ISOLATED FIXED POINT SETS FOR HOLOMORPHIC MAPS

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Abstract. We study discrete fixed point sets of holomorphic self-maps of complex manifolds. The main attention is focused on the cardinality of this set and its configuration. As a consequence of one of our observations, a bounded domain in $\mathbb{C}^n$ with no non-trivial holomorphic retractions is constructed.

Résumé. Nous étudions les ensembles discrets qui sont les ensembles de points fixes d'une application holomorphe d'une variété complexe dans elle-même. En particulier, nous étudions le nombre d'éléments de ces ensembles et leurs configurations. Comme application de ces résultats, nous contruisons un domaine borné de $\mathbb{C}^n$ sans rétraction holomorphe non triviale.

0. Introduction

In classical mechanics the following Euler's theorem is well known: the general displacement of a rigid body with one point fixed is a rotation about some axis. So, if one considers an orientation-preserving isometry of a domain in $\mathbb{R}^3$ fixing one point, the fixed point set of this isometry will necessarily contain at least a segment, so the fixed point set cannot be a discrete set. In the euclidean space $\mathbb{R}^n$, one can always find a domain which has a euclidean isometry with exactly *one* fixed point, however for any n, if an isometry of a domain in $\mathbb{R}^n$ has *two* fixed points it will force the existence of at least a segment to belong to the fixed point set, and so this set will be at least one dimensional.

Switching to complex analysis, we remark that any holomorphic automorphism of a bounded domain in $\mathbb{C}^n$ (or in general, hyperbolic manifold) is an isometry in an invariant metric, so an Euler type statement is certainly meaningful, that is if this automorphism has a discrete fixed point set one can inquire what its cardinality and structure might be. To describe this more precisely, let $f : M \to M$ be a holomorphic self-map of a complex manifold M. Let $Fix(f)$ denote the set of fixed points $\{x \in M \mid f(x) = x\}$ of f. Suppose that this set is discrete. In this paper we shall examine mostly two questions. First, how large

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this set can be for specific cases: M is a bounded domain in $\mathbb{C}^n$, a hyperbolic manifold, etc., while f is a holomorphic automorphism or endomorphism. Second, the structure of $Fix(f)$, namely which points of M could form such a set for some holomorphic self-map of M. Everywhere below we consider only holomorphic self-maps (automorphisms or endomorphisms) of various complex manifolds, and for the sake of compactness the word *holomorphic* may be omitted.

In examining the cardinality of a discrete fixed point set, let's first consider the situation in one dimension. For a bounded domain $D \subset\subset \mathbb{C}$ the discrete fixed point set of a holomorphic map $f : D \to D$ can have no more than two points. This follows from the following observation: any map fixing at least two points must be an automorphism (H. Cartan), and any automorphism fixing three points must be the identity [PL]. An annulus gives an example of a domain that has an automorphism with exactly two fixed points.

In $\mathbb{C}^n$ the situation is not yet completely clear. For a convex domain one has an Euler type theorem: the isolated fixed point set of any endomorphism consists of at most one point (see Prop. 1.1). For a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary the number of points in a discrete fixed point set of an automorphism is finite (see Thm 1.2). Must the cardinality of an isolated fixed point set of an automorphism or endomorphism be bounded by a number depending only on the dimension of the manifold under consideration? As one can see below (in section 1) for endomorphisms of bounded domains in $\mathbb{C}^n$ the answer is negative. It is also negative for automorphisms of a general hyperbolic manifold and the entire $\mathbb{C}^n$. However, for an automorphism of a bounded domain in $\mathbb{C}^n$ the answer is not yet clear (see more discussions on that in the collection of problems at the conclusion of this paper: section 5).

We then turn to the consideration of which *single* points of a domain can form $Fix(f)$ for a holomorphic f. In section 2 we prove that if every point of a hyperbolic manifold is $Fix(f)$ for some automorphism f then the manifold must be homogeneous; we also show an example of a one dimensional non-homogeneous domain with infinite number of such points. In section 3 we consider *pairs of points* as fixed point sets, and prove that for any domain "most" pairs of points, if fixed, force a whole analytic set of complex dimension one to be fixed (compare to an Euler type statement above for a domain in $\mathbb{R}^n$).

An application is given in section 4: we construct a bounded domain in $\mathbb{C}^n$ such that if any holomorphic endomorphism of it fixes two distinct points, it will necessarily be the identity. As a consequence, this domain will have no non-trivial holomorphic retractions.

1. General cardinality statements.

Below we use the following notation: if $f : M \to M$ is a holomorphic self-map of M, then $Fix(f)$ is its fixed point set, and if such a set is discrete then $\#(Fix(f))$ is its cardinality. We start with two positive statements (e.g. the cardinality $\#(Fix(f))$ is bounded).

Proposition 1.1. *Let D be a bounded convex domain in $\mathbb{C}^n$, $f : D \to D$ be a holomorphic endomorphism. Then if $Fix(f)$ is discrete and non-empty, it consists of one point only.*

Proof. Follows from the main theorem in [Vi1]: such a set has to be connected. $\square$

Remark. Description of some properties of fixed point sets in convex domains can be found in [Ab].

Theorem 1.2. *For any strictly pseudoconvex domain $D \subset \mathbb{C}^n$ with real analytic boundary, $n \geq 1$, the cardinality $\#(Fix(f))$ of the isolated fixed point set of an automorphism $f \in Aut(D)$ is finite. Moreover, there is a number $m = m(D)$ such that $\#(Fix(f)) \leq m$.*

Proof. If D is biholomorphic to the ball or if $n = 1$, then the statement is clear. Assume that $n \geq 2$ and D is not biholomorphic to the ball. By a theorem in [VEK], there is a neighborhood U_1 of $\overline{D}$ such that each automorphism of D extends to be an injective holomorphic map on U_1. Consider a $g \in Aut(D)$. Choose domains U_2, U_3 with smooth boundaries so that $D \subset\subset U_3 \subset\subset U_2 \subset\subset U_1$. For every $h \in Aut(D)$ in some neighborhood of g, $h(\partial U_2)$ is so close to $g(\partial U_2)$ that $h(\partial U_2) \cap g(\overline{U}_3) = \emptyset$. Since $h(U_2)$ is a connected component of $\mathbb{C}^n \backslash h(\partial U_2)$ and since $h(U_2) \supset D$, we see that $h(U_2) \supset g(U_3)$ for every $h \in Aut(D)$ in some neighborhood of g. Since $Aut(D)$ is compact, there is a neighborhood Q of $\overline{D}$ such that $Q \subset g(U_1)$ for each $g \in Aut(D)$. Let U be the interior of the intersection of the sets $g(U_1)$, $g \in Aut(D)$. Then $U \supset Q$ and $g(U) = U$ for each $g \in Aut(D)$. I.e., each automorphism of D is also an automorphism of U. There is a finite cover of open sets $\{V_j : j = 1, \dots, m\}$ of $\overline{D}$ such that each pair of points in a V_j is connected by a unique distance-minimizing geodesic with respect to the Bergman metric of U. Let $f \in Aut(D)$. If f fixes two points in a V_j, f must fix each point on the unique distance-minimizing geodesic connecting the two points. Consequently, each V_j contains at most one isolated fixed point of f. Therefore, the number of isolated fixed points of f is $\leq m$. $\square$

We now present counterexamples (e.g. the cardinality $\#(Fix(f))$ can be arbitrary, even infinity).

Proposition 1.3. *For any $k \subset \mathbb{N}$, there exists a bounded domain $D \subset \mathbb{C}^n$, $n \geq 2$, and a holomorphic endomorphism $f : D \to D$, such that $\#(Fix(f)) = k$.*

Proof. Without any loss of generality we can present an example for $n = 2$.

Let S be the open Riemann surface in $\mathbb{C}^2$:

$$S = \{(x, y) \in C^2 \mid y^2 = (x - a_1)...(x - a_k)\},$$

where $a_1, ..., a_k$ are k distinct points in $\mathbb{C}$. The restriction g of $(x, y) \mapsto (x, -y)$ to S has exactly k fixed points. Following [GR, VIII, C8, p.257] there exists a holomorphic retraction $\rho : V \to S$ of an open neighborhood V of S onto S. Now the mapping $f := g \circ \rho : V \to V$ has exactly k fixed points. Of course V is not bounded, but we can consider a bounded open set $W \subset V$, $W \ni (a_s, 0)$ for all $s = 1, ..., k$ and such that $g(W) = W$. This bounded domain will have the same property. ☐

Proposition 1.4. *There exists a hyperbolic manifold with a holomorphic automorphism whose fixed point set is discrete and consists of an infinite number of points.*

Proof. Consider the submanifold X of D^2 defined by $y^2 = B(x)$, where D is the open unit disc and B is a Blaschke product with an infinite number of zeroes, the restriction to X of the map $(x, y) \to (x, -y)$ is an automorphism of X and has an infinite number of isolated fixed points. ☐

Proposition 1.5. *For any $n \geq 2$ and any $k \in \mathbb{N}$, there exists a polynomial automorphism f of $\mathbb{C}^n$, such that $\#(Fix(f)) = k$.*

Proof. . Let $a_1, ..., a_k$ be k distinct complex numbers. Consider the map $H : \mathbb{C}^n \to \mathbb{C}^n$ given by
$w_1 = z_1 + z_2 + (z_1 - a_1)(z_1 - a_2)...(z_1 - a_k)$
$w_2 = z_2 + (z_1 - a_1)(z_1 - a_2)...(z_1 - a_k)$
$w_s = iz_s$ for all $s = 3, ..., n$
One can easily check that this map is an automorphism [$(z_1, ..., z_n)$ can be represented as polynomials of $(w_1, ..., w_n)$], whose fixed point set is the set of the following k points: $(a_1, 0, ..., 0), (a_2, 0, ..., 0), ..., (a_k, 0, ..., 0)$. ☐

Corollary 1.6. *Let $n \geq 2$; $p_1, p_2, ..., p_k$ are k distinct points in $\mathbb{C}^n$. Then there exists a polynomial automorphism $g \in Aut(\mathbb{C}^n)$ such that $Fix(g) = \{p_1, p_2, ..., p_k\}$.*

Proof. Let $p_j = (a_j, b_j)$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C}^{n-1}$. Without any loss of generality we assume that the a_j's are all distinct (in case they are not, one can first use an invertible linear transformation of $\mathbb{C}^n$ to achieve this). Consider the polynomial transformation $F : w_1 = z_1, w' = z' + f(z_1)$, where $f : \mathbb{C} \to \mathbb{C}^{n-1}$ is the Lagrange interpolation polynomial map satisfying $f(a_j) = b_j$. Then $F(a_j, 0) = p_j$, $j = 1, ..., k$, and $F \in Aut(\mathbb{C}^n)$. If $H \in Aut(\mathbb{C}^n)$ is the automorphism in the proof of the previous proposition, then the automorphism $g = F \circ H \circ F^{-1}$ is such that $Fix(g) = \{p_1, p_2, ..., p_k\}$. $\square$

2. Single points as fixed point sets

Here we consider some statements when the fixed point set of a holomorphic automorphism is one point, specifically: which points can be a fixed point set of an automorphism. First we show that if every point can be a fixed point set for an automorphism of a hyperbolic manifold then this manifold must be homogeneous. Second we provide an example when there are infinite number of points in the domain, each of which is a fixed point set for some holomorphic automorphism.

Theorem 2.1. *If every point of a hyperbolic manifold D is a fixed point set for some holomorphic automorphism of D, then D is a homogeneous manifold.*

For some concrete cases we have the following

Corollary 2.2. *(A) If in the above theorem $D \subset\subset \mathbb{C}^2$, then D is biholomorphic to the unit ball $\mathbb{B}^2$ or the polydisc $\mathbb{U}^2$.*

(B) If in the above theorem $D \subset\subset \mathbb{C}^n$ has a smooth C^2 boundary, then D is biholomorphic to the unit ball $\mathbb{B}^n$.

To prove the Corollary we note that *(A)* there are only two kinds of bounded homogeneous domains in $\mathbb{C}^2$: the unit ball and the polydisc, and *(B)* in $\mathbb{C}^n$ there is only one bounded homogeneous domain with a smooth boundary: the unit ball (this is a consequence of a Wong-Rosay theorem (see [Ro], [Wo]). We will now prove the theorem.

Proof. 1. First we note that the theorem will follow from a local statement: let $x \in D$, then there exists a neighborhood U_x of x such that for any $y \in U_x$ there is a $g \in Aut(D)$ such that $g(y) = x$.

Indeed, if this is true consider two arbitrary points $a, b \in D$, connect them by a compact path L, cover L by a finite number of U_x, $x \in L$, and one can obtain an $f \in Aut(D)$, such that $f(a) = b$.

2. We now prove the local statement. Let $x \in D$. By [FMV], for each point $x \in D$ there is an invariant Hermition metric in some

neighborhood of the orbit $G(x)$, where $G = Aut(D)$. Consider a small enough ball $b(x, \varepsilon)$ in that metric with center x and radius ε, $\varepsilon > 0$ will be determined by the construction later. Let $y \in b(x, \varepsilon)$; consider the orbit $O(y) = \{z \in D : \exists g \in Aut(D), g(y) = z\}$. Consider now a point $p \in O(y)$, such that $d(x, p) = d(x, O(y))$, where $d(\cdot, \cdot)$ denotes the distance function induced by the local invariant metric. Clearly, $p \in b(x, \varepsilon)$. If $p = x$, there is nothing to prove; otherwise consider a small ball b_1 of radius $< \frac{1}{4}d(x, p)$ that lies inside $b(x, d(x, p))$, and such that $\partial b_1 \cap \partial b(x, d(x, p)) = p$.

This construction is possible if ε is small enough, fixing such an $\varepsilon = \varepsilon(x)$, we denote $b(x, \varepsilon) = U_x$.

We observe that $O(y) \cap b(x, d(x, p)) = \emptyset$. Let q denote the center of the ball b_1. By the assumption of the theorem there exists an $h \in Aut(D)$ whose fixed point set is q. Now $h(p) \neq p$, and $h(p) \in \partial b_1$, since $h(\partial b_1) = \partial b_1$. We now conclude that $h(p) \in O(y) \cap b(x, d(x, p))$, which contradicts the previous observation that this intersection is empty. Therefore $x = p \in O(y)$, and the theorem has been proved. □

We now provide the following example.

Proposition 2.3. *There exists a domain D in $\mathbb{C}$ with infinite number of points each of which is the fixed point set for a holomorphic automorphism of D.*

Proof. Consider $D = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} \Delta(n, 1/3)$ where $\Delta(n, 1/3)$ is a disk with center at $n \in \mathbb{Z}$ and radius 1/3. Consider $f_k : z \mapsto (-z + (2k+1))$. Then for any $k \in \mathbb{Z}$, $f_k \in Aut(D)$, and its fixed point set consists of one point $Fix(f) = \{k + 1/2\}$. □

3. Pairs of points as fixed point sets

Here we examine the situation when the fixed point set consists of exactly two points. Though such domains exist, no domain can have too many pairs of distinct points as a fixed point set for an automorphism.

Theorem 3.1. *Let $D \subset\subset \mathbb{C}^n$. The set $N \subset D^2$ of all pairs, each of which cannot be a fixed point set for a holomorphic automorphism of D, contains a full measure set in D^2.*

It follows from the following

Lemma 3.2. *Let $D \subset\subset \mathbb{C}^n$, $a \in D$. Then there exists a complex analytic set $Z \subset D$ ($\dim Z < n$), such that if $b \in D \setminus Z$ then the two points $\{a, b\}$ are such that for any automorphism f fixing these two points, the fixed point set of f is at least one (complex) dimensional.*

First we need the following

Lemma 3.3 (H. Cartan)**.** *([Ca1, p.80]) Let $D \subset\subset \mathbb{C}^n$, let $z \in D$, and let $I_z = I_z(D)$ be the isotropy subgroup at z of the automorphism group of D. Then there exists a holomorphic map $\phi : D \to \mathbb{C}^n$ such that $\phi(z) = 0$, $\phi'(z) = id$, and for all $f \in I_z$ one has $\phi \circ f = f'(z) \circ \phi$.*

As in [Vi2, thm 2.3], for the proof of this Lemma, we define $\phi : D \to \mathbb{C}^n$ by

$$\phi(\zeta) = \int_{G_z} f'(z)^{-1}(f(\zeta) - z)\, d\mu(f),$$

where $d\mu$ is the Haar measure on I_z. Then $\phi(z) = 0$, $\phi'(z) = id$ (and therefore ϕ is locally biholomorphic), and $\phi \circ g = g'(z) \circ \phi$ for each $g \in I_z$.

We are now ready to prove Lemma 3.2

Proof. Let $Z = \{z \in D | \varphi(z) = 0\}$. If $b \in D \backslash Z$, then suppose $f \in Aut(D)$ and f fixes both points a and b. We have $f'(a) \cdot \varphi(b) = \varphi(f(b)) = \varphi(b)$. Since by choice $\varphi(b) \neq 0$, and φ is biholomorphic in the neighborhood U of a, for a number λ, $|\lambda| > 0$, small enough, there exists a point $c \in U \subset D$, $c \neq a$, $\varphi(c) = \lambda\varphi(b)$, and $f(c) \in U$. Now $\varphi(f(c)) = f'(a) \cdot \varphi(c) = f'(a) \cdot \lambda\varphi(b) = \lambda\varphi(b) = \varphi(c)$.

Since φ is biholomorphic in U we have $f(c) = c$. □

4. Application for holomorphic retractions

Obviously any one point in a domain can be the fixed point set for an *endomorphism* of this domain. The theorem below gives an example of a domain that no two distinct points can be the fixed point set for an endomorphism. Moreover:

Theorem 4.1. *There is a domain $D \subset\subset \mathbb{C}^n$ such that for any two distinct points $p \neq q \in D$, if a holomorphic endomorphism $f : D \to D$ fixes these two points ($f(p) = p, f(q) = q$) then $f = id$.*

Proof. We will construct the example in $\mathbb{C}^2$; $\mathbb{C}^n$ with $n > 2$ can be dealt with similarly.

We denote $B(z, r)$ - the euclidean ball with center at z and of radius r, $b(z, r) \subset D \subset\subset \mathbb{C}^2$ - ball in the Kobayashi metric (in D) with center at $z \in D$ and of (Kobayashi) radius r. $B = B(0, 1) = \mathbb{B}^2$ the unit ball in $\mathbb{C}^2$.

1. Statement. *Let $a, b \in B$ be two distinct points, L is the complex line through these points. Let $f \in H(B, B)$ fix these two points. Then f fixes all the points of $L \cap B$.*

Proof of this statement follows from Example 1, Section 4 in [Vi3].

2. Statement. *If three distinct points $a, b, c \in B$ do not belong to the same complex line, then if $f \in H(B, B)$ fixes these points then $f = id$.* Proof of this statement follows from example 1, sec. 4 in [Vi3].

3. Statement. *$\forall a \in B_1 = B(0, 1/2)$, $a \neq 0$ there exists a unique point $p \in \partial B_1$ that is closest to a in the Kobayashi metric of the larger ball B: $k(a, p) = \min\limits_{l \in \partial B_1} k(a, l)$, where $k(\cdot, \cdot)$ is the Kobayashi distance in B. Moreover, there exists a real number s such that $p = s \cdot a$.*
To prove this let σ be the Kobayashi distance from a to ∂B_1. There exists an r such that $b(0, \sigma) = B(0, r)$. Consider now $f \in Aut(B)$ such that $f(0) = a$. Then $f(B(0, r)) = f(b(0, \sigma)) = b(a, \sigma)$, and from this construction we conclude that $\partial b(a, \sigma)$ and ∂B_1 have only one common point p, moreover the vector p is the intersection of the real line $\{s \cdot a \mid s \in \mathbb{R}\}$ with ∂B_1.

4. *Example of a domain $D \subset\subset \mathbb{C}^2$ and two points $\{a, b\} \in D$ such that any endomorphism of D fixing those two points must be the identity.*

Consider $D = B \backslash \overline{B_1}$ and two distinct points $a, b \in D$ such that for the complex line L connecting them $L \cap B_1 \neq \emptyset$ and $0 \notin L$. Suppose $f \in H(D, D)$ fixes both points a, b. By Hartogs principle f can be extended to $F \in H(B, B)$, and therefore F fixes $L \cap B$. One can now pick a point $p \in L \cap B_1$ so that
(1) the boundary ∂B_1 has a unique point $c \in \partial B_1$ closest (in the Kobayashi metric of B) to p, and
(2) c is not lying on L.
Since the Kobayashi metric cannot increase under holomorphic maps, $F(c) = c$. Now the three points $a, b, c \in B$ do not lie on the same complex line and by the previous statement F (and therefore f) must be the identity.

5. We are now ready to prove the theorem by providing the main example: *a domain $D \subset\subset \mathbb{C}^2$ such that an endomorphism fixing any two given distinct points $\{a, b\} \in D$ is the identity.*

All we need to do is to take B and remove a (countable) number of closed neighborhoods of portions of spheres, so that any complex line intersecting B will intersect at least one of these removed spheres and then use the approach of the previous example.

For $k = 1, 2, \ldots$, let

$$\varepsilon_k = 1/2^{(4k)!}, \quad \delta_k = 1/2^{(4k)!+1},$$

$$\Omega_{2k+1} = \overline{B(0, 1 - \delta_k) \backslash B(0, 1 - \varepsilon_k)} \cap \{Im z_2 \geq -1/2\},$$

and

$$\Omega_{2k+2} = \overline{B(0,1-4\delta_k)\backslash B(0,1-4\varepsilon_k)} \cap \{Imz_2 \leq 1/2\}.$$

We will also need two more sets defined differently:

$$\Omega_1 = \overline{B(\alpha,1-\delta_1)\backslash B(\alpha,1-\varepsilon_1)} \cap \{Imz_2 \geq -1/2\};$$

$$\Omega_2 = \overline{B(\beta,1-4\delta_1)\backslash B(\beta,1-4\varepsilon_1)} \cap \{Imz_2 \leq 1/2\},$$

where $\alpha = (2^{-8!}, 0)$, $\beta = (0, 2^{-8!})$. And finally $D = B\backslash(\bigcup_{s=1}^{\infty} \Omega_s)$. Now D is a connected open set. Let $a, b \in D$ and $f \in H(D, D)$ fixes a, b. Then f can be extended to a holomorphic function $F : B \to B$. Let L be the complex line connecting a, b, then (see Statement 1 above) $F|_L = id$.

L intersects an infinite number of Ω_s. If $0 \notin L$ there will always be at least one of two possibilities: either for some $s = 2k+1$ there is a point $z \in L \cap \partial B(0, 1-\delta_k)$, and $Im(z) > -1/2$, or for some $s = 2k+2$ there is a point $z \in L \cap \partial B(0, 1-4\delta_k)$, and $Im(z) < 1/2$.

Similarly , if $0 \in L$ there will always be at least one of two possibilities: either for $s = 1$ there is a point $z \in L \cap \partial B(\alpha, 1-\delta_1)$, and $Im(z) > -1/2$, or for $s = 2$ there is a point $z \in L \cap \partial B(\beta, 1-4\delta_1)$, and $Im(z) < 1/2$. The above choice of s is restricted in the following two cases: if $\alpha \in L$, we pick $s = 2$, if $\beta \in L$, we pick $s = 1$. In any case we fix this point $z \in L \cap \partial\Omega_s$. If a point $p \in L$ is close enough to z, then the closest (in Kobayashi metric of the ball B) point to p in the boundary $\partial\Omega_s$ is a unique point $c \in \partial\Omega_s$ that does not lie on L. Since $F(p) = p$, F is a continuous non-increasing map in the Kobayashi metric, and $F(\overline{D}) \subset \overline{D}$, we conclude that $F(c) = c$.

Now F fixes three points in B that do not lie on the same complex line, and therefore $F = id$, so $f = id$. $\square$

A map $D \to D$ is a retraction, if $f \circ f = f$. A trivial retraction is either a constant map, or the identity.

Corollary 4.2. *The domain in the above theorem has no non-trivial holomorphic retraction.*

5. Final remarks, unsolved problems

5.1. **Some problems.** The main question that remains open is this.
1. Let D be a bounded domain in $\mathbb{C}^n$, $f \in Aut(D)$, and $Fix(f)$ is a discrete set. Can $\#(Fix(f)) = \infty$?

If one considers the domain $D \subset \mathbb{C}^n$ which is a direct product of n annuli, one can then find an $f \in Aut(D)$ with $\#(Fix(f)) = 2^n$. So, the next natural unsolved question is

2. Let $n \geq 2$, D be a bounded domain in $\mathbb{C}^n$, with a piecewise smooth boundary, $f \in Aut(D)$, and $Fix(f)$ is a set of isolated points. Can $\#(Fix(f)) \geq 2^n + 1$? (As noted earlier, for $n = 1$ the answer is negative [PL]).

A more restricted version of the above question is a generalization of Theorem 1.2.

3. Is there a number m such that for any strongly pseudoconvex domain $D \subset\subset \mathbb{C}^n$, $\partial D \in C^\infty$, and $f \in Aut(D)$, if $Fix(f)$ is a set of isolated points, then $\#(Fix(f)) \leq m$, where $m = m(n)$ (i.e. m depends on the dimension only)?

The next question, in case of a positive answer, would be a generalization of Proposition 1.1.

4. Let D be a bounded contractible domain in $\mathbb{C}^n$, $f \in Aut(D)$, and $Fix(f)$ is a non-empty set of isolated points. Is $\#(Fix(f)) = 1$?

5.2. **A (long) Remark.** We now turn to a connection of this paper with the notion introduced and studied in papers [FK1, FK2, FM, KK, FMV, Vi2, Vi3]: *determining sets.*

Let M be a complex manifold. Let $H(M, M)$ denote the set of holomorphic endomorphisms of M, and $Aut(M)$ the set of automorphisms of M.

Definition 5.1. *A set $K \subset M$ is called a determining subset of M with respect to $Aut(D)$ ($H(M, M)$ resp.) if, whenever g is an automorphism (endomorphism resp.) such that $Fix(g) \supseteq K$, then $Fix(g) = M$ (e.g. g is the identity map of M).*

One can now introduce a generalized notion of *quasi-determining set* for a complex manifold M:

Definition 5.2. *A set $K \subset M$ is called a quasi-determining subset of M with respect to $Aut(D)$ ($H(M, M)$ resp.) if, whenever g is an automorphism (endomorphism resp.) such that $Fix(g) \supseteq K$, then K is a proper subset of $Fix(g)$.*

Another way to state this definition: A set $K \subset M$ is called a quasi-determining subset of M with respect to $Aut(D)$ ($H(M, M)$ resp.) if it cannot be the fixed point set of any automorphism (endomorphism resp.) of M.

There is an obvious reformulation of a number of results in our paper by using this notion. For example, Proposition 1.1 means that any two points in a convex domain form a quasi-determining set; Theorem 1.2 states that any $m + 1$ points in D form a quasi-determining set, etc.

This definition obviously leads to a number of other open questions, which will be addressed in the future.

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