ODEs (Ordinary Differential Equations)

IVP (Initial Value Problem) for a 1st-order system

Find \( y = y(t) \) for given \( f, y_0 \) s.t.

\[
\begin{align*}
y' &= f(t, y) & \text{D.E.} \\
y(t_0) &= y_0 & \text{I.C.} = \text{Initial condition}
\end{align*}
\]

where \( t \geq t_0 \), \( y, y_0, f \in \mathbb{R}^d \), and \( t = \frac{d}{dt} \).

(Note: We'll write \( \mathbf{y} \) or \( \underline{y} \) for boldface vectors \( y \).

We'll try to stay close to notation in text.

Recall: Higher order ODEs can be reduced to 1st-order systems.

\( \| \cdot \| \) = vector norm — see appendix, e.g. \( \| \cdot \|_1, \| \cdot \|_2, \| \cdot \|_\infty \).

Def: \( f = f(t, y) \) satisfies a uniform **Lipschitz** condition with respect to \( y \) if there is a constant \( \Lambda \) (Lipschitz constant) s.t.

\[
\| f(t, x) - f(t, y) \| \leq \Lambda \| x - y \| \quad \text{for all } x, y \in \mathbb{R}^d, t \in [t_0, t + \Delta t].
\]

Eg. If \( \frac{df}{dy} \) is bounded, we may take \( \Lambda = \sup_{\| y \|} \| \frac{df}{dy} \| \).

Then (Existence and uniqueness)

Let \( f(t, y) \) be continuous w.r.t. \( t \) and uniformly Lipschitz continuous w.r.t. \( y \). Then there exists a unique differentiable function \( y(t) \) satisfying the IVP, for \( t \in [t_0, t + \Delta t] \).

Remarks:
- Standard proof by Picard iteration can be turned into numerical method
- Other results are possible e.g., \( f \) anal. in \((t, y) \Rightarrow y \) analytic.
Euler's method

Note \( f(t, y(t)) \approx f(t_0, y(t_0)) \quad t \in [t_0, t_0 + h] \), \( h > 0 \) small.

Then \[
y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(r, y(r)) \, dr \\
\approx y_0 + (t - t_0) f(t_0, y_0)
\]

Given \( t_0, t_1 = t_0 + h, t_2 = t_0 + 2h, \ldots \)
where \( h > 0 \) is the (small) time step
we denote \( \gamma_m \approx y(t_m) \quad m = 0, 1, 2, \ldots \)

\[
\begin{align*}
\gamma_1 &= y_0 + h f(t_0, y_0) \\
\gamma_{m+1} &= \gamma_m + h f(t_m, \gamma_m)
\end{align*}
\]

Euler's method \( \gamma_{m+1} \) is given explicitly

picture
(d=1)

see text (for better picture)

Main methods
- Multistep methods
- Runge-Kutta methods

\( h \to 0 \to \text{error}
\)

Q: How does error behave as \( h \to 0 \)?
Variable time step $h_m = t_{m+1} - t_m$

$$y_{m+1} = y_m + h_m f(t_m, y_m)$$

Many good software packages use variable time-steps.

Convergence in $[t_0, t_0 + t^*]$ as $h \to 0$ \( m = 0, 1, \ldots, \left\lfloor t^*/h \right\rfloor \) = greatest integer $\leq t^*/h$.

Let $y_m = y_{m, h}$. A method is convergent if for every $t^*>0$

$$\lim_{h \to 0^+} \max_{m=0,1,\ldots,\left\lfloor t^*/h \right\rfloor} \| y_{m, h} - y(t_m) \| = 0$$

i.e. the numerical solution converges to the true solution on the grid $t_0, t, \ldots, t_n$.

Note: Since all vector norms $\| \cdot \|$ are equivalent, it doesn't matter which we use.

(751 students?)
Theorem 1.1 Euler's method is convergent.

Proof: Assume \( f \) (and \( y \)) are smooth enough to use Taylor's Theorem.

Let \( e_{m,h} := y_{m,h} - y(t_m) = \) numerical error

Want to prove \( \lim_{h \to 0^+} \max_{n \in \mathbb{N}} \| e_{m,h} \| = 0 \)

Taylor's Theorem \( \Rightarrow \)

\[
g(t_{m+1}) = g(t_m) + h \cdot g'(t_m) + O(h^2)
\]

Recall

\[
y_{m+1,h} = y_{m,h} + h \cdot f(t_m, y_{m,h})
\]

\[
e_{m+1,h} = y_{m+1,h} - g(t_{m+1})
\]

\[
= y_{m,h} - g(t_m) + h \left[ f(t_m, y_{m,h}) - f(t_m, y(t_m)) \right] + O(h^4)
\]

\[
= e_{m,h} + h \left[ f(t_m, y(t_m) + e_{m,h}) - f(t_m, y(t_m)) \right] + O(h^2)
\]

And so by Lipschitz cond. on \( f \) \( \| f(t_m, y(t_m) + e_{m,h}) - f(t_m, y(t_m)) \| \leq \beta \| e_{m,h} \| \)

\[
\| e_{m+1,h} \| \leq \| e_{m,h} \| + h \beta \| e_{m,h} \| + O(h^2)
\]

i.e. \( \| e_{m+1,h} \| \leq (1 + h \beta) \| e_{m,h} \| + O(h^2) \) \( \text{for some } c > 0 \)

\[
m = 0, 1, \ldots, \left\lfloor \frac{t^*}{h} \right\rfloor - 1
\]

\[
c \leq \max_{y \in [y_0, y(t_0 + t^*)]} \frac{\| y'' \|}{\| f \|}
\]
Claim \( \| e_{m,h} \| \leq \frac{c}{n} h \left[ (1+h^2)^m - 1 \right] \) \( m=0,1, \ldots \)

Proof by induction on \( m \):

\( m=0 \) \( \| e_{0,h} \| \leq 0 \) since \( e_{0,h} = \nu_{0,h} = \beta / k \| = 0 \) by IC.

Induction step: Assume true for \( m \) and show true for \( m+1 \):

\( \| e_{m+1,h} \| \leq (1+h^2) \| e_{m,h} \| + c h^2 \) by above.

\[ \leq (1+h^2) \frac{c}{n} h \left[ (1+h^2)^m - 1 \right] + c h^2 \]

\[ = \frac{c}{n} h \left[ (1+h^2)^{m+1} - (1+h^2) \right] + \frac{c}{n} h (h^2) \]

\[ = \frac{c}{n} h \left[ (1+h^2)^{m+1} - 1 \right] \]

Result is true.

Next, note \( 1+h^2 \leq e^{h^2} \) since \( h^2 > 0 \).

\[ (1+h^2)^m \leq e^{mh^2} \leq e^{kh^2} \] for \( m=0,1, \ldots, [kh] \).

And so \( \| e_{m,h} \| \leq \frac{c}{n} (e^{kh^2} - 1) h \rightarrow 0, h \rightarrow 0 \) for all \( m=0,1, \ldots, [kh] \) independent of \( h \).

Note: This may be a great overestimate of the actual error as the test example shows.

So \( \lim_{h \rightarrow 0} \| e_{m,h} \| = 0 \)
Order of accuracy

Euler's method: \[ y_{m+1} - [y_m + h f(t_m, y_m)] = 0 \]

\[ \rightarrow \]
replace \( y_{m+1} \) by exact soln \( y(t_m), y(t_{m+1}) \)
+ use Taylor series

\[ y(t_{m+1}) = \left[ y(t_m) + h f(t_m, y(t_m)) \right] \]
\[ = y(t_m) + h y'(t_m) + O(h^2) \]

\[ = O(h^2) \text{ = local error} \]
\[ \uparrow \]
\[ h^2 = h^{p+1}, \quad p = 1 \]
Euler's method is of order 1

more generally given time stepping scheme

\[ y_{m+1} = y_m \left( f(t_m, y(t_m)), y(t_m), \ldots, y_m \right), \quad m = 0, 1, \ldots \]

if for suit. smooth \( f \) + \( y(t) = \text{exact soln.} \)

\[ y(t_{m+1}) = y_m \left( f(t_m, y(t)), y(t), \ldots, y(t_m) \right) = O(h^{p+1}) \]

The scheme is said to be of order \( p \)

\[ \text{Idea:} \]
\[ \text{local error} = O(h^{p+1}) \times \]
\[ \# \text{ of time steps} = O\left( \frac{1}{h} \right) \]
\[ \text{Final error} = O(h^p) \]

if method converges

\[ \text{as in case of Euler's method} \]
The trapezoidal rule

\[ y(t) = y(t_m) + \int_{t_m}^{t} f(t, y(t)) \, dt \]

\[ \approx y(t_m) + \frac{1}{2} (t - t_m) \left[ f(t_m, y(t_m)) + f(t_{m+1}, y(t_{m+1})) \right] \]

Trapezoidal rule:

\[ y_{m+1} = y_m + \frac{1}{2} h \left[ f(t_m, y_m) + f(t_{m+1}, y_{m+1}) \right] \]

This is given implicitly and a nonlinear eq. must be solved at each step.

Order - substitute exact solution if sufficiently smooth:

\[ y(t_{m+1}) - \left( y(t_m) + \frac{1}{2} h \left[ f(t_m, y(t_m)) + f(t_{m+1}, y(t_{m+1})) \right] \right) \]

\[ = y(t_{m+1}) - \left( y(t_m) + \frac{1}{2} h \left[ \frac{y'(t_m)}{t_m} + y'(t_{m+1}) \right] \right) \]

\[ = y(t_m) + h y'(t_m) + \frac{1}{2} h^2 y''(t_m) + O(h^3) \]

\[ - \left( y(t_m) + \frac{1}{2} h \left[ y'(t_m) + y'(t_m) + h y''(t_m) + O(h^2) \right] \right) \]

\[ = O(h^3) \]

The trapezoidal rule is order 2.
The trapezoidal rule is convergent.

**Proof:**

\[ e_{m+1}h = y_{m+1} - y(t_{m+1}) \]
\[ = y_m + \frac{h}{2} \left[ f(t_m, y_m) + f(t_{m+1}, y_{m+1}) \right] \]
\[ - \left( y(t_m) + \frac{h}{2} \left[ f(t_m, y(t_m)) + f(t_{m+1}, y(t_{m+1})) \right] \right) + O(h^3) \]
\[ = e_mh + \frac{h}{2} \left[ (f(t_m, y_m) - f(t_m, y(t_m))) + f(t_{m+1}, y_{m+1}) - f(t_{m+1}, y(t_{m+1})) \right] + O(h^3) \]
\[ \leq \|e_{m+1}\| \leq \|e_m\| + \frac{h^2}{2} (\|e_m\| + \|e_{m+1}\|) + ch^3 \]

Since \( n \to 0 \), assume \( 0 < h < 2 \), e.g. \( c = \sup \{ y'(x) \} \).

Then
\[ (1 - \frac{1}{2}h^2)\|e_{m+1}\| \leq (1 + \frac{1}{2}h^2)\|e_m\| + c h^3 \]
or
\[ \|e_{m+1}\| \leq \left( \frac{1 + \frac{1}{2}h^2}{1 - \frac{1}{2}h^2} \right) \|e_m\| + \frac{c}{1 - \frac{1}{2}h^2} h^3 \]

Again by induction on \( m \)
\[ \|e_{m+1}\| \leq \frac{c}{2} \left[ \left( \frac{1 + \frac{1}{2}h^2}{1 - \frac{1}{2}h^2} \right)^m - 1 \right] h^2 \quad (1.11) \]
\[ m=0 \quad e_{0,1} = y_0 - y'(y_0) = 0 \]

**Induction Step**

Assume \( \| e_{n+1} \| \leq \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right)^{m+1} \left( \frac{1}{1 - \frac{1}{2} \lambda h} \right) h^2 \)

\[
\| e_{n+1} \| \leq \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right) \| e_n \| + \left( \frac{\lambda}{1 - \frac{1}{2} \lambda h} \right) h^2
\]

\[
\leq \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right) \left( \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right)^m - 1 \right) h^2 + \left( \frac{\lambda}{1 - \frac{1}{2} \lambda h} \right) h^2
\]

\[
= \left( \frac{\lambda}{1 - \frac{1}{2} \lambda h} \right) \left( \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right)^{m+1} - \left( \frac{1 - \lambda h}{1 - \frac{1}{2} \lambda h} \right) \right) h^2
\]

\[
= \left( \frac{\lambda}{1 - \frac{1}{2} \lambda h} \right) \left( \left( \frac{1 + \frac{1}{2} \lambda h}{1 - \frac{1}{2} \lambda h} \right)^{m+1} - 1 \right) h^2
\]

\[ \text{QED} \]
\[
\frac{1 + \frac{1}{2} h \frac{d}{dt}}{1 - \frac{1}{2} h \frac{d}{dt}} = 1 + \frac{h \frac{d}{dt}}{1 - \frac{1}{2} h \frac{d}{dt}} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{h \frac{d}{dt}}{1 - \frac{1}{2} h \frac{d}{dt}} \right)^k = \exp \left( h \frac{d}{dt} \right)
\]

\[
\| e_m(t) \| \leq \frac{c h^2}{\lambda} \left( \frac{1 + \frac{1}{2} h \frac{d}{dt}}{1 - \frac{1}{2} h \frac{d}{dt}} \right)^m \leq \frac{c h^2}{\lambda} \exp \left( \frac{m h \frac{d}{dt}}{1 - \frac{1}{2} h \frac{d}{dt}} \right)
\]

\[
m h < t^+ \Rightarrow \| e_m(t) \| \leq \frac{c h^2}{\lambda} \exp \left( \frac{t^+}{1 - \frac{1}{2} h \frac{d}{dt}} \right) = O(h^2)
\]

and \( \lim_{h \to 0} \| e_m(t) \| = 0 \)

\[\text{Note we must solve nonlinear eqs.}
\]

\[
y_{m+1} - \frac{1}{2} h f(t_{m+1}, y_{m+1}) = y_m + \frac{1}{2} h f(t_m, y_m)
\]

for \( y_{m+1} \) at each step, (use iterative Newton solver with \( y_{m+1}^{(0)} = y_m \) as initial guess.)

- Second order accuracy is gained at higher cost.

- Usually evaluating \( f(t, y) \) is most expensive computation since \( f \) may be a complicated function.