote $l(x;\beta) = |x + \beta e_n|$.

Lemma 2.2 *Let $w(x) = exp(\tau(x_1^2 + ... + x_{n-1}^2 + (x_n + \beta)^2)$ where β is a real parameter. Then for some constant $C \leq C_0(n + sup\, l^3(x;\beta))$ over $x \in \Omega_1$ we have*

$$32\tau^3\|wlu\|^2(\Omega_1) + 5\tau\|w\nabla u\|^2(\Omega_1) \leq$$

$$(\|w(\Delta + k^2)u\|^2(\Omega_1) + C((\tau^3 + \tau k^2)\|wu\|^2(\partial\Omega_1) + \tau\|w\nabla u\|^2(\partial\Omega_1)) \tag{2.11}$$

for all functions $u \in H^2(\Omega_1)$ and all $\tau > 0$.

Proof.

We will use general ideas [2] of proving Carleman type estimates specifying them to our particular case.

The substitution $u = w^{-1}v$ reduces (2.11) to the following one

$$\tau^3\|lv\|^2(\Omega_1) + \tau\|\nabla v\|^2 \leq$$

$$(\|\Delta v - 4\tau(x + \beta e_n)\cdot\nabla v + (4\tau^2|x + \beta e_n|^2 - 2\tau n + k^2)v\|^2(\Omega_1) + \tag{2.12}$$
$$C((\tau^3 + \tau k^2)\|v\|^2(\partial\Omega_1) + \tau\|\nabla v\|^2(\partial\Omega_1)).$$

Obviously,

$$(\Delta v - 4\tau(x + \beta e_n)\cdot\nabla v + (4\tau^2|x + \beta e_n|^2 - 2\tau n + k^2)v)^2 \geq$$

$$(\Delta v - 4\tau(x + \beta e_n)\cdot\nabla v + (4\tau^2|x + \beta e_n|^2 - 2\tau n + k^2)v)^2 - \tag{2.13}$$
$$(\Delta v + 4\tau(x + \beta e_n)\cdot\nabla v + (4\tau^2|x + \beta e_n|^2 + 2\tau n + k^2)v)^2 =$$
$$-16\tau(\Delta v)(x + \beta e_n)\cdot\nabla v - 8\tau n v\Delta v -$$
$$16\tau(x + \beta e_n)\cdot\nabla v(4\tau^2|x + \beta e_n|^2 + k^2)v - 8\tau n(4\tau^2|x + \beta e_n|^2 + k^2)v^2.$$

Integrating by parts over Ω_1 the last expression in (2.13) and using the equalities

$$\nabla v \cdot \nabla((x+\beta e_n)\cdot\nabla v) = |\nabla v|^2 + \frac{1}{2}(x+\beta e_n)\cdot\nabla|\nabla v|^2,$$

$$(x+\beta e_n)\cdot\nabla v v = \frac{1}{2}(x+\beta e_n)\cdot\nabla v^2$$

yields

$$-16\tau\int_{\partial\Omega_1}(\partial_\nu v)(x+\beta e_n)\cdot\nabla v + 16\tau\int_{\Omega_1}|\nabla v|^2 + 8\tau\int_{\Omega_1}(x+\beta e_n)\cdot\nabla|\nabla v|^2 -$$

$$8\tau n\int_{\partial\Omega_1} v\partial_\nu v + 8\tau n\int_{\Omega_1}|\nabla v|^2$$

$$-8\tau\int_{\partial\Omega_1}(x+\beta e_n)\cdot\nu(4\tau^2|x+\beta e_n|^2+k^2)v^2 +$$

$$8\tau\int_{\Omega_1}(div((4\tau^2|x+\beta e_n|^2+k^2)(x+\beta e_n)) - 8\tau n(4\tau^2|x+\beta e_n|^2+k^2))v^2) =$$

$$\int_{\partial\Omega_1}(-16\tau\partial_\nu v(x+\beta e_n)\cdot\nabla v + 8\tau(x+\beta e_n)\cdot\nu|\nabla v|^2 - 8\tau n v\partial_\nu v -$$

$$8\tau(x+\beta e_n)\cdot\nu(4\tau^2|x+\beta e_n|^2+k^2)v^2) +$$

$$\int_{\Omega_1}(16\tau|\nabla v|^2 + 64\tau^3|x+\beta e_n|^2 v^2).$$

where we also used that

$$div((4\tau^2|x+\beta e_n|^2+k^2)(x+\beta e_n)) = 4\tau^2 2|x+\beta e_n|^2 + n(4\tau^2|x+\beta e_n|^2+k^2).$$

Summing up we conclude that

$$\int_{\Omega_1}(\Delta v - 4\tau x_n\partial_n v + (4\tau^2 - 2\tau + k^2)v)^2 \geq$$

$$-C\int_{\partial\Omega_1}(16\tau|\nabla v|^2 + (\tau^3 + k^2\tau)v^2) + \int_{\Omega_1}(16\tau|\nabla v|^2 + 64\tau^3|x+\beta e_n|^2 v^2)$$

Using that we obtain (2.12).

The proof is complete.

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.2. We will choose $\beta = \frac{2r^2}{d} - \frac{3}{8}d$ and we introduce the notation $\Omega(d) = \Omega \cap \{(d+\beta)^2 < |x'|^2 + (x_n+\beta)^2\}$. Using our choice of β and considering the intersection of level surface $|x'|^2+(x_n+\beta)^2 = (\frac{1}{2}d+\beta)^2$ with the lateral wall $\{|x'| = r\}$ of the cylindrical domain one can be convinced that the boundary layer $\{x_n < \frac{1}{4}d\}\cap\Omega$ is contained in $\Omega(\frac{d}{2})$. Hence there is a cut-off function χ which is 1 on $\Omega(\frac{d}{2})$, zero near $\partial\Omega \cap \{x_n = 0\}$ and which satisfy the bounds $|\nabla\chi| \leq Cd^{-1}, |\Delta\chi| \leq Cd^{-2}$.

Applying Lemma 2.2 to χu instead of u and shrinking the domain in the norms on the left side we get

$$32\tau^3\|wlu\|^(\Omega(d)) + 5\tau\|w\nabla u\|^2(\Omega(d)) \leq$$

$$(\|wf\|^2(\Omega) + \|w(2\nabla\chi\cdot\nabla u + \Delta\chi u)\|^2(\Omega \setminus \Omega(\frac{d}{2}))^2 +$$

$$C((\tau^3 + \tau k^2)\|wu\|^2(\Gamma) + \tau\|w\nabla u\|^2(\Gamma)) + \tau\|w\nabla\chi u\|^2(\Gamma))$$

where we used that $\chi = 1$ on $\Omega(\frac{d}{2})$ and utilized the triangle inequality. Observing that $w \leq exp(\tau(\frac{d}{2}+\beta)^2)$ on $\Omega \setminus \Omega(\frac{d}{2})$ and $exp(\tau(d+\beta)^2 \leq w$ on $\Omega(d)$ and replacing w by uts minimal value in the left side and by maximal values on the right side of the previous inequality between norms of u we yield

$$32\tau^3 e^{2\tau(d+\beta)^2}\|lu\|^2(\Omega(d)) + 5\tau e^{2\tau(d+\beta)^2}\|\nabla u\|^2(\Omega(d)) \leq$$

$$e^{2\tau(r^2+(h+\beta)^2)}(\|f\|^2(\Omega) + C(\tau^3+\tau k^2)\|u\|^2(\Gamma) + \tau\|\nabla u|^2(\Gamma)) +$$

$$\frac{C}{d^4}e^{2\tau(\frac{d}{2}+\beta)^2}(\|\nabla u\|^2(\Omega) + \|u\|^2(\Omega)) + \tau\|u\|^2(\Gamma)).$$

Dividing the both parts by $exp(2\tau(d+\beta)^2)$ and using that $\tau^3 exp(-2\tau(h+d+2\beta)d \leq C$ we obtain

$$32\tau^3\|lu\|^2(\Omega(d)) + 5\tau\|\nabla u\|^2(\Omega(d)) \leq$$

$$e^{2\tau((h+d+2\beta)h+r^2)}(\|f\|^2(\Omega) + (k^2+d^{-2})\|u\|^2(\Gamma) + \|\nabla u\|^2(\Gamma)) +$$

$$Ce^{-2\tau(2r^2+\frac{3}{8}d^2)}d^{-4}\|u\|^2_{(1)}(\Omega)) \tag{3.1}$$

where we regrouped terms and used that $(\frac{d}{2}+\beta)^2 - (d+\beta)^2 = -(d\beta + \frac{3}{4}d^2) = 2r^2 + \frac{3}{8}d^2$ due to the choice of β. Denoting the factor of the positive exponent in the right side by F^2 and the factor of the negative exponent by M^2 we will write this right side as

$$e^{2\tau(h^2+dh+2\beta h+r^2)}F^2 + e^{-2\tau(2r^2+\frac{3}{8})}M^2.$$

Now we will try to minimize this expression by equalizing the two terms, which leads to the choice

$$\tau = \frac{1}{h^2 + \frac{5}{4}dh + \frac{3}{8}d^2 + \frac{4r^2h}{d} + 3r^2} ln\frac{M}{F}$$

Hence the right side in (3.1) is getting

$$CF^{2\lambda}M^{2(1-\lambda)},\ \lambda = \frac{2r^2+\frac{3}{8}d^2}{h^2+\frac{5}{4}dh+\frac{3}{8}d^2+\frac{4r^2h}{d}+3r^2}$$

and we obtain (1.6).

The proof is complete.

Proof of Theorem 1.1.

Since Γ is Lipschitz, by known extension theorems there is a function u^* such that $u=u^*, \nabla u = \nabla u^*$ on Γ and

$$\|u^*\|_{(1)}(\Omega^*(0)) \leq C(\|u\|(\Gamma)+\|\nabla u\|(\Gamma)) \leq CF, \tag{3.2}$$

where we used the definition of F. Let $v=u-u^*$ on Ω and $v=0$ on $\Omega^*(0)\setminus\Omega$. It suffices to obtain the bound (1.4) for v instead of u. Observe that (in the weak sense)

$$\Delta v + k^2 v = f^* - k^2 u^* \text{ in } \Omega^*(0). \tag{3.3}$$

where f^* is the linear continous functional (an element of $H_{(-1)}(\Omega^*)$) on $H_{(1)}(\Omega^*(0)$ defined as

$$f^*(w) = -\int_{\partial\Omega}\partial_\nu u^* w + \int_\Omega (fw+\nabla u^*\cdot\nabla w).$$

Observe that due to known trace theorems

$$\|f^*\|_{(-1)}(\Omega^*) \leq C\|u^*\|_{(1)}(\Omega^*) \leq CF \tag{3.4}$$

Indeed,

$$\int_{\Omega^*(0)}(-\nabla u\cdot\nabla w + k^2 vw) + \int_{\Gamma_0}\partial_\nu v w =$$

$$\int_\Omega(-\nabla u\cdot\nabla w+k^2uw) - \int_\Omega(-\nabla u^*\cdot\nabla w + k^2u^*w) + \int_{\Gamma_0}(\partial_\nu uw - \partial_\nu u^* w) =$$

$$-\int_\Gamma \partial_\nu uw + \int_\Omega fw + \int_\Omega(\nabla u^*\cdot\nabla w - k^2u^*w) - \int_{\Gamma_0}\partial_\nu u^* w$$

where we used that u is a (weak) $H_{(1)}(\Omega)$-solution to the Helmholtz equation in Ω. Using that $\nabla u = \nabla u^*$ on Γ and regrouping the terms in the right side we obtain (3.3).

For technical reasons it is convenient to replace elements of negative Sobolev spaces by smooth functions. Let ϕ_ε be the standard mollyfying kernel and $v(;\epsilon) = v*\phi_\varepsilon$. We can choose positive ε so small that $\|v-v(;\varepsilon)\|_{(1)} \leq CF$. Since convolutions commute with differentiations from (3.3) it follows that

$$(\Delta+k^2)v(;\varepsilon) = f^*(;\varepsilon) - k^2u^*(;\varepsilon) \text{ on } \Omega^*(\frac{d}{2}) \tag{3.5}$$

From (3.4) and from well-known properties of mollifiers we have

$$\|f^*(;\varepsilon)\|_{(-1)}(\Omega^*(\frac{d}{2}) + \|u^*(;\varepsilon)\|_{(1)}(\Omega^*(\frac{d}{2}) \leq CF$$

From basic results about negative Sobolev spaces

$$f^*(;\varepsilon) = \partial_1 f_1^* + ... + \partial_n f_n^* + f_{n+1}^*,\ \|f_j^*\|(\Omega^*(\frac{d}{2}) \leq CF$$

Using millifiers again we can assume that

We will split $v(;\varepsilon)$ into the low frequency component v_1 and the high frequency component v_2 as follows. Let the cut off function $\chi_k(\xi') = 1$ when $|\xi| \leq \frac{k}{2}$ and zero for other $\xi' \in \mathbf{R}^{n-1}$. We let $v_1 = \mathcal{F}^{-1}\chi_k\mathcal{F}v(;\varepsilon)$ and $v_2 = v - v_1$. Since the Fourier transformation commutes with differentiations By (3.3) and Lemma 1.1

$$\|v_1\|(\mathbf{R}^{n-1} \times (0,h)) \leq C\|u^*\|_{(1)}(\mathbf{R}^{n-1} \times (0,h)) \leq CF \tag{3.6}$$

Due to the definition of v_2 and to the elementary properties of the Fourier transform,

$$\|v_2\|_{(1)}(\mathbf{R}^{n-1} \times (0,h)) \leq \|v\|_{(1)}(\mathbf{R}^{n-1} \times (0,h)) \leq \|u\|_{(1)}(\Omega) + \|u^*\|_{(1)}(\Omega) \tag{3.7}$$

$$\|v_2\|(\mathbf{R}^{n-1} \times (0,h)) \leq \frac{2}{k}\|v_2\|(\mathbf{R}^{n-1} \times (0,h)). \tag{3.8}$$

From (3.8), (3.7) and (3.2) we have

$$\|v_2\|(\Omega) \leq \frac{1}{k}(\|u\|_{(1)}(\Omega) + CF).$$

From this bound and from (1.6) we obtain the needed bound (1.4) for v and complete the proof.

4 Numerical evidence

To illustrate our theoretical developments, we have performed numerical case studies of reconstruction of vibration of a surface from its (nearfield) acoustical pressure in two dimensions. More detail and various reconstruction procedures for this problem are given e.g. in [1], [5], [10].

We consider three concentric arcs Γ_0, Γ_1 and Γ_2 defined as follows:

$$\Gamma_0 = \{x : \|x\| = r_0, \theta_1 < \arg(x) < \theta_2\}, \tag{4.1}$$

$$\Gamma_1 = \{x : \|x\| = r_1, \phi_1 < \arg(x) < \phi_2\}, \tag{4.2}$$

$$\Gamma_2 = \{x : \|x\| = r_2, \phi_1 < \arg(x) < \phi_2\}. \tag{4.3}$$

Figure 1 shows the arcs for an exterior problem with the parameters $r_0 = 1$, $\theta_1 = \frac{2\pi}{3}$, $\theta_1 = \frac{4\pi}{3}$, $r_1 = 1.1$, $r_2 = 1.2$, $\phi_1 = \frac{\pi}{2}$, $\phi_2 = \frac{3\pi}{2}$. Figure 2 shows the setup for an interior problem with the parameters $r_1 = 0.8$, $r_2 = 0.9$.

Our goal is to determine the (outward) normal velocity $v = \partial_\nu u$ on the arc Γ_0 from the acoustic pressure measurements u on the arcs Γ_1 and Γ_2. These measurements imitate either the Cauchy data (1.2) or the data of Corollary 1.2 where ω is the domain bounded by the arcs Γ_1, Γ_2 and by the intervals joining the endpoints of these arcs.

For $\|x\| \neq r_0$, we represent $u(x)$ via the single layer potential

$$u(x) = Sg(x) := \int\limits_{\|y\|=r_0} G(x,y)g(y)\, d\Gamma(y), \tag{4.4}$$

where $G(x,y)$ is the free space Green's function for the Helmholtz equation with the wave number k,

$$G(x,y) = \frac{i}{4}\, H_0^{(1)}(k|x-y|). \tag{4.5}$$

The vailidity of this representation is discussed in [1].

The charge density g is computed from the pressures $u(x)$ on the arcs Γ_1 and Γ_2, and then the normal velocity on the arc Γ_0 is computed for the interior problem from the formula

$$v(x) = \frac{\partial u}{\partial \nu}(x) = \frac{1}{2}\, g(x) + Kg(x), \tag{4.6}$$

where the operator K is defined by

$$Kg(x) = \int\limits_{\|y\|=r_0} \nabla_x G(x,y) \cdot \nu(x)\; g(y)\, d\Gamma(y). \tag{4.7}$$

In case of the exterior problem, we use the formula

$$v(x) = -\frac{1}{2}\, g(x) + Kg(x). \tag{4.8}$$

The operators S and K are approximated by the n-point Nyström method with spectral accuracy, which gives rise to matrices S_n and K_n.

We assume that u is contaminated with noise at the level $\delta > 0$. Thus in our experiments we do not use the exact pressures u, but rather noisy measurements $u^\delta = u + \xi$, where ξ is a random vector with $\|\xi\|_2 = \delta \|u\|_2$.

We first create a sequence of approximate charge densities φ_m, $(m = 1, 2, \ldots)$ by applying the Conjugate Gradient method on the Normal Equations (CGNE) to the system $S_n \varphi = u^\delta$. Then we compute a sequence of regularized normal velocities $v_m = K_n g_m$. We select the iteration number m_0, which minimizes the *relative* L^2 error of the normal velocities given by

$$\mathrm{err}_m = \frac{\|v - v_m\|_2}{\|v\|_2}. \tag{4.9}$$

Figure 1: The experimental setup for the exterior problems.

1.0	(-0.83149,	0.34442)
4.0	(-0.88271,	0.17558)
5.0	(-0.90000,	0.00000)
2.0	(-0.88271,	-0.17558)
3.0	(-0.83149,	-0.34442)

Table 1: The strengths and locations of the acoustic charges for the exterior problems.

This minimal error is denoted by err_0, i.e

$$\mathrm{err}_0 = \min_m \ \mathrm{err}_m. \tag{4.10}$$

Specific details are discussed in Examples 1 and 2 below.
Example 1. We first treat two exterior problems, where the arcs Γ_1 and Γ_2 are, respectively, described by equations (4.2), (4.3) with parameters $\phi_1 = \frac{\pi}{2}$, $\phi_2 = \frac{3\pi}{2}$. We consider two choices for the radii: $r_1 = 1.05$, $r_2 = 1.10$ or $r_1 = 1.10$, $r_2 = 1.15$. The normal velocities are reconstructed along the arc Γ_0 given by (4.1) with parameters $r = 1$, $\theta_1 = \frac{2\pi}{3}$, $\theta_1 = \frac{4\pi}{3}$, see Figure 1. The pressure u is generated by the acoustic charges $w_1, w_2, ..., w_p$ located, respectively, at the points $x_1, x_2, ..., x_p$ according to the formula

$$u(x) = \sum_{i=1}^{p} w_i \, G(x, x_i), \tag{4.11}$$

where G is the free space Green's function for the Helmholtz operator as in (4.5). We use five charges, whose strengths and locations are given in Table 1. The exact pressures are perturbed with noise at the level $\delta = 0.01$. Specifically, we compute

$$u^\delta = u + \delta \|u\|_2 \, \frac{\xi}{\|\xi\|_2}, \tag{4.12}$$

where the entries of vector ξ are independently sampled from the uniform distribution on the interval $(-1, 1)$.

Table 2 presents the minimal relative errors of the normal velocities v_m as defined in (4.10) for wave numbers $k = 2, 4, 8, 16$. The first column of the table contains the wave number k, followed by err_0^1, the minimal error corresponding to the choice $r_1 = 1.05$ and $r_2 = 1.10$. The third column shows err_0^2, the minimal error corresponding to $r_1 = 1.10$ and $r_2 = 1.15$. The last column is the quotient $\mathrm{err}_0^2/\mathrm{err}_0^1$. In order to compare the errors for different wave numbers, each relative error err_0^2, err_0^1 is the averge over 10 experiments.
Example 2. We now consider two interior problems, described by the radii: $r_1 = 0.90$, $r_2 = 0.95$ or $r_1 = 0.85$, $r_2 = 0.90$, see Figure 4. The arc Γ_0 is unchanged from the

k	err_0^1	err_0^2	$\mathrm{err}_0^2/\mathrm{err}_0^1$
2.0	0.053	0.062	1.17
4.0	0.042	0.054	1.29
8.0	0.031	0.051	1.65
16.0	0.016	0.030	1.88

Table 2: Relative errors of the normal velocities in the exterior problems.

Figure 2: The experimental setup for the interior problems.

previous examples. To create the pressure, we use five charges, whose strengths and locations are given in Table 3.

The minimal relative errors of the normal velocities are presentsed in Table 4. The second column contains err_0^1, the minimal error corresponding to the choice $r_1 = 0.90$ and $r_2 = 0.95$. The third column shows err_0^2, the minimal error corresponding to $r_1 = 0.85$ and $r_2 = 0.90$. As before, the errors are averged over 10 experiments.

We can immediately notice that the reconstruction of the normal velocities is more accurate for the exterior problems than for the interior ones. Specifically, the minimal relative errors err_0^1, err_0^2 for the exterior problems are smaller that their counterparts for the interior ones. In both cases $\mathrm{err}_0^1 > \mathrm{err}_0^2$, which simply confirms that the reconstruction quality deteriorates as the measurements are taken farther away from the arc Γ, where the reconstruction is attempted. Comparing the last columns of Tables 2 and 4, we note, that this deterioration is more pronounced for the interior problems. Most likely this effect would be even more visible when one is using a different method for the nearfield acoustical tomography like the HELS method of Wu [5]. The point is that the a single layer representation is always possible and unique for the interior problems [1], while for the exterior problem it holds only when the wave number k^2 is not the interior Dirichlet eigenvalue for Ω. In our examples the number of these interior eigenvalues is from 2 to 20 (when $k = 16$). While we can not explain so far the improved stability for the interior problem, the stability for the exterior problem is obviously better as predicted by the theory in sections 1

1.0	(-1.01627,	0.42095)
4.0	(-1.07886,	0.21460)
5.0	(-1.10000,	0.00000)
2.0	(-1.07886,	-0.21460)
3.0	(-1.01627,	-0.42095)

Table 3: The strengths and locations of the acoustic charges for the exterior problems.

k	err_0^1	err_0^2	$\text{err}_0^2/\text{err}_0^1$
2.0	0.096	0.117	1.22
4.0	0.073	0.096	1.32
8.0	0.039	0.078	2.00
16.0	0.022	0.053	2.40

Table 4: Relative errors of the normal velocities in the interior problems.

and 2.

5 Conclusion

At our opinion, difficulties in theory and applications of many imporant inverse problems are due to their notorious (exponential) instability. So any way to increase stability is indeed valuable. While increasing stability with the wave number is observed experimentally in several basic inverse problems, before there was no theoretical explanation.

We expect that this increased stability is more dramatic in the three-dimensional case , when the data are given at a larger distance from Γ_0, when singularities of the solution are distributed over Γ_0 and certainly for large frequencies. Accordingly ,the most stable solution (for the same space geometry as in section 1) is anticipated in the time domain (i.e. when the Helmholtz equation is replaced by the wave equation) provided the initial data are zero. This stability was already observed by Sean Wu on two important acoustical examples (a private communication at the International Acoustical Conference ITCA in Honolulu, Hawaii, in August 2003). In near future we plan to study this issue theoretically and to link it to the increased stability for the Helmholtz equation and to the (largely open) problem of the exact controllabity in a subdomain. Observe that the exact controllability in the whole domain is relatively well understood [4], [7].

The next natural step is to obtain similar estimates for the inverse scattering problems by obstacles and by the medium. It is clear that one has to impose some (pseudo)convexity condition on unknown obstacles.

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