On rigidity of Grauert tubes over locally symmetric spaces

Su-Jen Kan at Taipei and Daowei Ma at Wichita

1. Introduction

It is well-known that every paracompact real-analytic manifold $X$ admits a complexification $X_C$, a complex manifold that contains $X$ as the fixed point set of an antiholomorphic involution. One way to see this is as follows. The transition functions defining the manifold $X$ are real-analytic local diffeomorphisms of $\mathbb{R}^n$. The Taylor expansions of these transition functions can be considered as local biholomorphisms of $\mathbb{C}^n$, hence, they serve as transition functions of a complex manifold. The complexification of $X$ is unique at the level of germ in the sense that if $\Omega$ and $D$ are two complexifications of $X$ then there is a biholomorphic map $f$ from a neighborhood of $X$ in $\Omega$ to a neighborhood of $X$ in $D$ such that the restriction of $f$ to $X$ is the identity.

A sufficiently small neighborhood of $X$ in the tangent bundle $TX$ admits a complex structure with $X$ as a totally real submanifold, since such a neighborhood admits a real-analytic diffeomorphism into $X_C$ that fixes $X$. In general the complex structure on a neighborhood of $X$ is not unique since there are many ways to embed it into $X_C$.

In recent years there has been a lot of interest in finding and studying canonical complex structures on neighborhoods of $X$ in $TX$. With the additional datum of a real-analytic Riemannian metric $g$ on $X$, every sufficiently small neighborhood $\Omega$ of $X$ in $TX$ has a canonical complex structure (see [5], [12]). This canonical complex structure is the unique complex structure on $\Omega$ such that the map $f(\sigma + it) = (\tau y'(\sigma))_{(e)}$, $\sigma + it \in \mathbb{C}$, is holomorphic, wherever it is defined, for every geodesic $\gamma$ of $X$ (see [12]). It is called the adapted complex structure. With this complex structure the function

$$\rho(v) = ||v||^2 = g(v, v), \quad v \in TX,$$

is strictly plurisubharmonic on $\Omega$; the function $u = \sqrt{\rho}$ is plurisubharmonic on $\Omega \setminus X$.

The first author is partially supported by NSC 88-2115-M-001-009.
and satisfies the homogeneous complex Monge-Ampère equation \((\partial\bar{\partial}u)^n = 0\) there, where \(n\) is the dimension of \(X\) (see [12]).

The Grauert tube of radius \(r\) over a real-analytic Riemannian manifold \(X\) is the manifold \(X^r = T^rX = \{ v \in TX : \|v\| < r \}\) equipped with the adapted complex structure. The Riemannian manifold \(X\) is called the center of the Grauert tube \(\Omega = X^r\), and \(r\) is called its radius.

For a real-analytic Riemannian manifold \(X\), let \(r_{\text{max}}(X)\) be the maximal radius \(r\) such that the adapted complex structure is defined on \(T^rX\). It is clear that each point \(p \in X\) has a neighborhood \(U\) such that the adapted complex structure is defined on some \(T^rU\) with \(r > 0\). The maximal radius of \(X\) need not be positive, since as a point \(p \in X\) goes to “infinity” the “local allowable radius” may tend to 0. The maximal radius \(r_{\text{max}}(X)\) is positive, if \(X\) is compact, or \(X\) has a compact quotient, or \(X\) is homogeneous, or \(X\) contains a compact set \(K\) such that each of \(X\) is sent to \(K\) by some isometry of \(X\).

Suppose that \(\Omega\) is a Grauert tube of radius \(r\) over a Riemannian manifold \((X, g)\), it is natural to ask whether \(\Omega\) can also be a Grauert tube for some other Riemannian manifold \((Y, h)\). There is a trivial non-uniqueness coming from scaling: for each \(c > 0\), \(\Omega = \sqrt{c} \Omega\) over \((X, c g)\). Thus when considering the rigidity property of Grauert tubes one assumes the radii of Grauert tube representations of a given complex manifold are equal.

Burns proved a uniqueness theorem in [3] which states that if a complex manifold \(\Omega\) has two Grauert tube representations \(\Omega = X^r = Y^r\) of finite radius over compact Riemannian manifolds, then there exists an \(f \in \text{Aut}(\Omega)\), the group of automorphisms of \(\Omega\), such that \(f(\Omega) = Y\) and \(f|_X\) is an isometry. In other words, the center of a Grauert tube over a compact Riemannian manifold is unique up to an isometry. One could ask for a stronger version of uniqueness. Namely, one could ask whether the center of a Grauert tube \(\Omega = X^r\) is unique in the sense that \(\Omega = X^r = Y^r\) implies \(X = Y\) as subsets of \(\Omega\). In the following, the phrase “uniqueness of the center” will always refer to this stronger version of uniqueness.

If \(h\) is an isometry of \(X\), \(h \in \text{Isom}(X)\), then the differential \(dh\) maps \(\Omega = T^rX\) onto itself. Since the adapted complex structure is canonically associated with the Riemannian metric, \(dh\) is an element of \(\text{Aut}(\Omega)\). This gives a natural inclusion of \(\text{Isom}(X)\) into \(\text{Aut}(\Omega)\). A theorem of Szőke ([17]) states that if \(\Omega = X^r\) is a Grauert tube over the compact Riemannian manifold \(X\) and if \(f \in \text{Aut}(\Omega)\) satisfies \(f(X) = X\) then \(f = dh\) for some \(h \in \text{Isom}(X)\). Hence, for a compact \(X\) the uniqueness of the center is equivalent to the condition that the natural inclusion \(\text{Isom}(X) \hookrightarrow \text{Aut}(\Omega)\) is an isomorphism. Burns [3] proved that \(\text{Isom}(X)\) is of finite index in \(\text{Aut}(\Omega)\) when \(X\) is compact.

The uniqueness of the center has been known for a very short list of Grauert tubes. Burns [3] proved the rigidity for Grauert tubes of finite radii over compact homogeneous spaces. The uniqueness of the center for Grauert tubes covered by the ball and Grauert tubes with negatively curved Kähler-Einstein metrics have been proved by Kan [7], [8]. In [10], the authors identified the automorphism groups of Grauert tubes over Riemannian manifolds of constant curvature with the isometry group of the center manifold and hence proved the uniqueness of the center for such Grauert tubes.

In this article we will prove the following theorem.
Theorem 1.1. Let $X$ be a compact locally symmetric space, and let $\Omega = X_C^r$ be a Grauert tube of radius less than $r_{\text{max}}(X)$. Then $\Omega$ has a unique center.

Though the main result of [9] is contained in the above Theorem 1.1, the argument of [9] has the merit of giving a concrete realization of Grauert tubes over compact Riemannian manifolds of constant negative curvature as quotients of domains in $C^n$.

We will also prove the following two theorems about the rigidity of Grauert tubes over noncompact manifolds.

Theorem 1.2. Let $M$ be a symmetric space, and let $D = M_C^r$ be a Grauert tube of radius less than $r_{\text{max}}(M)$. Suppose that $D$ is not covered by the ball. Then the inclusion of $\text{Isom}(M)$ into $\text{Aut}(D)$ is an isomorphism and $D$ has a unique symmetric center.

Theorem 1.3. Let $Y$ be a complete locally symmetric space, and let $U = Y_C^r$ be a Grauert tube of radius less than $r_{\text{max}}(Y)$. Suppose that $U$ is not covered by the ball. Then the inclusion of $\text{Isom}(Y)$ into $\text{Aut}(U)$ is an isomorphism and $U$ has a unique complete locally symmetric center.

Although Theorems 1.1 and 1.2 are contained in Theorem 1.3, Theorems 1.1 and 1.3 are consequences of Theorem 1.2. Note that in Theorem 1.3 the manifold $Y$ need not be compact.

A closely related problem is the uniqueness of the radius. If $X$ is a real-analytic Riemannian manifold and two Grauert tubes $T'X$ and $T''X$ are biholomorphic, does it follow that $r = s$? The answer is in the affirmative if $X$ is compact. This is a clear consequence of Burns' uniqueness theorem mentioned above. For noncompact $X$, almost nothing is known. We will prove the following theorem.

Theorem 1.4. Let $Y$ be a non-Euclidean symmetric space. Suppose that the Grauert tubes $T'Y$ and $T''Y$ are biholomorphic. Then $r = s$.

In Section 2 we gather some background information. Section 3 contains a version of a generalized Rosay-Wong Theorem. In Section 4 we prove a theorem about "symmetric actions" (see §4 for the definition) of a compact connected Lie group on the unit ball of $R^n$. Namely, we prove that such an action must have the center as a fixed point. Section 5 contains the proof that the symmetric space $M$ is stable under the action of the identity component $\text{Aut}_0(D)$ of the automorphism group of the Grauert tube $D$. In Section 6 we prove the equality of $\text{Aut}_0(D)$ and $\text{Isom}_0(M)$. Proofs of the four theorems mentioned above are in Section 7.

We thank Lizhen Ji for informing us of the theorem that every simply-connected symmetric space admits a compact quotient. We thank the referee for making many suggestions to improve the paper.

2. Preliminaries

The Grauert tube of radius $r$ over a real-analytic Riemannian manifold $X$ is the set

$$X_C^r = T'X = \{ v \in TX : \|v\| < r \}$$
equipped with the adapted complex structure. The norm square function $\rho(v) = \|v\|^2$ is strictly plurisubharmonic in $X^C$. 

Let $M$ be the universal covering manifold of $X$, and $D$ be that of $\Omega = X^C$. Hence, $D = M^C_\mathcal{E}$. The maximal radius $r_{\text{max}} = r_{\text{max}}(X)$ for $X$ is defined to be the maximum of $r$ for which the adapted complex structure is defined on $X^C$. It is clear that $r_{\text{max}}(X) = r_{\text{max}}(M)$. For a compact manifold $X$ the maximal radius $r_{\text{max}}(X)$ is greater than 0. Therefore, a Riemannian manifold has a positive maximal radius if it admits a compact quotient. It was shown in [12] that $r_{\text{max}}(X) = \infty$ implies that all of the sectional curvatures of $X$ are non-negative. It was proved in [18] that $r_{\text{max}} = \infty$ for every compact normal homogeneous Riemannian manifold.

Let $X$ be a compact real-analytic Riemannian manifold and $Q = X^C_{\text{max}}$ be the Grauert tube over $X$ of the maximal radius $r_{\text{max}}(X)$. Then $Q$ is a Stein manifold since it has a strictly plurisubharmonic exhaustion function $\rho(v) = \|v\|^2$. Let $\Omega = X^C_\mathcal{E}$ be a Grauert tube of radius $r < r_{\text{max}}$, then $\Omega$ is relatively compact in $Q$ with smooth strictly pseudoconvex boundary, since $\tilde{\phi}(z) = \rho(z) - r^2$ is a smooth plurisubharmonic defining function of $\Omega = X^C_\mathcal{E}$. Let $\tilde{Q}$ denote the universal covering of $Q$, then $D \subset \tilde{Q}$ and the boundary $\partial D$ of $D$ in $\tilde{Q}$ is smooth and strictly pseudoconvex.

For complex manifolds $N$ and $W$, let $\text{Hol}(N, W)$ denote the set of holomorphic mappings from $N$ to $W$. Let $\Delta$ denote the unit disk in $\mathbb{C}$. As usual, $T_z W$ denotes the tangent space of $W$ at $z$. For $\xi \in T_z W$, the Kobayashi infinitesimal pseudometric $F^W_K(z, \xi)$ (length of the tangent vector $\xi$ at the point $z$) is defined by

$$F^W_K(z, \xi) = \inf \{ \|v\| : v \in T_0 \Delta, \text{there exists an } f \in \text{Hol}(\Delta, W) \text{ such that } f(0) = z \text{ and } df(0)v = \xi \}.$$ 

The Kobayashi pseudolength $l^W_K$ of a piecewise $C^1$-smooth curve in $W$ is the integral of the Kobayashi infinitesimal pseudometric along the curve. The Kobayashi pseudodistance $d^W_K(z, w)$ between two points $z, w \in W$ is defined to be the infimum of the Kobayashi pseudolength of a piecewise $C^1$-smooth curve connecting $z$ and $w$. The manifold $W$ is said to be hyperbolic if its Kobayashi pseudodistance is a metric, i.e., if $d^W_K(z, w) = 0$ implies $z = w$. In this case the Kobayashi pseudodistance is called the Kobayashi metric of $W$. A hyperbolic manifold $W$ is said to be complete hyperbolic if its Kobayashi metric is complete, i.e., if each Cauchy sequence in the Kobayashi metric has a limit. The automorphism group of a hyperbolic manifold is a Lie group ([11]).

It is well known that a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ is complete hyperbolic. One way to see this is through estimates of the Kobayashi metric of such a domain $D$ in [4]. The boundary is at infinite distance, i.e., $\lim_{t \to 0} d^D_K(q, \Gamma) = \infty$, for each compact $\Gamma \subset D$. The same estimates of the Kobayashi metric hold for a relatively compact strongly pseudoconvex domain in a Stein manifold, since the two main ingredients of the proof in [4], the local estimates and the localization, hold for the manifold case as well, see the remark on page 230 of [4]. It follows that $\Omega = X^C_\mathcal{E}$ is complete hyperbolic provided that the center manifold $X$ is compact and $r < r_{\text{max}}(X)$. (It is observed in [17] that the complete hyperbolicity of $\Omega$ also follows from the results of [10].)
The universal covering $D$ of $\Omega$ is the Grauert tube $M_\zeta$, where $M$ is the universal covering of $X$. The complete hyperbolicity of $\Omega$ implies that of $D$. This is so because each holomorphic map from $\Delta$ to $\Omega$ is lifted to one from $\Delta$ to $D$, hence, the Kobayashi metrics of $\Omega$ and $D$ coincide in the sense that for each $\zeta \in TD$, $F^D_k(z, \zeta) = F^\Omega_k(\pi(z), \pi_\zeta)$, where $\pi : D \to \Omega$ is the projection. 

A sequence $f_j \in \text{Hol}(N, W)$ is said to be \textit{compactly divergent} if for each compact subset $K \subset N$ and each compact $K' \subset W$ there is a $j_0 = j_0(K, K')$ such that the set $f_j(K) \cap K'$ is empty for each $j \geq j_0$. The family $\text{Hol}(N, W)$ is said to be \textit{normal} if each sequence in $\text{Hol}(N, W)$ contains a subsequence that either converges to an element of $\text{Hol}(N, W)$ in the compact open topology, or diverges compactly. A complex manifold $W$ is said to be \textit{taut} if $\text{Hol}(N, W)$ is normal for each complex manifold $N$ (see [20]). It is known that $W$ is taut if and only if $\text{Hol}(\Delta, W)$ is normal ([1]). A complete hyperbolic manifold is taut and a taut manifold is hyperbolic ([10]).

It also follows from the estimates in [4] that if $p \in \partial \Omega$, the boundary of $\Omega$ in $Q$, and if $V$ is a neighborhood of $p$ in $Q$, then

\begin{equation}
\lim_{q \in \Omega, q \to p} d^Q_k(q, \Omega \setminus V) = \infty.
\end{equation}

\textbf{Lemma 2.1.} Let $\Omega$ be a relatively compact domain with $C^4$-smooth strongly pseudoconvex boundary in a Stein manifold $Q$. Let $\bar{Q}$ be a covering manifold of $Q$, and let $D = \pi^{-1}(\Omega)$, where $\pi$ is the projection from $\bar{Q}$ to $Q$. Thus $D$ is a covering manifold of $\Omega$, and the boundary $\partial D$ of $D$ in $\bar{Q}$ is $C^4$-smooth strongly pseudoconvex. Let $p \in \partial D$ and let $U$ be a neighborhood of $p$ in $\bar{Q}$ such that $\pi|_U$ is injective. Then

\begin{equation}
\lim_{\bar{q} \in \partial D, \bar{q} \to p} d^\bar{D}_k(\bar{q}, D \setminus U) = \infty.
\end{equation}

\textbf{Proof.} Let $p = \pi(\bar{p})$ and $V = \pi(U)$. Since $\pi|_U$ is injective, each curve $y$ in $V \cap \Omega$ lifts to a unique curve $\bar{y}$ in $U \cap D$. We have, for $\bar{q} \in U \cap D$ and $q = \pi(\bar{q})$,

\[
d^\bar{D}_k(\bar{q}, D \setminus U) = \inf \{ d^\bar{Q}_k(\bar{y}) : \bar{y} \text{ lies in } U \cap D \text{ and connects } \bar{q} \text{ with a point in } \partial U \cap D \} = \inf \{ d^\bar{Q}_k(y) : y \text{ lies in } V \cap \Omega \text{ and connects } q \text{ with a point in } \partial V \cap \Omega \}
\]

\[
= d^\bar{Q}_k(q, \Omega \setminus V).
\]

Therefore, (2.2) follows from (2.1). \hfill \Box

\textbf{Remark.} In Lemma 2.1, we assume $C^4$-smoothness of the boundary. The reason for this is that we need to use the estimates in [4], and for the manifold case the $C^4$-smoothness is needed. See the remark on page 230 of [4].

Some explanations for notation are in order. Suppose that $K$ is a differentiable manifold and $U = TK$, the tangent bundle of $K$. It is a common practice to identify $K$ with the zero section in $U$. Hence, $K$ is considered as a submanifold of $U$. While the practice is
convenient, it may sometimes cause confusion. Let \( p \in K \). Since \( K \) is a submanifold of \( U \), \( T_pK \) is a subspace of \( T_pU \), consisting of vectors in \( T_pU \) tangent to the submanifold \( K \). On the other hand, the same notation \( T_pK \) also denotes the fiber of \( U \) at \( p \). In order to avoid possible ambiguity, we shall, when it is necessary, denote the zero section in \( TK \) by \( K_0 \). Therefore, \( K_0 \) is an embedded submanifold of \( U \) that is diffeomorphic to \( K \), and \( T_pU = T_pK_0 \oplus T_p(T_pK) \).

3. A generalization of the Rosay-Wong Theorem

The well-known theorem of Rosay-Wong states that a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with noncompact automorphism group is biholomorphic to the ball (see \([14], [19]\)). In this section we will prove the following generalization of the Rosay-Wong Theorem.

**Theorem 3.1.** Let \( \Omega \) be a relatively compact domain with \( C^4 \)-smooth strongly pseudoconvex boundary in a Stein manifold \( Q \). Let \( \tilde{Q} \) be a covering manifold of \( Q \) and let \( D = \pi^{-1}(\Omega) \), where \( \pi \) is the projection from \( \tilde{Q} \) to \( Q \). Let \( p \in \partial D \), the boundary of \( D \) in \( \tilde{Q} \), and \( q \in D \). Suppose there is a sequence \( \{f_j\} \subset \text{Aut}(D) \) such that \( \lim_{j \to \infty} f_j(q) = p \). Then \( D \) is biholomorphic to the ball.

**Remark.** Note that \( D \) is not necessarily relatively compact in \( \tilde{Q} \).

**Proof.** We will use Pinchuk's rescaling method (see \([13], [15]\)).

Let \( U \) be a connected neighborhood of \( p \) in \( \tilde{Q} \) such that \( \pi|_U \) is injective, and such that there is a biholomorphic map \( h \) from \( U \) to a neighborhood \( h(U) \) of 0 in \( \mathbb{C}^n \) which maps \( p \) to the origin. Since \( \partial D \) is strongly pseudoconvex, we can choose \( h \) and \( U \) so that \( h(U) = B^n \), the unit ball in \( \mathbb{C}^n \), and

\[
E := h(U \cap D) = \{ z \in B^n : \phi(z) < 0 \},
\]

where

\[
\phi(z) = \text{Re} z_1 + |z|^2 + O(|z|^3).
\]

Let

\[
q_j = f_j(q), \quad r_j = d^K_p(q_j, D \setminus U).
\]

From Lemma 2.1, we see that \( r_j \to \infty \) as \( j \to \infty \). Let

\[
B_K(q, r) = \{ w \in D : d^K_p(q, w) < r \}
\]

denote the Kobayashi ball of radius \( r \) centered at \( q \). Then

\[
f_j(B_K(q, r_j)) = B_K(q_j, r_j) \subset U \cap D.
\]
Hence,

\begin{equation}
B_K(q_j, r_j) \subset f_j^{-1}(U \cap D).
\end{equation}

Let \( \zeta_j \) denote the unique point on \( L = h(\partial D \cap U) \) closest to \( a_j = h(q_j) \) in the Euclidean distance. (Since \( L \) is smooth and \( 0 \in L \), there is a neighborhood \( N \) of \( 0 \) in \( \mathbb{C}^n \) such that for each point \( P \) in \( N \) there is a unique point on \( L \) closest to \( P \). Since \( a_j \to 0 \) as \( j \to \infty \), we assume, without loss of generality, that all \( a_j \) belong to \( N \).) Let \( l_j \) denote the composition of the translation that maps \( \zeta_j \) to \( 0 \) and a unitary transformation, taking the tangent plane \( T_{\zeta_j} L \) to the plane \( \text{Re} w_1 = 0 \). The map \( l_j \) maps \( E \) biholomorphically onto a bounded domain with the defining function given by

\begin{equation}
\phi \circ l_j^{-1}(w) = (1 + c_j) \text{Re} w_1 + |w|^2 + O(|\zeta_j| \cdot |w|^2 + |w|^3),
\end{equation}

where \( c_j = O(|\zeta_j|) \). Since \( a_j \) lies on the normal to \( L \) at \( \zeta_j \), the point \( l_j(a_j) \) has the form \( (-\delta_j, 0) \) where \( \delta_j > 0 \). By construction, \( a_j \to 0 \), and hence, \( \zeta_j \to 0 \) and \( \delta_j \to 0 \), as \( j \to \infty \).

We write the coordinates in \( \mathbb{C}^n \) as \( (w_1, w') \), where \( w_1 \in \mathbb{C}, w' \in \mathbb{C}^{n-1} \). Let \( u_j \) be a scaling function defined by

\[ u_j(w_1, w') = (w_1/\delta_j, w'/\sqrt{\delta_j}), \]

and let \( g_j = u_j \circ l_j \). Then \( g_j(a_j) = (-1, 0) \). The map \( g_j \) maps \( E \) biholomorphically to a domain \( E_j \) with the defining function

\begin{equation}
\psi_j(w) := (1/\delta_j) \phi \circ l_j^{-1}(\delta_j w_1, \sqrt{\delta_j} w')
\end{equation}

\[ = (1 + c_j) \text{Re} w_1 + (\delta_j |w_1|^2 + |w'|^2) + O(|\zeta_j| \cdot |w|^2 + \sqrt{\delta_j} |w|^3). \]

The defining functions \( \psi_j \) converge, uniformly on compact subsets of \( \mathbb{C}^n \), to

\[ \psi(w) := \text{Re} w_1 + |w'|^2 \]

which is the defining function of the Siegel half-space \( \mathcal{H} \), biholomorphically equivalent to the ball.

Let

\[ P_\epsilon = \{ w : \text{Re} w_1 + (1 + \epsilon)|w'|^2 < 0, |w| < 1/\epsilon \}, \]

\[ R_\delta = \{ w : \text{Re} w_1 + (1 - \delta)|w'|^2 < 0 \}. \]

These domains satisfy \( P_{\epsilon_2} \subset P_\epsilon \) for \( \epsilon_2 < \epsilon_1 \), \( R_{\delta_1} \subset R_\delta \) for \( \delta_1 < \delta_2 \), and

\[ \bigcup_{\epsilon > 0} P_\epsilon = \bigcap_{\delta > 0} R_\delta = \mathcal{H}. \]

Let

\[ \epsilon_j = \inf \{ \epsilon > 0 : P_\epsilon \subset E_j \}, \quad \delta_j = \inf \{ \delta > 0 : R_\delta \supset E_j \}. \]
Direct computations show that

\[(3.4) \quad \delta_j \to 0, \quad \delta_j \to 0 \quad \text{as} \quad j \to \infty.\]

Without loss of generality we assume that \(\delta_j < \frac{1}{j}\).

Consider the maps

\[F_j = g_j \circ h \circ f_j : f_j^{-1}(U \cap D) \to R_\delta.\]

By \((3.1)\), each compact subset of \(D\) is contained in \(f_j^{-1}(U \cap D)\) for sufficiently large \(j\). Since the domains \(R_\delta\) are contained in \(R_{1/2}\), which is biholomorphic to \(\mathbb{U}\) and therefore is taut, and since \(F_j(q) = (-1, 0)\), some subsequence \(\{F_{j_k}\}\) converges in the compact open topology to a holomorphic map \(F : D \to R_{1/2}\). Then \((3.4)\) tells us that \(F(D) \subseteq R_\delta\) for each \(0 < \delta < \frac{1}{j}\); hence, \(F : D \to \mathbb{U}\).

Now consider the maps \((F_j)^{-1} : R_{1/2} \to D\). Since \((F_j)^{-1}(-1, 0) = q\) and since \(\delta_j \to 0\), some subsequence of the sequence \(\{(F_{j_k})^{-1}\}\) converges to a holomorphic map \(\Phi : \mathbb{U} \to D\). Clearly \(F\) and \(\Phi\) are inverses of each other. Therefore, \(D\) is biholomorphically equivalent to the ball. \(\square\)

4. Symmetric actions of connected compact Lie groups on the ball

Let \(G\) be a Lie group acting on the unit ball \(B\) in \(\mathbb{R}^n\). We say \(G\) acts symmetrically on \(B\), or the action of \(G\) on \(B\) is symmetric, if for each \(g \in G\), there is a \(g' \in G\) such that \(g'(x) = -g(-x)\).

Let \(G\) be a Lie group acting on a manifold \(M\). For \(\xi \in \mathfrak{g}\), the Lie algebra of \(G\), we will use the same letter \(\xi\) to denote the vector field on \(M\) induced by \(\xi\). Thus, for each \(x \in M\),

\[\xi(x) = \frac{d}{dt}\bigg|_{t=0} [\exp(t\xi)](x).\]

Suppose that \(G\) acts symmetrically on \(B\). Then for each \(\eta \in \mathfrak{g}\), there is an \(\eta' \in \mathfrak{g}\) such that \(\eta'(x) = -\eta(-x)\) for each \(x \in B\).

**Theorem 4.1.** If a connected compact Lie group \(G\) acts on \(B\) symmetrically, then 0 is a fixed point of \(G\), i.e., \(g(0) = 0\) for each \(g \in G\).

**Proof.** Suppose that 0 is not a fixed point of the action. Then there is an \(\eta \in \mathfrak{g}\) such that the corresponding vector field \(\eta(x)\) satisfies \(\eta(0) \neq 0\). Let \(\eta' \in \mathfrak{g}\) be such that \(\eta'(x) = -\eta(-x)\), and let \(\zeta = \eta - \eta'\). Then \(\zeta \in \mathfrak{g}\) and

\[\zeta(x) = \eta(x) - \eta'(x) = \eta(x) + \eta(-x).\]

Thus \(\zeta(0) = 2\eta(0) \neq 0\) and
(4.3) \[ \zeta(-x) = \zeta(x) \]

for each \( x \in B \). Let

(4.4) \[ H = \{ \exp(t\zeta) : t \in \mathbb{R} \}, \]

and let \( L = H \) be the closure of \( H \) in \( G \). Now \( L \) is a connected compact Abelian Lie group, hence, it is a torus. Let \( \mathcal{L} \) be the Lie algebra of \( L \). Consider the subset \( \mathcal{S} \) of \( \mathcal{L} \) defined by

\[ \mathcal{S} = \{ \xi \in \mathcal{L} : \exp(t\xi) \text{ is a closed curve in } L \}. \]

Since \( \mathcal{S} \) is dense in \( \mathcal{L} \), and since \( \zeta \in \mathcal{L} \) satisfies \( \zeta(0) \neq 0 \), there exists a \( \xi_0 \in \mathcal{S} \) such that \( \xi_0(0) \neq 0 \).

We claim that (4.3) implies that

(4.5) \[ \exp(-t\xi)(0) = -\exp(t\xi)(0). \]

To see that, let

\[ \phi(t) = \exp(-t\xi)(0), \quad \psi(t) = -\exp(t\xi)(0). \]

By (4.1) and (4.3), we have

\[ \phi'(t) = \frac{d}{dt}[\exp(-t\xi)(0)] = \frac{d}{dh}_{h=0} [\exp(-(t+h)\xi)(0)] = \frac{d}{dh}_{h=0} [\exp(-h\xi)(\phi(t))], \]

\[ = -\zeta(\phi(t)) ; \]

\[ \psi'(t) = -\frac{d}{dt}[\exp(t\xi)(0)] = -\frac{d}{dh}_{h=0} [\exp((t+h)\xi)(0)] = -\frac{d}{dh}_{h=0} [\exp(h\xi)(-\psi(t))], \]

\[ = -\zeta(-\psi(t)) = -\zeta(\psi(t)). \]

Thus \( \phi \) and \( \psi \) are both integral curves of the vector field \(-\zeta(x)\). Since they satisfy the same initial condition: \( \phi(0) = 0 = \psi(0) \), \( \phi \) and \( \psi \) must be equal, establishing (4.5). It follows that \( g^{-1}(0) = -g(0) \) for each \( g \in H \) and, therefore, for each \( g \in L \). In particular,
exp(-t\xi_0)(0) = -exp(t\xi_0)(0)

for \( t \in \mathbb{R} \).

Now \( \gamma(t) := \exp(t\xi_0)(0) \) is a closed periodic curve in \( B \). This curve does not degenerate into a point since \( \xi_0(0) \neq 0 \). Let \( t_0 \) be the least positive period of this curve, i.e., the least positive number such that \( \gamma(t_0) = 0 \). Then

\[
\gamma(t_0 - t) = \exp(-t\xi_0)(\gamma(t_0)) = \exp(-t\xi_0)(0) = -\exp(t\xi_0)(0) = -\gamma(t).
\]

Letting \( t = t_0/2 \) gives \( \gamma(t_0/2) = -\gamma(t_0/2) \), whence \( \gamma(t_0/2) = 0 \). This contradicts the choice of \( t_0 \). Therefore, 0 is a fixed point of the action. \( \square \)

5. Stability of the center under \( \text{Aut}_0(D) \)

We first recall some definitions. Let \( B_R(0, r) \) denote the ball in \( \mathbb{R}^n \) of radius \( r \) centered at the origin. A connected Riemannian manifold \( X \) is said to be \textit{locally symmetric} if for each \( p \in X \) there is an \( r > 0 \) such that the exponential map

\[
\text{Exp}_p : B_R(0, r) \to \text{Exp}_p(B_R(0, r)) := B_p(r)
\]

is a diffeomorphism, and the \textit{geodesic symmetry} map \( \tau_p : B_p(r) \to B_p(r) \) defined by

\[
\tau_p(\text{Exp}_p(x)) = \text{Exp}_p(-x)
\]

is an isometry. A locally symmetric Riemannian manifold is also called a \textit{locally symmetric space}.

A connected Riemannian manifold \( M \) is said to be \textit{symmetric} if to each \( p \in M \) there is an involutive isometry \( \tau_p \) of \( M \) such that \( p \) is an isolated fixed point of \( \tau_p \). The mapping \( \tau_p \) is necessarily the geodesic symmetry with respect to \( p \), i.e., \( \tau_p \) reverses each geodesic through \( p \). A symmetric Riemannian manifold is also called a \textit{symmetric space}. It is well known that a complete connected Riemannian manifold is locally symmetric if and only if its universal covering is symmetric (see, e.g., [6]) and that a symmetric space is complete and homogeneous.

Another classical result about symmetric spaces we will make use of is that every simply-connected symmetric space admits a compact quotient (see [2]). That is, for a simply-connected symmetric space \( M \), there exists a discrete subgroup \( \Gamma \) of the isometry group of \( M \) such that \( M/\Gamma \) is a compact locally symmetric space.

**Lemma 5.1.** Let \( M \) be a symmetric space, and let \( D = M^c \) be a Grauert tube of radius \( r < r_{\max}(M) \). Then \( D \) is complete hyperbolic.
Proof. The universal covering $\tilde{M}$ of $M$ is a simply-connected symmetric space. Hence, it admits a compact quotient $X$. The Grauert tube $X^\rho_C$ is complete hyperbolic since it is relatively compact and strongly pseudoconvex in the Stein manifold $X^\rho_C$, where
\[ r_0 := r_{\max}(X) = r_{\max}(\tilde{M}) > r. \]
The manifold $\tilde{M}$ is complete hyperbolic since it is the universal covering of $X^\rho_C$. Being a quotient of $\tilde{M}$, the manifold $M^\rho_C$ is also complete hyperbolic.

Let $M$ be a symmetric space, and let $Q = M^\rho_C$ be the Grauert tube over $M$ of the maximal radius. The boundary $\partial D$ of $D = M^\rho_C$ in $Q$ is strongly pseudoconvex when $r < r_{\max}$. Let $F = T^\rho_q M$ denote the fiber of $D = M^\rho_C$ at a fixed point $q \in M$. The automorphism group $\text{Aut}(D)$ of $D$ is a Lie group since $D$ is hyperbolic by Lemma 5.1.

Lemma 5.2. Let $M$ be a symmetric space, and let $D = M^\rho_C$ be a Grauert tube of radius less than $r_{\max}(M)$ such that $D$ is not covered by the ball. Then
\[ H = \{ g \in \text{Aut}(D) : g(F) \subset F \} \]
is a compact Lie group.

Proof. We first assume that $M$ is simply-connected. Then $M$ has a compact quotient $X$, and $D$ is the universal covering manifold of $\Omega := X^\rho_C$. Since $\Omega$ is a relatively compact domain in $Q := X^\rho_C$ with real-analytic strongly pseudoconvex boundary, $D$ and $\Omega$ satisfy the hypotheses of Theorem 3.1.

We claim that no sequence in $H$ can diverge compactly. Seeking for a contradiction, suppose that a sequence $\{q_i\}$ in $H$ diverges compactly. The sequence $\{q_i(q)\}$ is a sequence in $F$ that diverges compactly. Passing to a subsequence if necessary, we can assume that $q_i(q) \to \in D$. By Theorem 3.1, $D$ is biholomorphically equivalent to the ball. This contradicts the hypothesis. Therefore, no subsequence of $H$ diverges compactly. Since $D$ is taut, each sequence in $H$ contains a subsequence that converges to some $h \in \text{Aut}(D)$. Clearly, $h(F) \subset F$ and $h \in H$. Therefore, $H$ is compact.

We now consider the general case without the condition that $M$ is simply-connected. Let $M$ and $D = M^\rho_C$ be the universal covering manifolds of $M$ and $D$. Choose $\tilde{q} \in \tilde{M}$ so that $\tilde{q}$ projects to $q$. Let $\tilde{F} = T^\rho_{\tilde{q}} \tilde{M}$, and let $\tilde{H} = \{ f \in \text{Aut}(\tilde{D}) : f(\tilde{F}) = \tilde{F} \}$. Since each $g \in H$ can be lifted to a unique element in $\tilde{H}$, we see that $H$ is naturally included in $\tilde{H}$. By what has been proved above, $\tilde{H}$ is compact. Therefore, $H$ is compact.

For each $\xi$ in the Lie algebra $\text{aut}(D)$ of $\text{Aut}(D)$ there exists a one-parameter subgroup $\exp t\xi$ of $\text{Aut}(0)$, the identity component of $\text{Aut}(D)$. Let $\xi$ also denote the corresponding vector field on $D$, so
\[ \xi(z) = \frac{d}{dt} \bigg|_{t=0} (\exp t\xi)(z). \]

Lemma 5.3. Let $M$ be a symmetric space, and let $D = M^\rho_C$ be a Grauert tube of radius less than $r_{\max}(M)$ such that $D$ is not covered by the ball. Then for each $\xi \in \text{aut}(D)$, the vector field $\xi(z)$ is tangent to $M$. 
Proof. Let $\xi \in \text{aut}(D)$ and $q \in M$. We shall show that $\xi(q)$ is tangent to $M$. Let $\tau_q$ be the geodesic symmetry at $q$. Then $\phi := d\tau_q$ is a holomorphic involution on $D = T_q'M$. If we choose a normal coordinate system $(x_1, \ldots, x_n)$ in a neighborhood of $q$ such that $\tau_q(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$, then in the corresponding complex coordinate system $\phi(z_1, \ldots, z_n) = (-z_1, \ldots, -z_n)$. Let $\sigma$ be the anti-holomorphic involution on $D = T_q'M$ defined by $\sigma(v) = -v$ on each fiber. In local coordinates $\sigma(z_1, \ldots, z_n) = (z_1, \ldots, z_n)$. Let $\omega = \sigma \circ \phi$. Then $\omega$ is an anti-holomorphic involution on $\Omega$. In local coordinates $\omega(z_1, \ldots, z_n) = (-z_1, \ldots, -z_n)$. Now $\omega \circ (\exp i\xi) \circ \omega \in \text{aut}(D)$. Let $\eta \in \text{aut}(D)$ be defined by

$$\eta = \xi + \frac{d}{dt}_{|t=0} (\omega \circ (\exp i\xi) \circ \omega).$$

If we write $\xi(z) = (\xi_1(z), \ldots, \xi_n(z))$, then

$$(5.1) \quad \eta(iy_1, \ldots, iy_n) = (2i \text{Im } \xi_1(iy), \ldots, 2i \text{Im } \xi_n(iy)).$$

It follows that $\eta$ is tangent to the fiber $F = T_q'M$ and

$$\exp(\eta) \in H := \{g \in \text{aut}(D) : g(F) = F\}.$$  

The group $H$ acts on $F$ symmetrically, since for each $g \in H$, the map $\sigma \circ g \circ \sigma$ is also in $H$. By Lemma 5.2 and Theorem 4.1, the point $q$ is a fixed point of $H$, the identity component of $H$. It follows that $\eta(q) = 0$. This, together with (5.1), implies that $\xi(q)$ is tangent to $M$. □

Lemma 5.4. Let $M$ be a symmetric space, and let $D = M_z$ be a Grauert tube over $M$ of radius less than $r_{\max}(M)$ such that $D$ is not covered by the ball. Then $f(M) = M$ for each $f \in \text{aut}_0(D)$.

Proof. Let $f \in \text{aut}_0(D)$. By Lemma 5.3, $f(M) \subset M$. Since $f(M)$ is a closed submanifold of $D$ and $M$ is connected, $f(M) = M$. □

Remark. Lemma 5.4 is an important step toward the proof of the uniqueness of the center of $D$. Since $\text{aut}_0(D)$ is a normal subgroup of $\text{aut}(D)$, we have, for $h \in \text{aut}(D)$ and $p \in M$,

$$h(M) = h(\text{aut}_0(D)(p)) = \text{aut}_0(D)(h(p)) \supset \text{Isom}_0(M)(h(p)).$$

This would force $h(M) = M$ if for each $q \in TM\setminus M$ the dimension of the orbit $\text{Isom}_0(M)(q)$ were greater than the dimension of $M$. Unfortunately, not all symmetric spaces satisfy the latter condition. The condition is satisfied by non-Euclidean irreducible symmetric spaces, by $\mathbb{R}^m$ with $m > 1$, and by their products. However, it is not satisfied by any symmetric space of the form $\mathbb{R} \times M_1 \times \cdots \times M_k$, where $M_j$ are non-Euclidean irreducible symmetric spaces.

6. Equality of $\text{Isom}_0(M)$ and $\text{aut}_0(D)$

Let $M$ be a Riemannian manifold. The Riemannian metric defines the Levi-Civita connection on $TM$, i.e., a splitting of $T_x(TM)$ into vertical and horizontal subspaces. The
vertical subspace is $T_z(T_{\pi(z)}M)$ and it is canonically identified with $T_{\pi(z)}M$, where $\pi$ is the projection $\pi : TM \to M$, while the horizontal subspace is the kernel of the connection map $K : T_z(TM) \to T_z(T_{\pi(z)}M)$. For $w, z \in T_pM$, the horizontal and vertical lifts of $w$ to $z$ are $\xi, \eta \in T_z(TM)$ determined by

$\pi_* \xi = w, \quad K \xi = 0, \quad \pi_* \eta = 0, \quad K \eta = w.$

Let $D = T'M$ and let $J$ be the almost complex tensor associated with the adapted complex structure. Let $p \in M$. Then $T_p(T'_pM)$ consists of the vertical vectors in $T_pD$, and $T_pM_0$ consists of the horizontal vectors in $T_pD$, where $M_0$ denotes the zero section in $D$ (see the last paragraph of §2). If $v \in T_pM$, then $\xi \in T_pM_0$ and $v_0 \in T_p(T'_pM)$ are horizontal and vertical lifts of $v$, then $v = Jv_0$ by the definition of the adapted complex structure.

**Lemma 6.1.** Let $Y$ be a locally symmetric space, and let $Y_z' = T'Y$ be a Grauert tube over $Y$. Let $J$ denote the almost complex tensor associated with the adapted complex structure on $Y_z'$. Suppose that $z \in T'Y$ and $\eta$ is a vertical vector in $T_z(T'Y)$. Then $\eta$ is a horizontal vector.

**Proof.** The lemma follows from the proof of Theorem 2.5 in [16]. For the convenience of the reader, let us indicate briefly why the assertion holds. If $z = 0$ the assertion follows from the definition of the adapted complex structure. We assume that $z \neq 0$. Let $p = \pi(z)$ and let $v_1, \ldots, v_n$ be a basis of $T_pY$. Let $\xi_1, \ldots, \xi_n$ be the horizontal lifts of $v_1, \ldots, v_n$ to $z$, and let $\eta_1, \ldots, \eta_n$ be the vertical lifts. Let $\gamma(t)$ be the geodesic in $Y$ such that $\gamma(0) = p$ and $\gamma'(0) = z$. The function $v + iu \mapsto \eta + \mu \gamma'(v)$ is defined and holomorphic on

$V := \{v + i\mu \in \mathbb{C} : \gamma(v) \text{ is defined; } |\mu| < r/||z||\},$

where $N_\mu$ is the multiplication by $\mu$ in the fibers of $TY$. Following (5.8) in [12], the tensor $J$ at $z$ is given by

$$J\xi_h = \sum_{k=1}^n e_{kh} \left( \eta_k - \sum_{j=1}^n \text{Re} f_{jk}(i) \xi_j \right), \quad (e_{kh}) = \left( \text{Im} f_{jk}(i) \right)^{-1},$$

where $(f_{jk})$ is a meromorphic matrix defined on $V$ and holomorphic on $V \setminus \mathbb{R}$ such that $(\text{Im} f_{jk})$ is invertible on $V \setminus \mathbb{R}$. When $Y$ is locally symmetric, by the formulae on p. 420 of [16], one can choose $v_1, \ldots, v_n$ so that $(\text{Re} f_{jk}(i)) = 0$. Hence, the assertion follows from (6.1). $\Box$

**Lemma 6.2.** Let $(Y, g)$ and $(Z, h)$ be two real-analytic Riemannian manifolds, at least one of which is locally symmetric. Let $f : T'Y \to T'Z$ be a biholomorphic map between the two Grauert tubes. Suppose that $f(Y) = Z$ and that $f$ maps each fiber to a fiber in the sense that $f(T_pY) = T_{f(p)}Z$. Let $u = f|_Y$. Then there is a constant $c > 0$ such that

$$r = cs, \quad g = c^2 u^* h.$$

Hence, $T'(Y, g) = T'(Y, u^* h)$. Furthermore, $f$ equals $du$, the differential of $u$.

**Proof.** Without loss of generality we assume that $(Z, h)$ is locally symmetric. Consider a geodesic $\gamma(t)$ in $Y$. The domain of $\gamma$ contains $(-\delta, \delta)$ for some $\delta > 0$. Let
\[ V := \{ t + i\mu \in \mathbb{C} : |t| < \delta, |\mu| < r/\|y'(0)\|_q \} . \]

The map \( \psi : V \to T'Y \) defined by

\[ \psi(t + i\mu) = N_{p,h}'(t) \]

is holomorphic. Let \( \omega = f \circ \psi \). Hence, \( \omega : V \to T'Z \) is holomorphic.

Let \( J_h \) denote the almost complex tensor on \( T'(Z, h) \). Since \( f \) maps a fiber to a fiber, we see that \( \omega(t + i\mu) \in T_{\alpha(t)}^Z Z \), where \( \alpha(t) := \omega(t + i0) = u \circ \gamma(t) \). Now \( \partial \omega / \partial \mu \) is a vertical vector in \( T(T'Z) \). Hence, \( \partial \omega / \partial t = -J_h(\partial \omega / \partial \mu) \) is a horizontal vector by Lemma 6.1. Thus the curve \( t \mapsto \omega(t + i\mu) \) is a horizontal curve in \( T'Z \) for each fixed \( \mu \). It follows that \( \nabla_{\alpha'(t)}^h \omega(t + i\mu) = 0 \), hence,

\[ \nabla_{\alpha'(t)}^h N_{1,\mu} \omega(t + i\mu) = 0, \quad \mu \neq 0. \]

The vector \( \lim_{\mu \to 0} N_{1,\mu} \omega(t + i\mu) \) in \( T_{\alpha(t)} Z \) is identified with the vector \( \left( \partial / \partial t \right) \omega(t + i0) \) in \( T_{\alpha(t)}(T_{\alpha(t)} Z) \). Since \( \left( \partial / \partial t \right) \omega(t + i0) \) is the horizontal lift of \( \alpha'(t) \), the vector

\[ \left( \partial / \partial \mu \right) \omega(t + i0) = J_h \left( \partial / \partial t \right) \omega(t + i0) \]

is the vertical lift of \( \alpha'(t) \) by the definition of the adapted complex structure. It follows that

\[ \lim_{\mu \to 0} N_{1,\mu} \omega(t + i\mu) = \alpha'(t). \]

Letting \( \mu \to 0 \) in (6.3), and using (6.4), we obtain

\[ \nabla_{\alpha'(t)}^h \alpha'(t) = 0. \]

Therefore, \( \alpha \) is a geodesic in \( Z \).

Let \( V' = \{ t + i\mu \in \mathbb{C} : |t| < \delta, |\mu| < s / \|\alpha'(0)\|_h \} , \) and let \( f : V' \to T'Z \) be the holomorphic map defined by \( f(t + i\mu) = N_{p,h} \alpha'(t) \). Since the holomorphic maps \( \omega \) and \( f \) coincide on the real axis, they are equal on \( V \cap V' \). This means that \( f(N_{p,h}'(t)) = N_{p,h} \circ \gamma'(t) \), i.e.,

\[ f = du. \]

Therefore, \( du(T_p T'Y) = T_{\alpha'(t)}^p Z \). The assertions of the lemma follow immediately.

**Lemma 6.3.** Let \((Y, g)\) and \((Z, h)\) be symmetric spaces, and let \( T'Y \) and \( T'Z \) be Grauert tubes satisfying either \( r < r_{\text{max}}(Y) \) or \( s < r_{\text{max}}(Z) \). Let \( f : T'Y \to T'Z \) be a biholomorphic map satisfying \( f(Y) = Z \). Then there is a constant \( c > 0 \) such that \( r = cs \), \( g = c^2 u^* h \), where \( u = f|_Y : Y \to Z \). Furthermore, \( f = du \).

**Proof.** By Lemma 5.1, either \( T'Y \) or \( T'Z \) is a taut manifold. Since they are both holomorphic, \( T'Y \) and \( T'Z \) are both taut manifolds. Let \( p \in Y \) and \( q = f(p) \). Let \( \tau_p : T'Y \to T'Y \) be the holomorphic involution of \( T'Y \) induced by the geodesic symmetry of \( Y \) at \( p \). Let \( \sigma_Y = N_{-1} : T'Y \to T'Y \) be the anti-holomorphic involution that is multiplication by \(-1\) on each fiber. Let \( \phi_p = \tau_p \circ \sigma_Y \). Then \( \phi_p \) is an anti-holomorphic involution.
The restriction of $\phi_p$ to $T'_p Y$ is the identity; while the restriction of $d\phi_p(p)$ to $T_p Y_0$ is multiplication by $-1$. Here $Y_0$ denotes the zero section in $T' Y$ (see the last paragraph of §2). The fiber $T'_p Y$ is the component of the fixed point set of $\phi_p$ that contains $p$. Similarly we define $\tau_0, \sigma Z$ and $\phi_q$.

Let $F = \phi_q \circ f \circ \phi_p \circ f^{-1}$. Then $F \in \text{Aut}(T' Z)$ and $F(Z) = Z$. We now denote by $T_q Z_0$ the tangent space at $q$ of the zero section in $T' Z$. Since the restrictions of $d\phi_q(q)$ and $d(f \circ \phi_p \circ f^{-1})(q)$ to $T_q Z_0$ are both multiplication by $-1$, we see that the restriction of $dF(q)$ to $T_q Z_0$ is the identity. By Cauchy-Riemann equations and by the fact that $T_q Z_0$ is a totally real subspace of maximal dimension of $T_q(T^4 Z)$, $dF(q)$ is the identity on $T_q(T^4 Z)$. Therefore, $F$ is the identity map by H. Cartan’s uniqueness theorem. Note that although H. Cartan’s uniqueness theorem is usually stated for bounded domains in $\mathbb{C}^n$, one can verify that its proof is valid for taut complex manifolds as well. It follows that $f \circ \phi_p \circ f^{-1} = \phi_q$.

Now $f(T'_p Y)$ is the component of the fixed point set of $f \circ \phi_p \circ f^{-1}$ that contains $q$, and $T'_q Z$ is the component of the fixed point set of $\phi_q$ that contains $q$. Since $f \circ \phi_p \circ f^{-1}$ and $\phi_q$ are equal, their fixed point sets are equal. Thus, $f(T'_p Y) = T'_q Z$. Therefore, $f$ maps each fiber to a fiber. Now the assertions of the lemma follow from Lemma 6.2.

**Theorem 6.4.** Let $(Y, g)$ and $(Z, h)$ be complete locally symmetric spaces, and let $T' Y$ and $T^4 Z$ be Grauert tubes with $r < r_{\text{max}}(Y)$ or $s < r_{\text{max}}(Z)$. Let $f : T' Y \to T^4 Z$ be a biholomorphic map satisfying $f(Y) = Z$. Then there is a constant $c > 0$ such that $r = cs$, $g = c^2 u^* h$, where $u = f|_Y : Y \to Z$. Furthermore, $f = du$.

**Proof.** Let $\tilde{Y}, \tilde{Z}$ be the universal coverings of $Y, Z$, respectively. Let

$$\pi_Y : T' \tilde{Y} \to T' Y \quad \text{and} \quad \pi_Z : T^4 \tilde{Z} \to T^4 Z$$

be projections. The map $f$ can be lifted to a biholomorphic map $\tilde{f} : T' \tilde{Y} \to T^4 \tilde{Z}$ such that $
\pi_Z \circ f_\ast = f \circ \pi_Y$. It is clear that $\tilde{f}(\tilde{Y}) = \tilde{Z}$. By Lemma 6.3, $\tilde{f} = d\tilde{u}$, where $\tilde{u} = \tilde{f}|_\tilde{Y}$. Hence, $f = du$. The assertions now follow immediately.

**Corollary 6.5.** Let $M$ be a symmetric space, and let $D = T'M$ be a Grauert tube of radius less than $r_{\text{max}}(M)$. Let $f \in \text{Aut}(D)$ be such that $f(M) = M$. Then $f$ is induced by an isometry of $M$.

**Lemma 6.6.** Let $M$ be a symmetric space, and let $D = T'M$ be a Grauert tube of radius less than $r_{\text{max}}(M)$ such that $D$ is not covered by the ball. Then the inclusion of $\text{Isom}_0(M)$ into $\text{Aut}(D)$ is an isomorphism.

**Proof.** This is a consequence of Lemma 5.4 and Corollary 6.5.

7. Proofs of the main theorems

In this section we let $M$ denote a simply-connected symmetric space of dimension $n$.
Let $D = T'M$ be a Grauert tube of radius less than $r_{\text{max}}(M)$ such that $D$ is not covered by the ball. Fix $p \in M$ and let $F = T'_p M$. Let $G = \text{Aut}(D)$. By Lemma 6.6,

$$\text{Aut}(D) = \text{Isom}_0(M).$$

We denote this group by $G_0$. Recall that $G_0$ acts transitively on $M$. 
Let \( f \in \text{Aut}(D) \), and \( N = f(M) \). The map \( f \) pushes the metric of \( M \) to \( N \). With this pushed-forward metric, \( N \) is a symmetric space isometric to \( M \), and \( D = T^*N \). We claim that \( N \cap F \) contains exactly one point \( q \).

To see this we note that \( N = f(G_0(p)) = G_0(f(p)) \) is a \( G_0 \)-orbit and it is \( n \)-dimensional. Let \( \pi_M : D = T^*M \to M \) be the projection. Since \( G_0 = \text{Isom}_0(M) \), \( \pi_M \) is \( G_0 \)-equivariant. By the \( G_0 \)-equivariance of \( \pi_M \) and by the fact that \( N \) is an \( n \)-dimensional \( G_0 \)-orbit, the map \( \pi_M|_N : N \to M \) is locally diffeomorphic. The \( G_0 \)-equivariance also implies that the above map has the following "lifting property": for each \( x \in N \) and each smooth curve \( \gamma(t) \) in \( M \) with \( \gamma(0) = \pi_M(x) \), there is a curve \( \tilde{y}(t) \) in \( N \) such that \( \gamma = \pi_M \circ \tilde{y} \) and \( \tilde{y}(0) = x \). It follows that \( \pi_M|_N : N \to M \) is a covering map and, since \( M \) is simply-connected, it is a diffeomorphism. Therefore, \( N \cap F \) contains exactly one point \( q \). Let \( u \in M \) be such that \( f(u) = q \). Then \( f(T^*_uM) = T^*_qN \). We will prove, in the following lemma, that \( T^*_qN = T^*_qM \) as subsets of \( D \), i.e., \( f^{-1}(F) = T^*_uM \). This means (since \( f \) is an arbitrary automorphism) that for each \( h \in G \), \( h^{-1}(F) \) is a fibre \( T^*_uM \) for some \( y \). It follows that for each \( g \in G \), \( g(F) \) is a fibre \( T^*_uM \) for some \( y \). Therefore, \( g(F) = F \) if and only if \( g(p) \in F \).

**Lemma 7.1.** With the above notation, \( T^*_qN = T^*_qM \).

**Proof.** Let \( \tau_q : T^*N \to T^*N \) be the holomorphic involution of \( D = T^*N \) that is induced by the geodesic symmetry of \( N \) at \( q \). Let \( \sigma_N : T^*N \to T^*N \) be the anti-holomorphic involution that maps each \( v \in T^*N \) to \(-v \). Let \( \phi_q = \tau_q \circ \sigma_N \). We similarly define

\[
\phi_p = \tau_p \circ \sigma_M.
\]

We shall prove that \( \phi_p = \phi_q \).

Let \( u \in M \) be such that \( f(u) = q \). Now

\[
N = f(M) = f(G_0(u)) = G_0(f(u)) = G_0(q),
\]

since \( G_0 \) is normal in \( \text{Aut}(D) \). We have \( \phi_p(z) = z \) for each \( z \in F = T^*_pM \). Hence,

\[
d\phi_p(q)|_{T^*_qF} = \text{id}.
\]

Since \( \phi_p \) is anti-holomorphic, \( d\phi_p(q)(v) = -v \) for each \( v \in J(T^*_qF) \), where \( J \) is the almost complex tensor associated with the adapted complex structure. It follows that \( J(T^*_qF) \) is the unique \( n \)-dimensional subspace of \( T^*_qD \) that is complementary with \( T^*_qF \) and invariant under \( d\phi_p(q) \).

Now

\[
\phi_p(N) = \phi_p(G_0(q)) = G_0(\phi_p(q)) = G_0(q) = N,
\]

i.e., \( \phi_p(N) = N \). Hence, \( T^*_qN_0 \) is invariant under \( d\phi_p(q) \), where \( N_0 \) is the zero section in \( D = T^*N \) and \( N_0 \) is identified with \( N \) (see the last paragraph in \( \S 2 \)). It is clear that \( T^*_qN_0 \) and \( T^*_qF \) are complementary in \( T^*_qD \). It follows that \( T^*_qN_0 = J(T^*_qF) \) and

\[
d\phi_p(q)|_{T^*_qN_0} = -\text{id}.
\]
We also have \( d\phi_q(q)|_{T^N_0} = -\text{id} \). Thus, \( d(\phi_q \circ \phi_p)(q)|_{T^N_0} = \text{id} \). This implies that \( d(\phi_q \circ \phi_p)(q) = \text{id} \) and, since \( D \) is taut, \( \phi_q \circ \phi_p = \text{id} \). Therefore, \( \phi_q = \phi_p \). Consequently, \( \phi_q \) and \( \phi_p \) have the same fixed point set. It follows that \( T^p_{\phi_q} N = T^p_{\phi_p} M \). \( \square \)

**Lemma 7.2.** \( G/G_0 \) is a finite group.

**Proof.** Let \( \{\tilde{g}_j\} \) be a sequence in \( G/G_0 \). Each \( \tilde{g}_j \) is a coset \( g_j G_0 \) for some \( g_j \in G \). Now \( g_j(M) = g_j G_0(n) = G_0 g_j(p) \) is a \( G_0 \)-orbit and it is \( n \)-dimensional. Hence, \( g_j(M) \cap F \) contains exactly one point \( q_j \) so \( g_j^{-1}(q_j) \in M \). Let \( h_j \in G_0 \) be such that \( h_j(p) = g_j^{-1}(q_j) \), and let \( h_j = g_j h_j \). Then \( h_j G_0 = g_j G_0 \) and \( h_j(p) = q_j \). The sequence \( \{h_j(p)\} \) lies in \( F \). As in the proof of Lemma 5.2, no subsequence of \( \{h_j\} \) can converge compactly. Since \( D \) is taut, some subsequence of \( \{h_j\} \) converges to some \( g_0 \in G \) in the topology of \( G \). Hence, some subsequence of \( \{g_j G_0\} \) converges to \( g_0 G_0 \) in the topology of \( G/G_0 \). Thus, \( G/G_0 \) is compact. Therefore, \( G/G_0 \) is finite. \( \square \)

Let \( \rho(z) = \|z_j\|_M^2 \) be the norm-square function on \( D = T^N M \). Then \( \rho \circ h = \rho \) for each \( h \in G_0 \). Let the order of \( G/G_0 \) be \( k \). We choose and fix \( \{g_1, \ldots, g_k\} \in G \) so that \( G/G_0 = \{g_j G_0 : j = 1, \ldots, k\} \). Let \( \psi \) be the function on \( D \) defined by

\[
\psi(z) = \sum_{j=1}^{k} \rho(g_j(z)).
\]

This definition is independent of the choice of the representatives \( g_j \); if \( h_j \in g_j G_0 \), then \( \rho(h_j(z)) = \rho(h_j g_j^{-1}(z)) = \rho(g_j(x)) \), since \( h_j g_j^{-1} \in G_0 \). It follows that

\[
\psi(z) = \sum_{j=1}^{k} \rho(h_j(z)), \quad \text{if} \quad \bigcup_{j=1}^{k} h_j G_0 = G.
\]

It is clear that \( \psi \) is real-analytic and strictly plurisubharmonic. The function \( \psi(z) \) is \( \leq kr^2 \) on \( D \), and tends to \( kr^2 \) as \( z \) approaches a point in \( \partial D \), the boundary of \( D \) in \( M_0^{\text{max}} \).

Let \( \tau : D \to D \) be the anti-holomorphic involution that maps each \( v \in T^N M \) to \( -v \), and let \( \pm \text{Aut}(D) = G \cup G \tau \) be the group of holomorphic or anti-holomorphic automorphisms of \( D \).

**Lemma 7.3.** \( \psi \) is \( \pm \text{Aut}(D) \)-invariant: \( \psi \circ f = \psi \) for each \( f \in \pm \text{Aut}(D) \).

**Proof.** Let \( g \in G \). Then

\[
G = Gg = \bigcup_{j=1}^{k} (g_j G_0 g) = \bigcup_{j=1}^{k} (g_j g G_0).
\]

By (7.1) and (7.2),

\[
\psi(g(z)) = \sum_{j=1}^{k} \rho(g_j g(z)) = \psi(z).
\]
Now suppose that $f = g \tau \in G \tau$. Then

$$G = \tau G g \tau$$
$$= \bigcup_{j=1}^{k} (\tau g_j G_0 g \tau)$$
$$= \bigcup_{j=1}^{k} (\tau g_j g \tau) (\tau g^{-1} G_0 g \tau)$$
$$= \bigcup_{j=1}^{k} (\tau g_j g \tau G_0).$$

By $\rho \circ \tau = \rho$, and by (7.2), we see that

$$\psi(f(z)) = \sum_{j=1}^{k} \rho(g_j f(z)) = \sum_{j=1}^{k} \rho(\tau g_j g \tau(z)) = \psi(z).$$

The lemma is proved. ☐

**Lemma 7.4.** Let $M$ be a symmetric space, and let $D = T' M$ be a Grauert tube of radius less than $r_{\max}(M)$ such that $D$ is not covered by the ball. Then the inclusion of $\text{Isom}(M)$ into $\text{Aut}(D)$ is an isomorphism.

**Proof.** We use Burns’ argument in the proof of Theorem 3 in [3]. Although the manifold $M$ is not compact, we have established all that is needed for the argument to work:

(a) $G_0 = \text{Isom}_0(M) = \text{Aut}_0(D)$ is transitive on $M$, and

(b) the function $\psi$ is strictly plurisubharmonic and $\pm \text{Aut}(D)$-invariant and, for the fiber $F = T_p^\tau M$, the restriction $\psi|_F : F \to [0, kr^2]$ is proper.

The proof of Theorem 3 in [3] can be copied here with obvious and slight modifications. We will sketch the proof. The reader is referred to [3] for details.

Fix $p \in M$ and let $F = T_p^\tau M$ and $f = \psi|_F$. For each $z \in D$,

$$T_z(G_0 \cdot z) + T_z(T_\pi(z) \cdot M) = T_z(T' M),$$

where $\pi : T' M \to M$ is the projection. Let $z$ be a critical point of $f$. Then $z$ is a critical point of $\psi$. This follows from (7.3), since $\psi$ is constant on $G_0 \cdot z$. Again by the $G_0$-invariance of $\psi$, the real Hessian $H$ of $\psi$ at $z$ satisfies $H(v, w) = 0$ for $v \in T_z D$ and $w \in T_z(G_0 \cdot z)$. Since the real Hessian of a strictly plurisubharmonic function has at least $n$ positive eigenvalues, it follows that the real Hessian of $f$ is positive definite at $z$. Therefore, each critical point of $f$ is a strict local minimum. Then the topology of $F$ implies that $f$ has exact one critical point. Since $\psi \circ \tau = \psi$, the point $p$ is the unique critical point of $f$. 


Suppose that \( g \in G \) and \( N = g(M) \). Let \( q \) be the point of intersection of \( N \) and \( F \), and let \( \tau_N = g \circ \tau \circ g^{-1} \). By Lemma 7.1, \( T_q^* N = F \), hence, \( \tau_N(F) = F \). Since \( \psi \circ \tau_N = \psi \) by Lemma 7.3, the point \( q \) is a critical point of \( f \). It follows that \( q = p \). Then \( N \) is the \( G_0 \)-orbit through \( p \), i.e., \( N = M \). By Corollary 6.5, \( g \) is induced by an isometry of \( M \). \( \square \)

We now prove the main theorems stated in Section 1. It is clear that Theorem 1.1 is a consequence of the main theorem in [7] and Theorem 1.3. Theorem 1.2 follows from Theorem 1.3. We first prove Theorem 1.2 with the additional condition that \( M \) is simply-connected.

**Lemma 7.5.** Let \( M \) and \( D = T'M \) satisfy the conditions in Theorem 1.2. We further assume that \( M \) is simply-connected. Then the inclusion of \( \text{Isom}(M) \) into \( \text{Aut}(D) \) is an isomorphism and \( D \) has a unique symmetric center.

**Proof.** By Lemma 7.4, \( \text{Aut}(D) = \text{Isom}(M) \). Denote this group by \( G \). Seeking for a contradiction suppose that \( D \) has another presentation as a Grauert tube \( D = T^*N \) with a symmetric center \( N \neq M \). Since \( D = T^*N \) and \( N \) is symmetric, \( G \) is transitive on \( N \). It follows that \( N \) is contained in a \( G \)-orbit other than \( M \). Thus, \( N \cap M = \emptyset \). Let \( p \in M \) and \( q = \pi_N(p) \), where \( \pi_N : D = T^*N \to N \) is the projection. Let \( H_0 = \text{Isom}_0(N) \). Then \( G \supset H_0 \) and \( M = G(M) \supset H_0(p) \). Since \( M \) is \( n \)-dimensional and \( H_0(p) \) is at least \( n \)-dimensional, we see that \( \dim(H_0(p)) = n \). By the \( H_0 \)-equivariance of \( \pi_N \) and by the fact that

\[
\dim(H_0(p)) = n,
\]

the map \( \pi_N|_{H_0(p)} : H_0(p) \to N \) is locally diffeomorphic. The \( H_0 \)-equivariance also implies that the above map has the following "lifting property": for each \( x \in H_0(p) \) and each smooth curve \( \gamma(t) \) in \( N \) with \( \gamma(0) = \pi_N(x) \), there is a curve \( \bar{\gamma}(t) \) in \( H_0(p) \) such that \( \gamma = \pi_N \circ \bar{\gamma} \) and \( \bar{\gamma}(0) = x \). In fact \( \lambda(t)(x) \) is such a \( \bar{\gamma}(t) \), where \( \lambda(t) \) is a smooth curve in \( H_0 \) satisfying \( \lambda(t)(\pi_N(x)) = \gamma(t) \) and \( \lambda(0) = \text{id} \). It follows that \( \pi_N|_{H_0(p)} : H_0(p) \to N \) is a covering map and, since \( N \) is simply-connected, it is a diffeomorphism. We see that \( H_0(p) \) is a closed submanifold of \( D \). Now \( H_0(p) \) is a closed submanifold of top dimension of \( M \), hence \( H_0(p) = M \). This implies that \( \pi_N : M \to N \) is a diffeomorphism and therefore

\[
M \cap T_q^*N = \{ p \}.
\]

Let \( \tau_M : D \to D \) be the anti-holomorphic involution that maps each \( v \in T'M \) to \( -v \), and let \( \tau_N \) be the similar map associated with \( N \). Then \( \tau_N \circ \tau_M \in G \). Hence,

\[
M = \tau_N \circ \tau_M(M) = \tau_N(M).
\]

This implies that \( \tau_N(p) \in M \), contradicting \( M \cap T_q^*N = \{ p \} \). Therefore, \( D \) has a unique symmetric center. \( \square \)

**Proof of Theorem 1.3.** Let \( M \) and \( D = T'M \) be the universal covering manifolds of \( Y \) and \( U \), and let \( p : D \to U \) be the projection.

Let \( f \in \text{Aut}(U) \). Then \( f \) lifts to an \( \tilde{f} \in \text{Aut}(D) \) such that \( f \circ p = p \circ \tilde{f} \). By Lemma 7.5, \( \tilde{f} = du \) for some \( u \in \text{Isom}(M) \). Hence \( f = du \) for some \( u \in \text{Isom}(Y) \). Therefore, \( \text{Aut}(U) = \text{Isom}(Y) \).
Suppose that \( Z \) is another complete locally symmetric center of \( U \). Let \( N = p^{-1}(Z) \). Then \( N \) is a symmetric center of \( D \). By Lemma 7.5, \( N = M \). Hence, \( Z = Y \). The proof is completed.

**Remark.** In Theorem 1.3, we do not know whether \( U \) can have another center which is not complete locally symmetric. Likewise, in Theorem 1.2, we do not know whether \( D \) can have another center that is not symmetric.

**Proof of Theorem 1.4.** If the universal covering of \( T^rY \) is biholomorphic to the ball, then, by the results in [7], the manifold \( Y \) has constant curvature \(-c\), for some \( c > 0 \), and \( r = s = \pi(2\sqrt{2c}) \).

We now assume that the universal covering of \( T^rY \) is not biholomorphic to the ball (hence, neither is that of \( T^sY \)). Suppose that \( r \neq s \). Then at least one of \( r, s \) is less than \( r_{\text{max}}(Y) \). Without loss of generality we assume \( s < r_{\text{max}} \). Let \( f : T^rY \to T^sY \) be a biholomorphic map, and let \( Z = f(Y) \). Since \( f \) pushes forward the metric of \( Y \) to \( Z \), \( Z \) is a symmetric center of \( T^sY \). By Theorem 1.2, \( Z = Y \). It then follows from Lemma 6.3 that there is a \( c > 0 \) such that \( r = cs \), and \( g = c^2(f|_Y)^*g \), where \( g \) is the metric of \( Y \). This means that \((Y, g)\) is isometric to \((Y, c^2g)\). Since \( Y \) is non-Euclidean, \( c \) must be 1. Therefore, \( r = s \). The theorem is proved.

**References**


Institute of Mathematics, Academia Sinica, Taipei, Taiwan
e-mail: kan@math.sinica.edu.tw

Department of Mathematics and Statistics, Wichita State University, Wichita, Kansas 67260, USA
e-mail: dma@math.twu.edu