TESTING HOLOMORPHY ON CURVES

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ABSTRACT. For a domain $D \subset \mathbb{C}^n$ we construct a continuous foliation of D into one real dimensional curves such that any function $f \in C^1(D)$ which can be extended holomorphically into some neighborhood of each curve in the foliation will be holomorphic on D.

This paper complements the study of the following general question. Let f be a function on a domain D in complex n-dimensional space, and its restrictions on each element of a given family of subsets of D is holomorphic. When can one claim that f has to be holomorphic in D?

This is a natural question arising from the fundamental Hartogs theorem stating that a function f in $\mathbb{C}^n$, $n > 1$, is holomorphic if it is holomorphic in each variable separately, that is, f is holomorphic in $\mathbb{C}^n$ if it is holomorphic on every complex line parallel to an axis. The complex lines parallel to an axis form a continuous foliation of $\mathbb{C}^n$ into two real dimensional planes. So if a function is holomorphic along each component of these n foliations, then it is holomorphic on $\mathbb{C}^n$. We are interested in finding a one family of one real dimensional curves forming a foliation such that a similar theorem will hold. There is a body of interesting work on testing the holomorphy property on curves: see [AG, E, G1-G4, T1, T2] and references in those articles. Some of these results assume a holomorphic extension into the inside of each closed curve in a given family, others a "Morera-type" property.

Below we use the following definition. Let $S \subset \mathbb{C}^n$. We say that $f : S \to \mathbb{C}$ is holomorphic if f is a restriction on S of a function holomorphic in some open neighborhood of S. We prove the following

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be a domain. Then there exists a continuous foliation E of D into one (real) dimensional curves, such that any C^1 function on D which is holomorphic on each of the curves of E, is holomorphic on D.*

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The needed foliation will be constructed as a homeomorphic image of the natural continuous foliation of $\overline{D}$ by segments parallel to the axis Rez_1.

First we need the following notion (see [FM]). Let $S \subset \mathbb{C}$, $p \in S$. A point t in $T := \{z \in \mathbb{C} : |z| = 1\}$ is said to be a limit direction of S at p if there exists a sequence (q_j) in S such that $\lim_j q_j = p$ and $\lim_j \tau(p, q_j) = t$, where $\tau(p, q_j) := (q_j - p)/|q_j - p|$.

Lemma 1.2. *Let $U \subset \mathbb{C}$ be an open set, $p \in U \cap S$ and there are at least two limit directions t_1, t_2 of S at p. Suppose a function $f \in C^1(U)$ is holomorphic on $S \cap U$. If $t_1 \neq \pm t_2$ then $\frac{\partial f}{\partial \bar{z}} = 0$ at p.*

Proof. The derivatives of f along linearly independent directions t_1 and t_2 coincide with derivatives of a holomorphic function in the neighborhood of p. The statement now follows from the Cauchy-Riemann equations. □

Example 1.3. Consider a set $\gamma \subset \mathbb{C}$, which is an angle ($\gamma = \angle$) in a neighborhood of a point $p \in \gamma$ formed by two linear segments. If this angle θ satisfies $0 < \theta < \pi$, then at the tip of the angle $p \in \gamma$, γ has two linearly independent directions.

In general if in a neighborhood of a point $p \in \gamma$ the curve $\gamma \subset \mathbb{R}^m$ lies in a two real dimensional plane M and forms an angle $0 < \theta < \pi$ there, we will say that γ has an angular point at p.

For the construction of the continuous foliation in the Theorem1.1 we also need the following general statement.

Lemma 1.4. *Let M be a two-dimensional plane in $\mathbb{R}^m$ with $m \geq 2$, let p be a point in M, let U be a neighborhood of p in $\mathbb{R}^m$, and let γ be a C^∞ curve passing through p and relatively closed in U. Then there is a homeomorphism $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that (a) the function Φ is a (C^∞) diffeomorphism on $\mathbb{R}^m - \{p\}$, (b) the restriction of Φ to $\mathbb{R}^m - U$ is the identity map, (c) a neighborhood of $\Phi(p)$ in $\Phi(\gamma)$ lies in M, and (d) the curve $\Phi(\gamma)$ has an angular point at $\Phi(p)$.*

Proof. Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^m$, *i.e.*, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, etc. Choose a vector v parallel to M such that v and the tangent vector of γ at p are linearly independent. Without loss of generality, we assume that $p = 0$, $v = e_2$, and M is spanned by e_1 and e_2. Let $S = \mathbb{R}e_2 = \{te_2 \in \mathbb{R}^m : t \in \mathbb{R}\}$. For $r > 0$ let B_r denote the open ball in $\mathbb{R}^m$ of center 0 and radius r. There is a neighborhood V of 0, a $\delta > 0$, and a diffeomorphism $G : V \to B_{3\delta}$ such that $V \subset\subset U$, $G(\gamma \cap V) = \mathbb{R}e_1 \cap B_{3\delta}$, and $G(S \cap V) = \mathbb{R}e_2 \cap B_{3\delta}$.

Let $\omega : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $0 \le \omega(t) \le 1$ for all t, $\omega(t) = 0$ for $|t| \ge 2$, and $\omega(t) = 1$ for $|t| \le 1$. Define a vector field X on $\mathbb{R}^m$ by $X(y) = \omega(|y|/\delta)(y_1 + y_2)(e_2 - e_1)$, where $y = \sum_{j=1}^n y_j e_j$. Let $\theta : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be the associated action. Define $\lambda : \mathbb{R}^m \to \mathbb{R}^m$ by $\lambda(y) = \theta(1, y)$. Then λ is a diffeomorphism, and the restriction of λ to $\mathbb{R}^m - B_{2\delta}$ is the identity map, since $X = 0$ there. We claim that for $-\delta < s < \delta$, $\lambda(se_1) = se_2$. Indeed, it is straightforward to verify that the curve $\tau(t) = s(1-t)e_1 + ste_2$ satisfies $\tau(0) = se_1$, $\tau(1) = se_2$, and $\tau'(t) = X(\tau(t))$ for $0 \le t \le 1$, and hence $\tau|_{[0,1]}$ is a segment of an integral curve of X.

Define a diffeomorphism $g : \mathbb{R}^m \to \mathbb{R}^m$ by

$$g(x) = \begin{cases} G^{-1} \circ \lambda \circ G(x), & \text{if } x \in G^{-1}(B_{2\delta}), \\ x, & \text{if } x \notin G^{-1}(B_{2\delta}). \end{cases}$$

Then $g(0) = 0$, and $V_1 \cap g(\gamma) \subset S \subset M$, where $V_1 := G^{-1}(B_\delta)$.

Choose a $K > 0$ so that the function $\psi(t) := K\omega(2t)(1 - |t|)$ satisfies the Lipschitz condition $|\psi(t_1) - \psi(t_2)| \le |t_1 - t_2|/2$. It is clear that for each $\eta > 0$ the function $\psi_\eta(t) := \eta\psi(t/\eta)$ satisfies the same Lipschitz condition.

Choose an $\eta > 0$ such that $B_\eta \subset\subset V_1$. Define $h : \mathbb{R}^m \to \mathbb{R}^m$ by $h(x) = x + \psi_\eta(|x|)e_1$. Then h is a homeomorphism, and it is a diffeomorphism away from the origin. For $x \in \mathbb{R}^m - B_\eta$, $h(x) = x$. The set $h(g(\gamma) \cap B_{\eta/2})$ lies in M and equals $\{K(\eta - |t|)e_1 + te_2 : -\eta/2 < t < \eta/2\}$, which is the union of two line segments forming an angle $2\tan^{-1}(1/K)$ at the point $K\eta e_1$.

Let $\Phi = h \circ g$. Then Φ has all the prescribed properties. $\square$

We now proceed with the construction of E and proof of the Theorem 1.1.

Proof. Consider E_0 a natural continuous foliation of D by segments parallel to the axis Rez_1.

1. Pick a sequence $\{w_k\} \subset D$, such that $\overline{\{w_m\}}_{m \equiv l \pmod n} = \overline{D}$ for every $l = 1, ..., n$. We also can choose the sequence in such a way that no two points lie on the same line segment of E_0, so we assume that each of these points w_k lies on a unique segment L_k.

2. We now proceed by induction on k.

(1). For $k = 1$ pick $\varepsilon_1 > 0$ so small that the ball $\overline{B}(w_1, \varepsilon_1) \subset D$. Use Lemma 1.4 to create a homeomorphism $\Phi_1 : \overline{D} \to \overline{D}$ which is a diffeomorphism on $D \backslash \{w_1\}$ with the following properties. At the point $\Phi_1(w_1)$ the image $\Phi_1(L_1)$ has an angle $0 < \alpha_1 < \pi$, and that angle (as a portion of $\Phi_1(L_1)$) lies in the plane parallel to z_1. Let

$d_1 = \min_{d(z,w)\geq 1/2} d(\Phi_1(z), \Phi_1(w))$, where d is the Euclidean distance between two points in $\mathbb{C}^n$.

(2). Consider now step $k = s+1$. By now we have constructed a homeomorphism $\Phi_j : \overline{D} \to \overline{D}$ which is diffeomorphic on $D \backslash \{w_1, ..., w_j\}$, $\varepsilon_j > 0$, and $d_j = \min_{d(z,w)\geq 1/(j+1)} d(\Phi_j(z), \Phi_j(w))$ for all $j \leq s$. Also for all $j \leq s$ we assume that $\Phi_s(L_j) = \Phi_j(L_j)$, and that $\Phi_j(L_j)$ has an angular point at $\Phi_j(w_j)$ in the plane parallel to z_l, axis, where $l \equiv j \pmod{n}$.

Pick now $\varepsilon_{s+1} > 0$ such that the following four conditions hold:

(a) $\varepsilon_{s+1} < \frac{1}{2}\varepsilon_s$.

(b)$B(\Phi_s(w_{s+1}), \varepsilon_{s+1}) \subset D \backslash \{\cup_{j\leq s} \Phi_s(L_j)\}$.

(c) $\varepsilon_{s+1} < \frac{1}{16} d_s$.

(d) ε_{s+1} is so small that one can use Lemma 1.4 to create the specific perturbation $\widetilde{\Phi}_{s+1} : \overline{D} \to \overline{D}$ inside $B(\Phi_s(w_{s+1}), \varepsilon_{s+1})$ that makes an angle $0 < \alpha_{s+1} < \pi$ at the point $\widetilde{\Phi}_{s+1}(\Phi_s(w_{s+1}))$ in the plane parallel to z_l axis, where $l \equiv s+1 \pmod{n}$, and is the identity map outside $B(\Phi_s(w_{s+1}), \varepsilon_{s+1})$.

Consider now $\Phi_{s+1} : \overline{D} \to \overline{D}$, which is defined the following way: $\Phi_{s+1} = \widetilde{\Phi}_{s+1} \circ \Phi_s$.

One can see that $\Phi_{s+1}(z)$ is a well defined homeomorphism which is diffeomorphic on $D \backslash \{w_1, w_2, ..., w_{s+1}\}$. Also for all $j \leq s{+}1$, $\Phi_{s+1}(L_j) = \Phi_j(L_j)$. Let $d_{s+1} = \min_{d(z,w)\geq 1/(s+2)} d(\Phi_{s+1}(z), \Phi_{s+1}(w))$.

Consider now $\Phi_0 = \lim_j \Phi_j$. The limit exists since $\|\Phi_{s+1} - \Phi_s\| < \frac{1}{2^{s-1}}\varepsilon_1$ for all s. We shall prove that Φ_0 is a homeomorphism from $\overline{D}$ onto $\overline{D}$. All we need to check is that for two points $z \neq w$ in D, $\Phi_0(z) \neq \Phi_0(w)$. Indeed, find the smallest s, such that $d(z,w) \geq \frac{1}{s+1}$. By the construction $d(\Phi_s(z), \Phi_s(w)) \geq d_s$, $\|\Phi_{j+1} - \Phi_j\| \leq 2\varepsilon_{j+1}$ for all j; considering the last inequality for $j \geq s$, we have $d(\Phi_s(z), \Phi_0(z)) \leq 2\sum_{j\geq s+1} \varepsilon_j < \sum_{j\geq 0} \frac{1}{2^{j-1}} \varepsilon_{s+1} = 4\varepsilon_{s+1} < \frac{1}{4} d_s$. Same inequality holds for the point w. So, $d(\Phi_0(z), \Phi_0(w)) > \frac{1}{2} d_s > 0$, and therefore $\Phi_0(z) \neq \Phi_0(w)$.

We now check that the continuous foliation $E = \Phi_0(E_0)$ satisfies the theorem. First we notice that for all j by construction $\Phi_0(L_j) = \Phi_j(L_j)$, and therefore $\Phi_0(L_j)$ has an angular point at $\Phi_0(w_j)$ in the plane parallel to z_l axis, where $l \equiv j \pmod{n}$.

If a function $f \in C^1(D)$ is holomorphic on each of the curves in E, then by Lemma 1.2, $\frac{\partial f}{\partial \bar{z}_l} = 0$ at an everywhere dense set in D, and therefore on all of D, and for each l. By Hartogs theorem, f is holomorphic on D. $\square$

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