UPPER SEMICONTINUITY OF THE DIMENSIONS OF AUTOMORPHISM GROUPS OF DOMAINS IN $\mathbb{C}^n$

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Abstract. Let $\mathcal{H}^n$ be the metric space of all bounded domains in $\mathbb{C}^n$ with the metric equal to the Hausdorff distance between boundaries of domains. We prove that the dimension of the group of automorphisms of domains is an upper semicontinuous function on $\mathcal{H}^n$. We also provide theorems and examples regarding the change in topological structure of these groups under small perturbation of a domain in $\mathcal{H}^n$.

0. Introduction

The automorphism group $\mathrm{Aut}(D)$ (the group of biholomorphic self-maps of D) of a bounded domain D in $\mathbb{C}^n$ is, in general, difficult to describe and little is known about it. However, it is known (see [SZ, BD]) that any compact Lie group can be realized as the group of automorphisms of a smooth strictly pseudoconvex domain, and (see [ShT]) that any linear Lie group can be realized as the group of automorphisms of a bounded domain. So, if we consider the group $\mathrm{Aut}(D)$ as a function of D, the set of values is quite large.

If one considers this function on the metric space $\mathcal{H}^n$ of all bounded domains in $\mathbb{C}^n$ with the metric equal to the Hausdorff distance between boundaries of domains, one can expect that small perturbation of the boundary may only "decrease" the group, i.e., the function $\mathrm{Aut}(D)$ is "upper semicontinuous". Indeed, in [GK], [Ma] and [FP] the authors, using topologies different from $\mathcal{H}^n$, proved the upper semicontinuity of the function $\mathrm{Aut}(D)$ in the sense that $\mathrm{Aut}(\widetilde{D})$ is isomorphic to a subgroup of $\mathrm{Aut}(D)$ when $\widetilde{D}$ is "close" to D. But, in general, this idea is not true according to the following theorem ([FP]).

Theorem 0.1. *Let M be a domain in $\mathbb{C}^n$. Then there exists an increasing sequence of bounded domains $M_k \subset M_{k+1} \subset\subset M$ such that $M = \cup M_k$ and $\mathrm{Aut}(M_k)$ contains a subgroup isomorphic to Z_k.*

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This shows that domains in $\mathbb{C}^n$ with an automorphism group containing Z_k are everywhere dense in $\mathcal{H}^n$, and it is well known that domains without non-trivial automorphisms are dense in $\mathcal{H}^n$. So arbitrarily small perturbation of a domain in $\mathcal{H}^n$ may create a domain with a larger automorphism group. But, for all known examples, this group is discrete, so it is of dimension zero. The natural question arises: can small perturbation in $\mathcal{H}^n$ create domains with larger dimensions of automorphism groups?

In this paper we answer this question in the negative. Namely, we prove the following

Theorem 0.2. *The function* $\dim \mathrm{Aut}(D)$ *is upper semicontinuous on* $\mathcal{H}^n$.

An immediate consequence is the following

Corollary 0.3. *For each $k > 0$ the set of all domains in $\mathcal{H}^n$ whose groups of automorphisms have dimensions greater than or equal to k is closed and, therefore, nowhere dense.*

Thus a domain cannot be approximated by domains whose automorphism groups have strictly larger dimensions.

To prove Theorem 0.2 we consider a sequence of domains D_j converging in $\mathcal{H}^n$ to a domain D. The identity components $\mathrm{Aut}_0(D_j)$ of $\mathrm{Aut}(D_j)$ have the same dimensions as $\mathrm{Aut}(D_j)$. Also the dimensions of the Lie algebras of holomorphic vector fields generated by all one-parameter groups in $\mathrm{Aut}_0(D_j)$ coincide with $\dim \mathrm{Aut}_0(D_j)$. Lemma 2.4 states that the uniform norm of such fields on a compact set is bounded by its norm on an arbitrarily selected ball times a constant that, basically, depends on the size of the ball and the distance from the ball and the compact set to the boundary of a domain. This allows us to normalize bases in Lie algebras of $\mathrm{Aut}_0(D_j)$ and apply Theorem 2.5, which asserts the existence of non-trivial limits of those vector fields. The limits belong to the Lie algebra of $\mathrm{Aut}_0(D)$ and this gives us the proof.

It is reasonable to ask whether $\mathrm{Aut}_0(D_j)$ are always isomorphic to a subgroup of $\mathrm{Aut}_0(D)$ when j is large. An example in Section 3 shows that the answer is negative.

If K_j is a maximal compact subgroup of $\mathrm{Aut}_0(D_j)$, then $\mathrm{Aut}_0(D_j)$ is homeomorphic to $K_j \times \mathbb{R}^{k_j}$ (see [MZ, p. 188]). The groups K_j may decrease or even disappear in the limit (see Example 3.2), while non-compact parts never vanish (see Theorem 3.3).

1. Some basic facts

Let D be a bounded domain in $\mathbb{C}^n$. If the Lie group $\mathrm{Aut}(D)$ has positive dimension, then it has one-parameter subgroups $g(\cdot, t)$, $-\infty < t < \infty$, i.e., $g(z, t+s) = g(g(z,t), s)$. Such subgroups generate vector fields

$$X(z) = \frac{\partial g}{\partial t}(z, 0)$$

that are holomorphic. Also, if X is a holomorphic vector field on D that is $\mathbb{R}$-complete, i.e., the initial value problem $g'(z,t) = X(g(z,t))$, $g(z, 0) = z$, has a solution on $D \times \mathbb{R}$, then $g(z,t)$ is a one-parameter group.

The vector field X has the following group property:

$$X(g(z,t)) = \frac{\partial g}{\partial z}(z,t) X(z). \tag{1}$$

For every two points z and w in D among all holomorphic mappings of D into the unit disk Δ we choose holomorphic functions f such that $f(w) = 0$ and $f(z)$ is real and the maximal possible. Such functions f exist and are called Carathéodory extremal functions for z and w on D. The quantity

$$\rho(0, f(z)) = \frac{1}{2} \ln \frac{1 + f(z)}{1 - f(z)} \tag{2}$$

is called the Carathéodory distance $c_D(z, w)$ on D. (Note that the formula for $\rho(0, a)$ gives the Poincaré distance between 0 and a in the unit disc.) When D is bounded this distance is non-degenerate and invariant, i.e., $c_D(g(z), g(w)) = c_D(z, w)$ for every $g \in \mathrm{Aut}(D)$ (see [Sh, Ch. 5, §18]).

For a point $w \in D$ and a vector Y in $\mathbb{C}^n$, among all holomorphic mappings of D into the unit disk Δ we choose holomorphic functions f such that $f(w) = 0$ and $(f'(w), Y)$ is real and the maximal possible. (Here $(Z, Y) = \sum_{j=1}^n z_j y_j$.) These functions are Carathéodory extremal functions for Y at w in D. It follows from [Sh, Ch. 5, §18] that if $w(t)$ is a smooth curve in D with $w(0) = w$ and $Y = w'(0)$, then

$$c_D(w, w(t)) = C_D(w, Y)t + o(t), \tag{3}$$

where

$$C_D(w, Y) = \sup\{|g'(w)Y| : g(D) \subset \Delta, g(w) = 0\}$$

is the Carathéodory infinitesimal metric. Let $B(w, r)$ be the ball of radius r centered at w and let $|Y|$ be the Euclidean norm of Y. If

$B(w,r) \subset D \subset B(w,R)$, then

$$\frac{|Y|}{R} \le C_D(w,Y) \le \frac{|Y|}{r}. \tag{4}$$

2. Proof of Theorem 0.2

Lemma 2.1. *Let D be a domain in $\mathbb{C}^n$ and let $d(z,w)$ be an invariant metric on D satisfying the triangle inequality. If $g(z,t)$ is a group action on D, then for any $w,z \in D$*

$$|d(g(w,t),z) - d(w,z)| \le d(z, g(z,t)).$$

Proof. Apply the identity $d(w,z) = d(g(w,t), g(z,t))$ and the triangle inequality. □

Lemma 2.2. *Let $w \in B(w,r) \subset\subset D \subset\subset B(w,R) \subset\subset \mathbb{C}^n$. Then for any $Y \in \mathbb{C}^n$, $|Y| = 1$,*

$$\mathbf{Re}\,(\nabla f_s(w), Y) > \frac{1}{4R},$$

where $f_s(z)$ is a Carathéodory extremal function for w and $w+sY$ in D, and s is a real number such that

$$0 < s \le \varepsilon = \frac{r^2}{16R}.$$

Proof. Let us fix Y and introduce $D_Y = \{\xi \in \mathbb{C} : w + \xi Y \in D\}$. Clearly, $\Delta(0,r) \subset\subset D_Y \subset\subset \Delta(0,R)$, where $\Delta(0,s)$ is the disk of radius s centered at 0.

Let $g_Y(z)$ be a Carathéodory extremal function for Y at w in D. For $\xi \in D_Y$ we introduce the functions $u(\xi) = \mathbf{Re}\, F(\xi)$, where $F(\xi) = f_s(w+\xi Y)$, and $v(\xi) = \mathbf{Re}\, G(\xi)$, where $G(\xi) = g_Y(w+\xi Y)$. All these functions are well-defined on D_Y and $F(0) = G(0) = 0$, $G'(0) = C_D(w,Y)$ and $u(s) = F(s) \ge v(s)$. Let us prove that $v(t) \ge t/(2R)$ when $t \in [0,\varepsilon]$. Since

$$|v(t) - v'(0)t| \le \frac{1}{2} \sup_{0 \le x \le \varepsilon} |v''(x)| \cdot t^2,$$

$\varepsilon \le r^2/(16R) < r/2$ and by Cauchy estimate $|v''(x)| \le 2/(r-\varepsilon)^2$ when $x < \varepsilon$, we see that

$$v(t) \ge v'(0)t - \frac{1}{(r-\varepsilon)^2} t^2 \ge v'(0)t - \frac{4}{r^2} t^2.$$

Since $t \le r^2/(16R)$ and by (4)

$$v'(0) = G'(0) = C_D(w,Y) \ge \frac{1}{R},$$

$$v(t) \geq \frac{t}{R} - \frac{t}{4R} \geq \frac{t}{2R}.$$

for $0 \leq t \leq \varepsilon$. In particular,

$$v(s) = \mathbf{Re}\, G(s) \geq \frac{s}{2R}.$$

Applying to the function $u(t)$ the same analysis as above we obtain

$$u(s) \leq u'(0)s + \frac{4}{r^2}s^2.$$

Hence

$$\frac{s}{2R} \leq v(s) \leq u(s) \leq u'(0)s + \frac{4}{r^2}s^2 \leq u'(0)s + \frac{s}{4R}.$$

Thus

$$\mathbf{Re}\, (\nabla f_s(w), Y) = \mathbf{Re}\, F'(0) = u'(0) \geq \frac{1}{4R}.$$

□

Lemma 2.3. *Let $B(0, r+a) \subset\subset D \subset\subset B(0, R)$, $r, a > 0$. Then there exists a positive $\delta = \delta(a, r, R) < a$ such that*

$$\|X\|_{B(0,r+\delta)} \leq \frac{32R}{a}\|X\|_{B(0,r)}$$

for every holomorphic vector field X generated by a one-parameter group action $g(z, t)$ on D.

Proof. Let w belong to $B(0, r + a/2)$. Since

$$w \in B(w, a/2) \subset\subset D \subset\subset B(w, 2R),$$

by Lemma 2.2 there is an $\varepsilon = \varepsilon(a, R) > 0$ such that for every $w \in B(0, r + a/2)$, every $Y \in \mathbb{C}^n$, $|Y| = 1$, and every $s \in (0, \varepsilon]$

$$\mathbf{Re}\, (\nabla f(w), Y) \geq \frac{1}{8R}, \tag{5}$$

where f is a Carathéodory extremal function for w and $w + sY$.

Let us take a positive number $\delta < a/2$ so small that for every $w \in B(0, r + \delta)$ and every unit vector V there is a unit vector Y such that $w + sY \in B(0, r)$ for some real s with $|s| < \varepsilon$ and

$$|V - Y| < b = \frac{a}{32R}.$$

Clearly, the choice of this δ depends only on a, r and R.

The lemma needs a proof only for non-trivial group actions when $X \not\equiv 0$. Let $w \in \partial B(0, r + \delta)$, $X(w) \neq 0$ and let $V = X(w)/|X(w)|$. We choose a vector Y and a real s satisfying the above conditions.

Let f be a Carathéodory extremal function for w and $z = w + sY$. Since $B(w, a/2) \subset\subset D$, by Schwarz inequality,

$$|\mathbf{Re}\,(\nabla f(w), Y - V)| \leq b|\nabla f(w)| \leq \frac{2b}{a}.$$

Hence by (5),

$$\mathbf{Re}\,(\nabla f(w), V) \geq \mathbf{Re}\,(\nabla f(w), Y) - \frac{2b}{a} \geq \frac{1}{8R} - \frac{1}{16R} = \frac{1}{16R}.$$

Let $\zeta(t) = f(g(w,t))$ and $p = f(z)$. We introduce

$$m(t) = \left| \frac{\zeta(t) - p}{1 - p\zeta(t)} \right|.$$

If $\rho(\zeta, \xi)$ is the Poincare metric on U, then $\rho(0,p) = c_D(z,w)$ and

$$\rho(\zeta(t), p) = \frac{1}{2} \ln \frac{1 + m(t)}{1 - m(t)}.$$

A straightforward calculation shows that

$$\frac{dm^2}{dt}(0) = -2p(1-p^2)\mathbf{Re}\,(\nabla f(w), X(w)),$$

and, by using this calculation, we obtain

$$\frac{d}{dt}\rho(\zeta(0), p) = -\mathbf{Re}\,(\nabla f(w), X(w)).$$

Hence

$$\rho(\zeta(-t), p) \geq \rho(0,p) + t\mathbf{Re}\,(\nabla f(w), X(w)) \geq c_D(z,w) + \frac{t}{16R}|X(w)|$$

for small positive t. Since the Carathéodory metric decreases under the holomorphic mapping f,

$$c_D(z, g(w,-t)) \geq \rho(\zeta(-t), p) \geq c_D(z,w) + \frac{t}{16R}|X(w)|. \qquad (6)$$

By (6) and Lemma 2.1,

$$\frac{t}{16R}|X(w)| \leq c_D(z, g(w,-t)) - c_D(z,w) \leq c_D(z, g(z,-t)).$$

By (3), $c_D(z, g(z,-t)) = C_D(z, X(z))t + o(t)$. Note that $B(z,a) \subset D$ and, therefore, $C_D(z, X(z)) \leq 1/a$. Hence

$$c_D(z, g(z,-t)) \leq 2C_D(z, X(z))t \leq \frac{2}{a}|X(z)|t$$

for small positive t. Thus

$$|X(w)| \leq \frac{32R}{a}|X(z)|$$

and

$$\|X\|_{B(0,r+\delta)} \leq \frac{32R}{a}\|X\|_{B(0,r)}.$$

$\square$

Lemma 2.4. *Let $R > 2r > 2s > 0$. Let K be a connected compact set containing 0 in $\mathbb{C}^n$. Let D be a domain in $\mathbb{C}^n$ such that $B(0,2r) \subset D \subset B(0,R)$ and such that the $3s$-neighborhood of K is contained in D. Then there exists a positive constant $C = C(K,R,s)$ such that $\|X\|_K \leq C\|X\|_{B(0,r)}$ for each holomorphic vector field X generated by a one-parameter group action $g(z,t)$ on D.*

Proof. Let X be such a vector field on D. By the previous lemma there exist positive numbers $\delta = \delta(s,R) < s$ and $c = c(s,R)$ such that

$$\|X\|_{B(z,s+\delta)} \leq c\|X\|_{B(z,s)}$$

whenever $z \in D$ is at least $3s$ away from ∂D. There is a positive integer $N = N(K,\delta)$ such that for each $z \in K$ there is a set of N points $\{z_1,\ldots,z_N\} \subset K$ with $z_1 = 0$, $z_N = z$, and $|z_{k+1} - z_k| < \delta$ for $k = 1,\ldots,N-1$. Since $B(z_{k+1},s) \subset B(z_k,s+\delta)$, we see that

$$\|X\|_{B(z_{k+1},s)} \leq c\|X\|_{B(z_k,s)}$$

for $k = 1,\ldots,N-1$. Thus,

$$\|X\|_{B(z,s)} \leq c^{N-1}\|X\|_{B(0,s)}.$$

In particular, $|X(z)| \leq c^{N-1}\|X\|_{B(0,s)} \leq c^{N-1}\|X\|_{B(0,r)}$. Therefore, $\|X\|_K \leq c^{N-1}\|X\|_{B(0,r)}$. $\square$

Theorem 2.5. *Suppose a sequence of domains D_j converge in $\mathcal{H}^n$ to a domain D and a ball $B(p,r+a)$, $r,a > 0$, belongs to all D_j. Also suppose that $g_j(z,t)$ are non-trivial one-parameter group actions on D_j generating the holomorphic vector fields X_j. If $\|X_j\|_B = 1$, $B = B(p,r)$, then there is a subsequence of the group actions $g_{j_k}(z,t)$ that converges to a non-trivial group action $g(z,t)$ on D uniformly on compacta in $D \times \mathbb{R}$ and*

$$\lim_{j\to\infty} X_j(w) = X(w)$$

uniformly on compacta in D, where X is the holomorphic vector field generated by g.

Proof. Let $K \subset\subset D$. Choose $\delta > 0$ so that the 3δ-neighborhood of K is contained in D and in each D_j. Let $\tilde{K}$ and $\hat{K}$ denote the 2δ-neighborhood and the δ-neighborhood of K respectively. By Lemma 2.4 there exists $A > 0$ such that $\|X_j\|_{\tilde{K}} \leq A$.

Let $\tau = \delta/(2A)$. Define the mapping $h_j : \hat{K} \times (-\tau, \tau) \to D_j$ as the solution of the initial value problem

$$\frac{\partial}{\partial t} h_j(z,t) = iX_j(h_j(z,t)), \quad h_j(z,0) = z.$$

Since $\tau |X_j| < \delta$ in $\tilde{K}$, it follows from the ODE's theory that the mapping h_j is well-defined.

For $M = \{\zeta \in \mathbb{C} : |\mathbf{Im}\, \zeta| < \tau\}$ we define $G_j : \hat{K} \times M \to D_j$ by $G_j(z, t+is) = g_j(h_j(z,s), t)$. Since X_j is holomorphic, the mapping G_j is holomorphic in z. We now prove that it is holomorphic in $\zeta = t + is$. It is clear that

$$\frac{\partial G_j}{\partial t}(z, t+is) = X_j(G_j(z, t+is)). \tag{7}$$

It follows immediately from the fact that the Poisson brackets $[X_j, iX_j] \equiv 0$, that

$$\frac{\partial G_j}{\partial s}(z, t+is) = iX_j(G_j((z, t+is)). \tag{8}$$

This fact also can be proved by a straightforward reasoning:

$$\begin{aligned}\frac{\partial G_j}{\partial s}(z, t+is) &= \frac{\partial g_j}{\partial z}(h_j(z,s), t) \cdot iX(h_j(z,s)) \\ &= iX_j(g_j(h_j(z,s),t)) = iX_j(G_j(z, t+is));\end{aligned}$$

the middle equality is by the infinitesimal group property (1). The equations (7) and (8) are the Cauchy-Riemann equations for G_j in ζ. So G_j is holomorphic.

Passing to a subsequence, if necessary, we may assume that the mappings G_j converge to a mapping G uniformly on compacta in $\hat{K} \times M$. Consequently, the mappings $g_j(z,t)$ converge to $g(z,t)$ uniformly on compacta in $\hat{K} \times \mathbb{R}$, and the vector fields X_j converge to

$$X(z) = \frac{\partial g}{\partial t}(z, 0)$$

uniformly on compacta in $\hat{K}$.

It follows that some subsequence of the sequence $\{g_j(z,t)\}$ converges to a mapping $g(z,t) = G(z,t)$ uniformly on compacta in $D \times \mathbb{R}$. Thus, $g(z,t)$ is a group action. Since $\|X\|_B = 1$, this group action is nontrivial. $\square$

Proof of Theorem 0.2. Let D_j be a sequence of domains converging in $\mathcal{H}^n$ to a domain D. Let us choose a ball $B(p, r+a)$, $r, a > 0$, belonging to all D_j for sufficiently large j and take $\delta > 0$ from Lemma 2.3. Let $B = B(p,r)$ and $\hat{B} = B(p, r+\delta)$. We may assume that the dimensions of all groups $G_j = \mathrm{Aut}_0(D_j)$ are the same and equal to k. Since the Lie algebra A_j of all holomorphic vector fields on D_j generated by one-parameter subgroups in G_j has the same dimension as G_j, we can choose $X_j^m \in A_j$, $1 \le m \le k$, such that

$$\int_{\hat{B}} (X_j^m, \overline{X}_j^l)\, dV = \delta_{ml},$$

where δ_{ml} is Kronecker's delta.

Clearly, $\|X_j^m\|_{\hat{B}} \ge \mathrm{Vol}(\hat{B})^{-1}$. On the other hand, by Cauchy estimates and Lemma 2.4, for some constants we have

$$1 \ge C_1 \|X_j^m\|_B \ge C_2 \|X_j^m\|_{\hat{B}}.$$

Let g_j^m be the one-parameter groups generated by X_j^m. By Theorem 2.5 one can choose a subsequence $\{j_k\}$ such that $g_{j_k}^m$ converge, uniformly on compacta in $D \times \mathbb{R}$, to a one-parameter group $g^m(z,t)$ on D, and $X_{j_k}^m$ converge to a vector field X^m uniformly on compacta in D. Since

$$\int_{\hat{B}} (X^m, \overline{X}^l)\, dV = \delta_{ml},$$

the dimension of $\mathrm{Aut}_0(D)$ is at least k.

3. Structural theorems

By Iwasawa's theorem (see [MZ, p. 188]) the group $\mathrm{Aut}_0(D)$ is homeomorphic to $K \times \mathbb{R}^k$, where K is a maximal compact subgroup and k is the *characteristic number* of $\mathrm{Aut}(D)$. It is interesting to find out what happens with K and $\mathbb{R}^k$ under small perturbations of domains. Let us look at maximal compact subgroups first. The argument of Corollary 4.1 in [FP] provides the following theorem.

Theorem 3.1. *Let D be a bounded domain in $\mathbb{C}^n$, let z_0 be a point in D, and let W be a compact set in D. If $\widetilde{D}$ is sufficiently close to D in $\mathcal{H}^n$ and for some maximal compact subgroup $\widetilde{K}$ in $\mathrm{Aut}_0(\widetilde{D})$ the orbit $\widetilde{K}(z_0) \subset W$, then $\widetilde{K}$ is isomorphic to a subgroup of $\mathrm{Aut}_0(D)$.*

Next example shows that without the condition in the above theorem of orbits being contained in a fixed compact set, it is possible that $\mathrm{Aut}(D)$ does not contain a compact subgroup while close domains have $\mathrm{Aut}_0(\widetilde{D})$ isomorphic to S^1. Let Δ denote the unit disc in $\mathbb{C}$.

Example 3.2. There is a sequence $\{D_j\}$ of bounded pseudoconvex domains in $\mathbb{C}^2$ converging to a domain D such that $\mathrm{Aut}(D_j) \cong S^1$ for each j, and $\mathrm{Aut}(D) \cong \mathbb{R}$.

Construction. Let $Q_j = \{z \in \Delta : |z - 2^{-1} + 2^{-j}| > 1/2\}$, $Q = \{z \in \Delta : |z - 2^{-1}| > 1/2\}$, $D_j = \{(z, w) : z \in Q_j, w \in \Delta, w \neq z\}$, $D = \{(z, w) : z \in Q, w \in \Delta, w \neq z\}$.

1. One can see that $D_j \to D$.

2. The domains D_j and D are bounded and pseudoconvex.

3. We now prove that $\mathrm{Aut}(D) \cong \mathbb{R}$. Let $F \in \mathrm{Aut}(D)$. On each fiber $(z, \cdot)$, F is bounded and has an isolated singularity, so F extends to be an automorphism of $Q \times \Delta$. Thus, F has the form $F(z, w) = (f(z), g(w))$, or $F(z, w) = (g(w), f(z))$. For both cases, one has, by the definition of D, that

$$f(z) = g(z), \quad z \in Q. \tag{9}$$

The second case is impossible, since implies that $f(Q) = \Delta$, $g(\Delta) = Q$, and $f(Q) = g(Q)$, which leads to a contradiction that Δ coincides with a subset of Q. Therefore, F has the form $F(z, w) = (f(z), g(w))$, where $f \in \mathrm{Aut}(Q)$, $g \in \mathrm{Aut}(\Delta)$. By (9), $f = g|_Q$. Let $\phi(w) = -i(w+1)/(w-1)$. Then ϕ is a biholomorphic map from Δ to the upper half-plane $\Pi = \{\zeta \in \mathbb{C} : \mathbf{Im}\,\zeta > \mathbf{0}\}$, and $\phi(Q) = \Lambda \equiv \{\zeta \in \mathbb{C} : 0 < \mathbf{Im}\,\zeta < \mathbf{1}\}$. Now $\phi \circ g \circ \phi^{-1}$ is an automorphism of Π, and its restriction to Λ is an automorphism of Λ. Thus $\phi \circ g \circ \phi^{-1}(\zeta) = \zeta + t$ for some $t \in \mathbb{R}$. It follows that $\mathrm{Aut}(D) = \{F_t : t \in \mathbb{R}\} \cong \mathbb{R}$, where $F_t(z, w) = (g_t(z), g_t(w))$, and

$$g_t(w) = \phi^{-1}(\phi(w) + t) = \frac{2w + i(w-1)t}{2 + i(w-1)t}.$$

4. In a way very similar to the above argument, one can prove that $\mathrm{Aut}(D_j) \cong S^1$ for each j. $\square$

By Theorem 0.2 the creation of compact subgroups with larger dimensions by small perturbations must be compensated by an elimination of some non-compact subgroups so that the total dimension will not go up. It seems to us that the other way around is impossible: characteristic numbers are upper semicontinuous on $\mathcal{H}^n$. While we cannot prove this statement, the following theorem certifies that non-compact parts cannot be created from nothing.

Theorem 3.3. *Let $D \subset \mathbb{C}^n$ be a bounded domain such that $\mathrm{Aut}_0(D)$ is compact. Then for all $\widetilde{D}$ sufficiently close in $\mathcal{H}^n$ to D the group $\mathrm{Aut}_0(\widetilde{D})$ is also compact.*

Proof. If the statement is not true, then there is a sequence $\{D_j\}$ of domains converging to D such that for each j the identity component $G_j = \mathrm{Aut}_0(D_j)$ is noncompact. Write $G = \mathrm{Aut}_0(D)$. Fix a $z_0 \in D$. The orbit $G(z_0)$ is compact. We may assume that $G(z_0) \subset D_j$ for each j. For each connected component H of $\mathrm{Aut}(D)$, either the set

$H(z_0)$ coincides with $G(z_0)$ or $G(z_0) \cap H(z_0) = \emptyset$. Indeed, if $h \in H$ and $h(z_0) \in G(z_0)$, then $H(z_0) = Gh(z_0) = G(z_0)$, since $H = Gh$. Now we claim that there exists a positive number a such that $a < d(H(z_0), G(z_0))$ for each component H of $\mathrm{Aut}(D)$ with $H(z_0) \neq G(z_0)$, where d is the euclidean distance. Otherwise, there is a sequence $\{H_k\}$ of distinct components of $\mathrm{Aut}(D)$ with $H_k(z_0) \neq G(z_0)$ such that $d(H_k(z_0), G(z_0)) \to 0$. Passing to a subsequence if necessary, we may assume that there are $h_k \in H_k$ such that $h_k(z_0)$ tends to a point in $G(z_0)$. It follows that some subsequence of $\{h_k\}$ converges in the compact-open topology to a $g \in \mathrm{Aut}(D)$; but this is impossible because h_k belong to different components of the Lie group $\mathrm{Aut}(D)$. Therefore, such an a exists. Decreasing a if necessary, we see that the open set

$$V = \{z \in D : d(z, G(z_0)) < a\}$$

is relatively compact in D and in each D_j, and satisfies $\overline{V} \cap \mathrm{Aut}(D)(z_0) = G(z_0)$. This implies that $\partial V \cap \mathrm{Aut}(D)(z_0) = \emptyset$. Since G_j is noncompact, $G_j(z_0)$ is noncompact, hence $G_j(z_0) \cap \partial V \neq \emptyset$. It follows that for each j there is a $g_j \in G_j$ with $g_j(z_0) \in \partial V$. Some subsequence of the sequence $\{g_j\}$ converges uniformly on compacta to a $g \in \mathrm{Aut}(D)$. It is clear that $g(z_0) \in \partial V$, contradicting $\partial V \cap \mathrm{Aut}(D)(z_0) = \emptyset$. $\square$

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