UPPER SEMICONTINUITY OF THE DIMENSIONS OF AUTOMORPHISM GROUPS OF DOMAINS IN \( \mathbb{C}^n \)

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Abstract. Let \( \mathcal{H}^n \) be the metric space of all bounded domains in \( \mathbb{C}^n \) with the metric equal to the Hausdorff distance between boundaries of domains. We prove that the dimension of the group of automorphisms of domains is an upper semicontinuous function on \( \mathcal{H}^n \). We also provide theorems and examples regarding the change in topological structure of these groups under small perturbation of a domain in \( \mathcal{H}^n \).

0. Introduction

The automorphism group \( \text{Aut}(D) \) (the group of biholomorphic self-maps of \( D \)) of a bounded domain \( D \) in \( \mathbb{C}^n \) is, in general, difficult to describe and little is known about it. However, it is known (see [?, ?]) that any compact Lie group can be realized as the group of automorphisms of a smooth strictly pseudoconvex domain, and (see [?]) that any linear Lie group can be realized as the group of automorphisms of a bounded domain. So, if we consider the group \( \text{Aut}(D) \) as a function of \( D \), the set of values is quite large.

If one considers this function on the metric space \( \mathcal{H}^n \) of all bounded domains in \( \mathbb{C}^n \) with the metric equal to the Hausdorff distance between boundaries of domains, one can expect that small perturbation of the boundary may only “decrease” the group, i.e., the function \( \text{Aut}(D) \) is “upper semicontinuous”. Indeed, in [GK], [Ma] and [FP] the authors, using topologies different from \( \mathcal{H}^n \), proved the upper semicontinuity of the function \( \text{Aut}(D) \) in the sense that \( \text{Aut}(\tilde{D}) \) is isomorphic to a subgroup of \( \text{Aut}(D) \) when \( \tilde{D} \) is “close” to \( D \). But, in general, this idea is not true according to the following theorem ([?]).

**Theorem 0.1.** Let \( M \) be a domain in \( \mathbb{C}^n \). Then there exists an increasing sequence of bounded domains \( M_k \subset M_{k+1} \subset M \) such that \( M = \cup M_k \) and \( \text{Aut}(M_k) \) contains a subgroup isomorphic to \( \mathbb{Z}_k \). 

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This shows that domains in $\mathbb{C}^n$ with an automorphism group containing $\mathbb{Z}_k$ are everywhere dense in $\mathcal{H}^n$, and it is well known that domains without non-trivial automorphisms are dense in $\mathcal{H}^n$. So arbitrarily small perturbation of a domain in $\mathcal{H}^n$ may create a domain with a larger automorphism group. But, for all known examples, this group is discrete, so it is of dimension zero. The natural question arises: can small perturbation in $\mathcal{H}^n$ create domains with larger dimensions of automorphism groups?

In this paper we answer this question in the negative. Namely, we prove the following

**Theorem 0.2.** The function $\dim \text{Aut}(D)$ is upper semicontinuous on $\mathcal{H}^n$.

An immediate consequence is the following

**Corollary 0.3.** For each $k > 0$ the set of all domains in $\mathcal{H}^n$ whose groups of automorphisms have dimensions greater than or equal to $k$ is closed and, therefore, nowhere dense.

Thus a domain cannot be approximated by domains whose automorphism groups have strictly larger dimensions.

To prove Theorem 0.2 we consider a sequence of domains $D_j$ converging in $\mathcal{H}^n$ to a domain $D$. The identity components $\text{Aut}_0(D_j)$ of $\text{Aut}(D_j)$ have the same dimensions as $\text{Aut}(D_j)$. Also the dimensions of the Lie algebras of holomorphic vector fields generated by all one-parameter groups in $\text{Aut}_0(D_j)$ coincide with $\dim \text{Aut}_0(D_j)$. Lemma 2.2 states that the uniform norm of such fields on a compact set is bounded by its norm on an arbitrarily selected ball times a constant that, basically, depends on the size of the ball and the distance from the ball and the compact set to the boundary of a domain. This allows us to normalize bases in Lie algebras of $\text{Aut}_0(D_j)$ and apply Theorem 2.2, which asserts the existence of non-trivial limits of those vector fields. The limits belong to the Lie algebra of $\text{Aut}_0(D)$ and this gives us the proof.

It is reasonable to ask whether $\text{Aut}_0(D_j)$ are always isomorphic to a subgroup of $\text{Aut}_0(D)$ when $j$ is large. An example in Section 2.2 shows that the answer is negative.

If $K_j$ is a maximal compact subgroup of $\text{Aut}_0(D_j)$, then $\text{Aut}_0(D_j)$ is homeomorphic to $K_j \times \mathbb{R}^{k_j}$ (see [MZ, p. 188]). The groups $K_j$ may decrease or even disappear in the limit (see Example 2.2), while non-compact parts never vanish (see Theorem 2.2).
1. SOME BASIC FACTS

Let $D$ be a bounded domain in $\mathbb{C}^n$. If the Lie group $\text{Aut}(D)$ has positive dimension, then it has one-parameter subgroups $g(\cdot, t), -\infty < t < \infty$, i.e., $g(z, t + s) = g(g(z, t), s)$. Such subgroups generate vector fields

$$X(z) = \frac{\partial g}{\partial t}(z, 0)$$

that are holomorphic. Also, if $X$ is a holomorphic vector field on $D$ that is $\mathbb{R}$-complete, i.e., the initial value problem $g'(z, t) = X(g(z, t)), g(z, 0) = z$, has a solution on $D \times \mathbb{R}$, then $g(z, t)$ is a one-parameter group.

The vector field $X$ has the following group property:

$$X(g(z, t)) = \frac{\partial g}{\partial z}(z, t)X(z). \tag{1}$$

For every two points $z$ and $w$ in $D$ among all holomorphic mappings of $D$ into the unit disk $\Delta$ we choose holomorphic functions $f$ such that $f(w) = 0$ and $f(z)$ is real and the maximal possible. Such functions $f$ exist and are called Carathéodory extremal functions for $z$ and $w$ on $D$. The quantity

$$\rho(0, f(z)) = \frac{1}{2} \ln \frac{1 + f(z)}{1 - f(z)} \tag{2}$$

is called the Carathéodory distance $c_D(z, w)$ on $D$. (Note that the formula for $\rho(0, a)$ gives the Poincaré distance between 0 and $a$ in the unit disc.) When $D$ is bounded this distance is non-degenerate and invariant, i.e., $c_D(g(z), g(w)) = c_D(z, w)$ for every $g \in \text{Aut}(D)$ (see [?, Ch. 5, §18]).

For a point $w \in D$ and a vector $Y$ in $\mathbb{C}^n$, among all holomorphic mappings of $D$ into the unit disk $\Delta$ we choose holomorphic functions $f$ such that $f(w) = 0$ and $(f'(w), Y)$ is real and the maximal possible. (Here $(Z, Y) = \sum_{j=1}^{n} z_j y_j$.) These functions are Carathéodory extremal functions for $Y$ at $w$ in $D$. It follows from [?, Ch. 5, §18] that if $w(t)$ is a smooth curve in $D$ with $w(0) = w$ and $Y = w'(0)$, then

$$c_D(w, w(t)) = C_D(w, Y)t + o(t), \tag{3}$$

where

$$C_D(w, Y) = \sup\{|g'(w)Y| : g(D) \subset \Delta, g(w) = 0\}$$

is the Carathéodory infinitesimal metric. Let $B(w, r)$ be the ball of radius $r$ centered at $w$ and let $|Y|$ be the Euclidean norm of $Y$. If
$B(w,r) \subset D \subset B(w,R)$, then
\[ \frac{|Y|}{R} \leq C_D(w,Y) \leq \frac{|Y|}{r}. \tag{4} \]

2. Proof of Theorem 0.2

Lemma 2.1. Let $D$ be a domain in $\mathbb{C}^n$ and let $d(z,w)$ be an invariant metric on $D$ satisfying the triangle inequality. If $g(z,t)$ is a group action on $D$, then for any $w,z \in D$
\[ |d(g(w,t), z) - d(w,z)| \leq d(z,g(z,t)). \]

Proof. Apply the identity $d(w,z) = d(g(w,t), g(z,t))$ and the triangle inequality. □

Lemma 2.2. Let $w \in B(w,r) \subset \subset D \subset \subset B(w,R) \subset \subset \mathbb{C}^n$. Then for any $Y \in \mathbb{C}^n$, $|Y| = 1$,
\[ \text{Re} \left( \nabla f_s(w), Y \right) > \frac{1}{4R}, \]
where $f_s(z)$ is a Carathéodory extremal function for $w$ and $w+sY$ in $D$, and $s$ is a real number such that
\[ 0 < s \leq \varepsilon = \frac{r^2}{16R}. \]

Proof. Let us fix $Y$ and introduce $D_Y = \{ \xi \in \mathbb{C} : w + \xi Y \in D \}$. Clearly, $\Delta(0,r) \subset \subset D_Y \subset \subset \Delta(0,R)$, where $\Delta(0,s)$ is the disk of radius $s$ centered at 0.

Let $g_Y(z)$ be a Carathéodory extremal function for $Y$ at $w$ in $D$. For $\xi \in D_Y$ we introduce the functions $u(\xi) = \text{Re} F(\xi)$, where $F(\xi) = f_s(w + \xi Y)$, and $v(\xi) = \text{Re} G(\xi)$, where $G(\xi) = g_Y(w + \xi Y)$. All these functions are well-defined on $D_Y$ and $F(0) = G(0) = 0$, $G'(0) = C_D(w,Y)$ and $u(s) = F(s) \geq v(s)$. Let us prove that $v(t) \geq t/(2R)$ when $t \in [0,\varepsilon]$. Since
\[ |v(t) - v'(0)t| \leq \frac{1}{2} \sup_{0 \leq x \leq \varepsilon} |v''(x)| \cdot t^2, \]
$\varepsilon \leq r^2/(16R) < r/2$ and by Cauchy estimate $|v''(x)| \leq 2/(r-\varepsilon)^2$ when $x < \varepsilon$, we see that
\[ v(t) \geq v'(0)t - \frac{1}{(r-\varepsilon)^2} t^2 \geq v'(0)t - \frac{4}{r^2} t^2. \]

Since $t \leq r^2/(16R)$ and by (??),
\[ v'(0) = G'(0) = C_D(w,Y) \geq \frac{1}{R}. \]
\[ v(t) \geq \frac{t}{R} - \frac{t}{4R} \geq \frac{t}{2R}. \]

for \( 0 \leq t \leq \varepsilon \). In particular,
\[ v(s) = \Re G(s) \geq \frac{s}{2R}. \]

Applying to the function \( u(t) \) the same analysis as above we obtain
\[ u(s) \leq u'(0)s + \frac{4}{r^2}s^2. \]

Hence
\[ \frac{s}{2R} \leq v(s) \leq u(s) \leq u'(0)s + \frac{4}{r^2}s^2 \leq u'(0)s + \frac{s}{4R}. \]

Thus
\[ \Re (\nabla f_s(w), Y) = \Re F'(0) = u'(0) \geq \frac{1}{4R}. \]

Lemma 2.3. Let \( B(0, r + a) \subset D \subset B(0, R) \), \( r, a > 0 \). Then there exists a positive \( \delta = \delta(a, r, R) < a \) such that
\[ \|X\|_{B(0, r+\delta)} \leq \frac{32R}{a}\|X\|_{B(0, r)} \]
for every holomorphic vector field \( X \) generated by a one-parameter group action \( g(z, t) \) on \( D \).

Proof. Let \( w \) belong to \( B(0, r + a/2) \). Since
\[ w \in B(w, a/2) \subset D \subset B(w, 2R), \]
by Lemma ?? there is an \( \varepsilon = \varepsilon(a, R) > 0 \) such that for every \( w \in B(0, r + a/2) \), every \( Y \in \mathbb{C}^n, |Y| = 1 \), and every \( s \in (0, \varepsilon] \)
\[ \Re (\nabla f(w), Y) \geq \frac{1}{8R}, \quad (5) \]
where \( f \) is a Carathéodory extremal function for \( w \) and \( w + sY \).

Let us take a positive number \( \delta < a/2 \) so small that for every \( w \in B(0, r + \delta) \) and every unit vector \( V \) there is a unit vector \( Y \) such that \( w + sY \in B(0, r) \) for some real \( s \) with \( |s| < \varepsilon \) and
\[ |V - Y| < b = \frac{a}{32R}. \]

Clearly, the choice of this \( \delta \) depends only on \( a \), \( r \) and \( R \).

The lemma needs a proof only for non-trivial group actions when \( X \neq 0 \). Let \( w \in \partial B(0, r + \delta), X(w) \neq 0 \) and let \( V = X(w)/|X(w)| \). We choose a vector \( Y \) and a real \( s \) satisfying the above conditions.
Let \( f \) be a Carathéodory extremal function for \( w \) and \( z = w + sY \).

Since \( B(w, a/2) \subset D \), by Schwarz inequality,

\[
|\text{Re}(\nabla f(w), Y - V)| \leq b|\nabla f(w)| \leq \frac{2b}{a}.
\]

Hence by (??),

\[
\text{Re}(\nabla f(w), V) \geq \text{Re}(\nabla f(w), Y) - \frac{2b}{a} \geq \frac{1}{8R} - \frac{1}{16R} = \frac{1}{16R}.
\]

Let \( \zeta(t) = f(g(w, t)) \) and \( p = f(z) \). We introduce

\[
m(t) = \left| \frac{\zeta(t) - p}{1 - p\zeta(t)} \right|.
\]

If \( \rho(\zeta, \xi) \) is the Poincaré metric on \( U \), then \( \rho(0, p) = c_D(z, w) \) and

\[
\rho(\zeta(t), p) = \frac{1}{2} \ln \frac{1 + m(t)}{1 - m(t)}.
\]

A straightforward calculation shows that

\[
\frac{dm^2}{dt}(0) = -2p(1 - p^2)\text{Re}(\nabla f(w), X(w)),
\]

and, by using this calculation, we obtain

\[
\frac{d}{dt}\rho(\zeta(0), p) = -\text{Re}(\nabla f(w), X(w)).
\]

Hence

\[
\rho(\zeta(-t), p) \geq \rho(0, p) + t\text{Re}(\nabla f(w), X(w)) \geq c_D(z, w) + \frac{t}{16R}|X(w)|
\]

for small positive \( t \). Since the Carathéodory metric decreases under the holomorphic mapping \( f \),

\[
c_D(z, g(w, -t)) \geq \rho(\zeta(-t), p) \geq c_D(z, w) + \frac{t}{16R}|X(w)|.
\]

(6)

By (??) and Lemma ??,

\[
\frac{t}{16R}|X(w)| \leq c_D(z, g(w, -t)) - c_D(z, w) \leq c_D(z, g(z, -t)).
\]

By (??), \( c_D(z, g(z, -t)) = C_D(z, X(z))t + o(t) \). Note that \( B(z, a) \subset D \) and, therefore, \( C_D(z, X(z)) \leq 1/a \). Hence

\[
c_D(z, g(z, -t)) \leq 2C_D(z, X(z))t \leq \frac{2}{a}|X(z)|t
\]

for small positive \( t \). Thus

\[
|X(w)| \leq \frac{32R}{6a}|X(z)|
\]
Lemma 2.4. Let $R > 2r > 2s > 0$. Let $K$ be a connected compact set containing 0 in $\mathbb{C}^n$. Let $D$ be a domain in $\mathbb{C}^n$ such that $B(0, 2r) \subset D \subset B(0, R)$ and such that the $3s$-neighborhood of $K$ is contained in $D$. Then there exists a positive constant $C = C(K, R, s)$ such that $\|X\|_K \leq C\|X\|_{B(0, r)}$ for each holomorphic vector field $X$ generated by a one-parameter group action $g(z, t)$ on $D$.

Proof. Let $X$ be such a vector field on $D$. By the previous lemma there exist positive numbers $\delta = \delta(s, R) < s$ and $c = c(s, R)$ such that $\|X\|_{B(z,s+\delta)} \leq c\|X\|_{B(z,s)}$ whenever $z \in D$ is at least $3s$ away from $\partial D$. There is a positive integer $N = N(K, \delta)$ such that for each $z \in K$ there is a set of $N$ points $\{z_1, \ldots, z_N\} \subset K$ with $z_1 = 0$, $z_N = z$, and $|z_{k+1} - z_k| < \delta$ for $k = 1, \ldots, N - 1$. Since $B(z_{k+1}, s) \subset B(z_k, s + \delta)$, we see that $\|X\|_{B(z_{k+1}, s)} \leq c\|X\|_{B(z_k, s)}$ for $k = 1, \ldots, N - 1$. Thus,

$$\|X\|_{B(z,s)} \leq c^{N-1}\|X\|_{B(0,s)}.$$

In particular, $|X(z)| \leq c^{N-1}\|X\|_{B(0,s)} \leq c^{N-1}\|X\|_{B(0,r)}$. Therefore,$n_s\|X\|_K \leq c^{N-1}\|X\|_{B(0,r)}$. □

Theorem 2.5. Suppose a sequence of domains $D_j$ converge in $\mathcal{H}^\alpha$ to a domain $D$ and a ball $B(p, r + a)$, $r, a > 0$, belongs to all $D_j$. Also suppose that $g_j(z, t)$ are non-trivial one-parameter group actions on $D_j$ generating the holomorphic vector fields $X_j$. If $\|X_j\|_B = 1$, $B = B(p, r)$, then there is a subsequence of the group actions $g_{j_k}(z, t)$ that converges to a non-trivial group action $g(z, t)$ on $D$ uniformly on compacta in $D \times \mathbb{R}$ and

$$\lim_{j \to \infty} X_j(w) = X(w)$$

uniformly on compacta in $D$, where $X$ is the holomorphic vector field generated by $g$.

Proof. Let $K \subset \subset D$. Choose $\delta > 0$ so that the $3\delta$-neighborhood of $K$ is contained in $D$ and in each $D_j$. Let $\tilde{K}$ and $\hat{K}$ denote the $2\delta$-neighborhood and the $\delta$-neighborhood of $K$ respectively. By Lemma ?? there exists $A > 0$ such that $\|X_j\|_{\tilde{K}} \leq A$. 

and

$$\|X\|_{B(0,r+\delta)} \leq \frac{32R}{a}\|X\|_{B(0,r)}.$$

□
Let \( \tau = \delta/(2A) \). Define the mapping \( h_j : \tilde{K} \times (-\tau, \tau) \to D_j \) as the solution of the initial value problem
\[
\frac{\partial}{\partial t} h_j(z, t) = iX_j(h_j(z, t)), \quad h_j(z, 0) = z.
\]

Since \( \tau |X_j| < \delta \) in \( \tilde{K} \), it follows from the ODE’s theory that the mapping \( h_j \) is well-defined.

For \( M = \{ \zeta \in \mathbb{C} : |\text{Im } \zeta| < \tau \} \) we define \( G_j : \hat{K} \times M \to D_j \) by \( G_j(z, t + is) = g_j(h_j(z, s), t) \). Since \( X_j \) is holomorphic, the mapping \( G_j \) is holomorphic in \( z \). We now prove that it is holomorphic in \( \zeta = t + is \).

It is clear that
\[
\frac{\partial G_j}{\partial t}(z, t + is) = X_j(G_j(z, t + is)).
\] (7)

It follows immediately from the fact that the Poisson brackets \([X_j, iX_j] \equiv 0\), that
\[
\frac{\partial G_j}{\partial s}(z, t + is) = iX_j((z, t + is)).
\] (8)

This fact also can be proved by a straightforward reasoning:
\[
\frac{\partial G_j}{\partial s}(z, t + is) = \frac{\partial g_j}{\partial z}(h_j(z, s), t) \cdot iX(h_j(z, s))
\]
\[
= iX_j(g_j(h_j(z, s), t)) = iX_j(G_j(z, t + is));
\]
the middle equality is by the infinitesimal group property (??). The equations (??) and (??) are the Cauchy-Riemann equations for \( G_j \) in \( \zeta \). So \( G_j \) is holomorphic.

Passing to a subsequence, if necessary, we may assume that the mappings \( G_j \) converge to a mapping \( G \) uniformly on compacta in \( \tilde{K} \times M \). Consequently, the mappings \( g_j(z, t) \) converge to \( g(z, t) \) uniformly on compacta in \( \tilde{K} \times \mathbb{R} \), and the vector fields \( X_j \) converge to
\[
X(z) = \frac{\partial g}{\partial t}(z, 0)
\]
uniformly on compacta in \( \tilde{K} \).

It follows that some subsequence of the sequence \( \{g_j(z, t)\} \) converges to a mapping \( g(z, t) = G(z, t) \) uniformly on compacta in \( D \times \mathbb{R} \). Thus, \( g(z, t) \) is a group action. Since \( \|X\|_B = 1 \), this group action is non-trivial. \( \square \)

**Proof of Theorem ??**. Let \( D_j \) be a sequence of domains converging in \( \mathcal{H}^n \) to a domain \( D \). Let us choose a ball \( B(p, r + a) \), \( r, a > 0 \), belonging to all \( D_j \) for sufficiently large \( j \) and take \( \delta > 0 \) from Lemma ???. Let \( B = B(p, r) \) and \( \tilde{B} = B(p, r + \delta) \). We may assume that the dimensions
of all groups $G_j = \text{Aut}_0(D_j)$ are the same and equal to $k$. Since the Lie algebra $A_j$ of all holomorphic vector fields on $D_j$ generated by one-parameter subgroups in $G_j$ has the same dimension as $G_j$, we can choose $X_j^m \in A_j$, $1 \leq m \leq k$, such that
\[
\int_B (X_j^m, \overline{X}_j^l) dV = \delta_{ml},
\]
where $\delta_{ml}$ is Kronecker’s delta.

Clearly, $\|X_j^m\|_B \geq \text{Vol}(B)^{-1}$. On the other hand, by Cauchy estimates and Lemma ??, for some constants we have
\[
1 \geq C_1\|X_j^m\|_B \geq C_2\|X_j^m\|_{\overline{B}}.
\]

Let $g_j^m$ be the one-parameter groups generated by $X_j^m$. By Theorem ?? one can choose a subsequence $\{j_k\}$ such that $g_{j_k}^m$ converge, uniformly on compacta in $D \times \mathbb{R}$, to a one-parameter group $g^m(z, t)$ on $D$, and $X_{j_k}^m$ converge to a vector field $X^m$ uniformly on compacta in $D$. Since
\[
\int_B (X^m, \overline{X}) dV = \delta_{ml},
\]
the dimension of $\text{Aut}_0(D)$ is at least $k$.

3. Structural theorems

By Iwasawa’s theorem (see [MZ, p. 188]) the group $\text{Aut}_0(D)$ is homeomorphic to $K \times \mathbb{R}^k$, where $K$ is a maximal compact subgroup and $k$ is the characteristic number of $\text{Aut}(D)$. It is interesting to find out what happens with $K$ and $\mathbb{R}^k$ under small perturbations of domains. Let us look at maximal compact subgroups first. The argument of Corollary 4.1 in [?] provides the following theorem.

Theorem 3.1. Let $D$ be a bounded domain in $\mathbb{C}^n$, let $z_0$ be a point in $D$, and let $W$ be a compact set in $D$. If $\tilde{D}$ is sufficiently close to $D$ in $\mathcal{H}^n$ and for some maximal compact subgroup $\tilde{K}$ in $\text{Aut}_0(\tilde{D})$ the orbit $\tilde{K}(z_0) \subset W$, then $\tilde{K}$ is isomorphic to a subgroup of $\text{Aut}_0(D)$.

Next example shows that without the condition in the above theorem of orbits being contained in a fixed compact set, it is possible that $\text{Aut}(D)$ does not contain a compact subgroup while close domains have $\text{Aut}_0(D)$ isomorphic to $S^1$. Let $\Delta$ denote the unit disc in $\mathbb{C}$.

Example 3.2. There is a sequence $\{D_j\}$ of bounded pseudoconvex domains in $\mathbb{C}^2$ converging to a domain $D$ such that $\text{Aut}(D_j) \cong S^1$ for each $j$, and $\text{Aut}(D) \cong \mathbb{R}$. 

Construction. Let $Q_j = \{ z \in \Delta : |z - 2^{-j}| > 1/2 \}$, $Q = \{ z \in \Delta : |z - 2^{-1}| > 1/2 \}$, $D_j = \{(z,w) : z \in Q_j, w \in \Delta, w \neq z \}$, $D = \{(z,w) : z \in Q, w \in \Delta, w \neq z \}$.

1. One can see that $D_j \to D$.

2. The domains $D_j$ and $D$ are bounded and pseudoconvex.

3. We now prove that $\operatorname{Aut}(D) \cong \mathbb{R}$. Let $F \in \operatorname{Aut}(D)$. On each fiber $(z,\cdot)$, $F$ is bounded and has an isolated singularity, so $F$ extends to be an automorphism of $Q \times \Delta$. Thus, $F$ has the form $F(z,w) = (f(z),g(w))$, or $F(z,w) = (g(w),f(z))$. For both cases, one has, by the definition of $D$, that

$$f(z) = g(z), \quad z \in Q. \quad (9)$$

The second case is impossible, since implies that $f(Q) = \Delta, g(\Delta) = Q$, and $f(Q) = g(Q)$, which leads to a contradiction that $\Delta$ coincides with a subset of $Q$. Therefore, $F$ has the form $F(z,w) = (f(z),g(w))$, where $f \in \operatorname{Aut}(Q), g \in \operatorname{Aut}(\Delta)$. By (??), $f = g|Q$. Let $\phi(w) = -i(w+1)/(w-1)$. Then $\phi$ is a biholomorphic map from $\Delta$ to the upper half-plane $\Pi = \{ \zeta \in \mathbb{C} : \text{Im} \, \zeta > 0 \}$, and $\phi(Q) = \Lambda \equiv \{ \zeta \in \mathbb{C} : 0 < \text{Im} \, \zeta < 1 \}$. Now $\phi \circ g \circ \phi^{-1}$ is an automorphism of $\Pi$, and its restriction to $\Lambda$ is an automorphism of $\Lambda$. Thus $\phi \circ g \circ \phi^{-1}(\zeta) = \zeta + t$ for some $t \in \mathbb{R}$. It follows that $\operatorname{Aut}(D) = \{ F_t : t \in \mathbb{R} \} \cong \mathbb{R}$, where $F_t(z,w) = (g_t(z),g_t(w))$, and

$$g_t(w) = \phi^{-1}(\phi(w) + t) = \frac{2w + i(w-1)t}{2 + i(w-1)t}. \quad (\ast)$$

4. In a way very similar to the above argument, one can prove that $\operatorname{Aut}(D_j) \cong S^1$ for each $j$. \hfill \Box

By Theorem ?? the creation of compact subgroups with larger dimensions by small perturbations must be compensated by an elimination of some non-compact subgroups so that the total dimension will not go up. It seems to us that the other way around is impossible: characteristic numbers are upper semicontinuous on $\mathcal{H}_n$. While we cannot prove this statement, the following theorem certifies that non-compact parts cannot be created from nothing.

**Theorem 3.3.** Let $D \subset \mathbb{C}_n$ be a bounded domain such that $\operatorname{Aut}_0(D)$ is compact. Then for all $D$ sufficiently close in $\mathcal{H}_n$ to $D$ the group $\operatorname{Aut}_0(D)$ is also compact.

**Proof.** If the statement is not true, then there is a sequence $\{D_j\}$ of domains converging to $D$ such that for each $j$ the identity component $G_j = \operatorname{Aut}_0(D_j)$ is noncompact. Write $G = \operatorname{Aut}_0(D)$. Fix a $z_0 \in D$. The orbit $G(z_0)$ is compact. We may assume that $G(z_0) \subset D_j$ for
each \( j \). For each connected component \( H \) of \( \text{Aut}(D) \), either the set 
\( H(z_0) \) coincides with \( G(z_0) \) or \( G(z_0) \cap H(z_0) = \emptyset \). Indeed, if \( h \in H \) and \( h(z_0) \in G(z_0) \), then \( H(z_0) = Gh(z_0) = G(z_0) \), since \( H = Gh \). Now we claim that there exists a positive number \( a \) such that 
\( a < d(H(z_0), G(z_0)) \) for each component \( H \) of \( \text{Aut}(D) \) with \( H(z_0) \neq G(z_0) \), where \( d \) is the euclidean distance. Otherwise, there is a sequence \( \{H_k\} \) of distinct components of \( \text{Aut}(D) \) with 
\( H_k(z_0) \neq G(z_0) \) such that 
\( d(H_k(z_0), G(z_0)) \to 0 \). Passing to a subsequence if necessary, we may assume that there are \( h_k \in H_k \) such that \( h_k(z_0) \) tends to a point 
\( \partial V \cap \text{Aut}(D)(z_0) \) is relatively compact in \( D \) and in each \( D_j \), and satisfies 
\( \partial V \cap \text{Aut}(D)(z_0) = G(z_0) \). This implies that \( \partial V \cap \text{Aut}(D)(z_0) = \emptyset \). Since \( G_j \) is noncompact, \( G_j(z_0) \) is noncompact, hence \( G_j(z_0) \cap \partial V \neq \emptyset \). It follows that for each \( j \) there is a \( g_j \in G_j \) with \( g_j(z_0) \in \partial V \). Some subsequence of the sequence \( \{g_j\} \) converges uniformly on compacta to a \( g \in \text{Aut}(D) \). It is clear that \( g(z_0) \in \partial V \), contradicting 
\( \partial V \cap \text{Aut}(D)(z_0) = \emptyset \). □

References


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