# ON DETERMINING SETS FOR HOLOMORPHIC AUTOMORPHISMS

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**Abstract:** *We study sets $K$ in the closure of a domain $D \subset \mathbb{C}^n$ such that, if an automorphism $\varphi$ of $D$ fixes each point of $K$, then $\varphi$ is the identity mapping. A separate result is proved for the case that $K$ lies entirely in the boundary of $D$.*

## 1. Principal Ideas

We begin by defining the central idea in this paper.

**Definition 1.1.** Let $K$ be a subset of the closure $\overline{D}$ of a bounded domain $D \subset \mathbb{C}^n$. The set $K$ is said to be a *determining subset* of $\overline{D}$ if, whenever $g$ is an automorphism of $D$ that is continuous up to the boundary of $D$ and $g(k) = k \ \forall k \in K$, then $g$ is the identity map of $D$.

In our previous paper [FKKM] we considered $K$ lying entirely in $D$ (and automorphisms were not necessarily continuous up to the boundary). Our goal now is to study the situation when some, or all, of the points of $K$ lie in the boundary of the domain $D$. Recall that, when the domain $D$ is the disc in the plane, then two interior points are a determining set for automorphisms; but it takes three boundary points to be a determining set (the difference between "two" and "three" here is best understood in terms of the Iwasawa decomposition of the automorphism group). Just examining the situation in one dimension, one might surmise that if the topology of the domain is more complicated (higher connectivity) then it takes fewer points to make a determining set. We would like to explore all these aspects in the discussions below.

We begin our work by investigating more generally what happens if some, but not all, points of $K$ lie on the boundary $\partial D$. First we prove that a similar theorem to the one in [FKKM] holds in the case that $K$ consists of the same number $n+1$ points and one of these is in $D$ while the rest are on the boundary. In Theorem 1.2 we note that the Euclidean dimension of $D \times (\partial D)^n$ is $2n + n\cdot(2n-1) = 2n^2+n$, and we use $(2n^2+n)$-dimensional Hausdorff-Lebesgue measure on subsets of $D \times (\partial D)^n$.

**Theorem 1.2.** *Let $D \subset \mathbb{C}^n$ be a bounded domain with $C^1$ boundary such that the automorphisms of $D$ are continuous up to the boundary. Then there exists a subset $W$ of $D \times (\partial D)^n$ of full Hausdorff-Lebesgue measure (i.e., $(D \times (\partial D)^n)\backslash W$ has $(2n^2+n)$-dimensional Hausdorff-Lebesgue measure 0) with this property: if $(p_0, p_1, \ldots, p_n) \in W$ then $\{p_0, p_1, \ldots, p_n\}$ is a determining subset of $\overline{D}$.*

**Remark.** The number of points $(n+1)$ for a general domain in $\mathbb{C}^n$ in the above theorem is the least possible. To verify this assertion consider the unit ball $B \subset \mathbb{C}^n$ and $n$ arbitrary points $(p_0, p_1, ..., p_{n-1})$, where $p_0 \in B, p_i \in \partial B$ for $i = 1, ..., n-1$. For a domain $D$ let $\mathrm{Aut}(D)$ denote the automorphism group of $D$.

Consider $g \in \mathrm{Aut}(B)$ such that $g(p_0) = 0$. Consider now $n-1$ vectors $g(p_i)$, and the complex linear space $\pi$ spanned by these vectors. Since $\dim(\pi) \le n-1$, there is a rotation $f \in \mathrm{Aut}(B)$ that is not the identity and keeps all the points of $\pi$ fixed. Now the automorphism $h = g^{-1} \circ f \circ g \in \mathrm{Aut}(B)$ is not the identity and it fixes all $n$ points $(p_0, p_1, ..., p_{n-1})$.

To prove Theorem 1.2 we will need the following four statements. In these, we will use the notation $f'$ to denote the complex Jacobian matrix of the mapping $f$.

**Lemma 1.3** (H. Cartan). *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, $a \in D$, $f : D \to D$ a holomorphic map such that $f(a) = a$ and $f'(a) = \mathrm{id}$. Then $f = \mathrm{id}$.*

**Lemma 1.4** (H. Cartan). *Let $D \subset\subset \mathbb{C}^n$, let $z \in D$, and let $G_z$ be the isotropy subgroup at $z$ of the automorphism group of $D$. Then there exists a holomorphic map $\phi : D \to \mathbb{C}^n$ such that $\phi(z) = 0$, $\phi$ is a local coordinate system at $z$, and for all $f \in G_z$ one has $\phi \circ f = f'(z) \circ \phi$.*

As in [VIG], for the proof of Lemma 1.4, we define $\phi : D \to \mathbb{C}^n$ by

$$\phi(\zeta) = \int_{G_z} f'(z)^{-1}(f(\zeta) - z)\, d\mu(f),$$

where $d\mu$ is the Haar measure on $G_z$ (note that $G_z$ is, perforce, compact). Then $\phi(z) = 0$, $\phi$ gives a local coordinate system at $z$, and $\phi \circ g = g'(z) \circ \phi$ for each $g \in G_z$. For a complete proof see [VIG]. It may be worth noting that Bergman representative coordinates may serve the same purpose as the mapping $\phi$; see [GRK].

The following is a one-dimensional version of the Privalov uniqueness theorem ([GOL, Chap. 10, §2]).

**Lemma 1.5** (Privalov). *If $f$ is holomorphic on a bounded domain $D \subset \mathbb{C}$ with a rectifiable boundary, $f$ is continuous up to the boundary, and is equal to zero on a set $S \subset \partial D$ of positive one-dimensional Hausdorff-Lebesgue measure, then $f \equiv 0$.*

**Corollary 1.6.** *Let $D$ be a bounded domain in $\mathbb{C}^n$ with $C^1$ boundary, $m, k$ integers, $m > 0$, $m \ge k \ge 0$, $\psi : \overline{D}^m \to \mathbb{C}$, such that $\psi$ is holomorphic on $D^m$ and continuous up to the boundary. If $\psi = 0$ on a subset $M \subset D^k \times (\partial D)^{m-k}$ of positive $((2nk + (2n-1)(m-k))$-dimensional) Hausdorff-Lebesgue measure then $\psi \equiv 0$.*

*Proof of the Corollary.*

**1.** First let $k = 0$. We use induction on $m$. For $m = 1$ the conclusion follows from the above Privalov uniqueness theorem by considering the slicing of $D$ by one-dimensional complex lines, and applying Lemma 1.5 on the slices.

Now suppose that the statement has been verified for $m = s$. Let $m = s + 1$. There is an $M_1 \subset \partial D$ with positive ($(2n-1)$-dimensional) Hausdorff-Lebesgue measure such that, for each $z \in M_1$, the set $N_z = \{w \in (\partial D)^s : \psi(z, w) = 0\}$ has positive ($(2n-1)s$-dimensional) Hausdorff-Lebesgue measure. By the induction hypothesis we see that $\psi = 0$ on $M_1 \times \overline{D}^s$. Now $\psi = 0$ on a positive-measure subset of the boundary $\partial(D^{s+1})$, and the statement now follows from the very first observation in this proof.

**2.** Now consider the case $k > 0$. There is an $\widetilde{M}_1 \subset D^k$ with positive ($2nk$-dimensional) Hausdorff-Lebesgue measure such that for each $z \in \widetilde{M}_1$ the set $\widetilde{N}_z = \{w \in (\partial D)^{m-k} : \psi(z, w) = 0\}$ has positive ($(2n-1)(m-k)$-dimensional) Hausdorff-Lebesgue measure. By the case already proved, $\psi = 0$ on $\widetilde{M}_1 \times \overline{D}^{m-k}$. Now $\psi = 0$

on a set of full measure in $D^m$ and the statement follows from the standard uniqueness theorem for holomorphic functions. □

*Proof of Theorem 1.2.* We follow Vigué's idea [VIG] to use Lemma 1.4 for determining sets. Fix $z \in D$. We consider the mapping $\phi$ from the proof of Lemma 1.4. Let $\psi : \overline{D}^n \to \wedge^n \mathbb{C}^n \cong \mathbb{C}$ be defined by $\psi(w_1, \ldots, w_n) = \phi(w_1) \wedge \cdots \wedge \phi(w_n)$. Then $\psi$ is a submersion near $(z, \ldots, z)$, so $\psi \not\equiv 0$. Let $S_z \subset (\partial D)^n$ be defined as $S_z = \{w = (w_1, ..., w_n) \in (\partial D)^n : \exists g_w \in G_z, g_w \neq \mathrm{id}, g_w(w_i) = w_i, i = 1, ..., n\}$. Let $w \in S_z$. So $g'_w(z)\phi(w_j) = \phi(w_j)$, $j = 1, \ldots, n$. By Lemma 1.3, $g'_w(z) \neq \mathrm{id}$. Thus $\phi(w_1), \ldots, \phi(w_n)$ belong to a proper complex linear subspace of $\mathbb{C}^n$, hence $\psi(w) = 0$. Therefore $\psi = 0$ on $S_z$. By Corollary 1.6 (where $k = 0, m = n$), $S_z$ has $(2n-1)n$-dimensional measure 0. Let $V = \{(z, w) : z \in D, w \in S_z\}$ and $W = (D \times (\partial D)^n) \backslash V$. Then $W$ has the specified properties. □

The following is a more general result.

**Theorem 1.7.** *Let $D \subset \mathbb{C}^n$ be a bounded domain with $C^1$ boundary such that the automorphisms of $D$ are continuous up to the boundary. Let $1 \le k \le n+1$. Then there exists a subset $W$ of $D^k \times (\partial D)^{n+1-k}$, of full Hausdorff-Lebesgue measure, such that if $(p_0, p_1, \ldots, p_n) \in W$ then $\{p_0, p_1, \ldots, p_n\}$ is a determining subset of $\overline{D}$.*

*Proof.* Fix $z \in D$. Now one can proceed in a manner similar to the proof of Theorem 1.2 by constructing a holomorphic function that will be equal to zero on the set $\widetilde{S}_z \subset D^{k-1} \times (\partial D)^{n+1-k}$ , $\widetilde{S}_z = \{w = (w_1, ..., w_n) \in D^{k-1} \times (\partial D)^{n+1-k} : \exists g_w \in \mathrm{Aut}(D); g_w \neq \mathrm{id}; g_w(w_j) = w_j \ \forall j\}$. By Corollary 1.6, $\widetilde{S}_z$ has zero measure. As before the set $W = (D^k \times (\partial D)^{n+1-k}) \backslash \{(z, w) : z \in D, w \in \widetilde{S}_z\}$ has the specified properties. □

The next lemma is a boundary version of the classical Schwarz lemma.

**Lemma 1.8.** *Let $D \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain with $C^\infty$ smooth boundary. Further, assume that the automorphism group of $D$ is compact. If $g \in \mathrm{Aut}(D)$, and there exists a point $z \in \partial D$ such that $g(z) = z$, and $g'(z) = \mathrm{id}$, then $g = \mathrm{id}$.*

EXAMPLE 1.9. The compactness of the group is necessary for the lemma, since the conclusion is not true for the ball $B$ in $\mathbb{C}^n$, $n \ge 1$. [Of course $B$ does *not* have compact automorphism group.] Let $(z, w)$ denote the coordinates in $\mathbb{C}^n$, where $z \in \mathbb{C}$, $w \in \mathbb{C}^{n-1}$. Define $g : \bar{B} \to \mathbb{C}^n$ by

$$g(z, w) = \left( -i \frac{z - a}{1 - \bar{a} z} \, , \, e^{-\pi i/4} \frac{\sqrt{1 - |a|^2}}{1 - \bar{a} z} w \right) ,$$

where $a = (-1 + i)/2$. Then $g$ is an automorphism of $B$, $g(-1, 0) = (-1, 0)$, $g'(-1, 0) = \mathrm{id}$. But of course $g$ is not the identity map. It should be stressed that, by an important theorem of Bun Wong [WON], the ball is the *only* strongly pseudoconvex domain with non-compact automorphism group.

*Proof of Lemma 1.8:* By a result of [GRK], there is a smooth metric $g$ on $\overline{D}$ such that any automorphism of $D$ is an isometry in $g$. Now observe that, since $\partial D$ is smooth, there is a convex cone, with nonempty interior, of directions at $z$ such that the geodesics emanating from $z$ in those directions will travel (for a short time) into the interior of $D$. Of course the mapping $g$, being an isometry, will preserve each of those geodesics. Hence $g$ will be the identity on an open set in $D$. As a result, $g = \mathrm{id}$. □

Now we can prove our theorem when all points are on the boundary.

**Theorem 1.10.** *Let $D \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain with $C^\infty$ boundary and compact automorphism group. Then there exists a subset $W$ of $(\partial D)^{n+1}$ such that $W$ has full ($(2n-1)(n+1)$-dimensional) Hausdorff-Lebesgue measure and such that if $(p_0, p_1, \ldots, p_n) \in W$ then $\{p_0, p_1, \ldots, p_n\}$ is a determining subset of $\overline{D}$.*

*Proof.* The proof of Theorem 1.2, with Cartan's theorem (Lemma 1.3) replaced by Lemma 1.8, applies here. □

In the above theorem the assumption of compact automorphism group is crucial. For the unit ball $B$ in $\mathbb{C}^n$ with $n \geq 2$, the only strongly pseudconvex domain that has non-compact automorphism group, the following is true: there is an open dense subset $W$ of $(\partial B)^{n+1}$ (of full measure) such that if $(p_0, p_1, \ldots, p_n) \in W$ then $\{p_0, p_1, \ldots, p_n\}$ is a determining subset of $\overline{B}$.

The above statement does not hold for $n = 1$ since one needs three points on the boundary of the unit disc to make a determining subset.

## 2. Examples

The four examples presented here indicate in what sense our results are sharp, and in what sense something more may be true (especially in the case of more complicated topology of the domain $D$).

Example 2.1. Let

$$D = B(0,2) \setminus \overline{B(0,1)} \subseteq \mathbb{C}^2 .$$

Then the only automorphisms of $D$ are unitary rotations. Observe that this domain $D$ has non-trivial topology.

We note that any rotation in the $z_2$ variable will fix *all* points of $\mathbf{d} \equiv \{(z_1, 0) \in \overline{D}\}$. Then $\mathbf{d} \cap D$ is uncountable and also $\mathbf{d} \cap \partial D$ is uncountable. But of course $\mathbf{d}$ is *not* generic, in the sense that it is a complex variety (hence has no real component). □

Example 2.2. Let $D \subset \mathbb{C}^2$ be defined as

$$D = B(0,1) \setminus \left( \overline{B\big((-1/2,0),1/4\big)} \cup \overline{B\big((1/2,0),1/4\big)} \cup \overline{B\big((0,-1/2),1/4\big)} \cup \overline{B\big((0,1/2),1/4\big)} \right).$$

Then the automorphism group of $D$ is generated by the maps $(z_1, z_2) \mapsto (-z_1, z_2)$, and $(z_1, z_2) \mapsto (z_2, z_1)$, because the only possible automorphisms are unitary rotations that preserve the excised balls. Of course the mapping $(z_1, z_2) \mapsto (z_1, -z_2)$ fixes any point of the form $(z_1, 0)$ in $\overline{D}$. This set contains uncountably many points in the boundary and uncountably many points in the interior.

By contrast, any single point of the form $(-1 + \epsilon, \sqrt{2\epsilon - \epsilon^2})$, some small $\epsilon > 0$, is a determining set for automorphisms. □

EXAMPLE 2.3. Let $U = B(0,1) \subseteq \mathbb{C}^2$ and let $E$ be a small, asymmetric (i.e., having no unitary symmetries) perturbation of the small ball $\overline{B(0,1/10)} \subseteq \mathbb{C}^2$. Define

$$\varphi(z_1,z_2) = \left( \frac{z_1 - 1/2}{1 - z_1/2}, \frac{\sqrt{1-[1/2]^2}z_2}{1-z_1/2} \right) ,$$

Finally, set

$$D = U \setminus \left( \bigcup_{j=-\infty}^{\infty} \varphi^j(E) \right) .$$

Here $\varphi^0 = \mathrm{id}$, $\varphi^j = \varphi \circ \varphi \circ \cdots \circ \varphi$ ($j$ times) when $j > 0$, and $\varphi^j \equiv (\varphi^{-1})^{|j|}$ when $j < 0$. Observe that, by the Hartogs extension phenomenon, any automorphism of $D$ is the restriction to $D$ of an automorphism of the unit ball. Because $E$ is asymmetric, the automorphism group therefore contains no unitary rotations. Therefore it consists only of Möbius transformation of the form

$$(z_1,z_2) \longmapsto \left( \frac{z_1 - a}{1 - az_1}, \frac{\sqrt{1-a^2}z_2}{1-az_1} \right) . \tag{$*$}$$

The possible choices of $a$ in $(*)$ are uniquely determined by our construction. (One way to see it is to use the Poincaré-Bergman metric on the unit ball. Since its curvature is negative, each hole $\varphi^j(E)$ has a unique center of mass. Since every automorphism of $D$ extends to an automorphism of the unit ball, it must transport the aforementioned centers of mass. This determines the possible values for $a$, and in fact leads us to conclude that the entire automorphism group is generated by the map $\varphi$.)

Then any of the mappings $\varphi^j$ fixes both $(1,0)$ and $(-1,0)$ in the boundary of $D$. Any third point will give a determining set for automorphisms of $D$—just by the explicit given structure of the automorphism group. But note that the boundary of $D$ is not smooth at $(-1,0)$ and $(1,0)$. □

EXAMPLE 2.4. Let $\eta \in C_c^\infty(\mathbb{C}^2)$ be non-negative such that $\eta(z_1,z_2) = \eta(-z_1,z_2)$ and $\eta$ has support in $B((0,1),1/100)$. Define

$$\varphi(z_1,z_2) = \left( \frac{z_1 - 1/2}{1 - z_1/2}, \frac{\sqrt{1-[1/2]^2}z_2}{1-z_1/2} \right) .$$

Now set

$$V = \{ z \in \mathbb{C}^2 : -1 + |z|^2 + \eta(z) < 0 \} .$$

Define

$$D = \bigcap_{j=-\infty}^{\infty} \varphi^j(V) .$$

Note that $D$ is open because $\phi^j(V)$ perturbs $\cap_{j=-\infty}^{j-1} \varphi^j(V)$ in a region of the boundary that is disjoint from all the previous perturbations (so the intersection is locally finite). The domain $D$ has as automorphisms all the mappings $\varphi^j$ together with the mapping $(z_1,z_2) \mapsto (-z_1,z_2)$ (see [LER]). Certainly the two-point set $\{(-1,0),(1,0)\}$ is *not* a determining set for automorphisms. The addition of any third point will make this a determining set. By contrast, any single point of the form $(-1+\epsilon, \sqrt{2\epsilon - \epsilon^2})$, $\epsilon > 0$ small, will be a determining set. The interest of this example is that the domain $D$ is topologically trivial. Note, however, that the boundary of $D$ is not $C^1$ (at the points $(-1,0),(1,0)$).

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