A COMPARISON OF SOME NUMERICAL CONFORMAL MAPPING METHODS FOR EXTERIOR REGIONS*

THOMAS K. DELILLO† AND ALAN R. ELCRAT‡‡

Abstract. Several methods for conformally mapping the exterior of the unit disk onto the exterior of a smooth curve are compared. These methods, which are known as the Timman, Friberg, Wegmann, and Theodorsen methods, are based on Fourier series and conjugation and are implemented using Fast Fourier Transforms. The computations reported here indicate that Wegmann's method converges faster and is more robust than the others.

Key words. numerical conformal mapping, conjugation, the Theodorsen method, the Timman method, the Friberg method, the Wegmann method

AMS(MOS) subject classifications. 30C30, 65E05

1. Introduction. This paper is concerned with a comparison among several methods for computing the conformal mapping of the exterior of the unit disk onto the exterior of a simple closed curve in the complex plane. The methods studied are often associated with the names Timman, Friberg, Wegmann, and Theodorsen [3], [10]. Timman's method is also known as James' method and appears to be the method of choice in the aerodynamics community [1], [8], [9], [12]. Wegmann's method is based on solving a sequence of Riemann-Hilbert problems for the approximate mapping functions; it has been shown theoretically to converge quadratically [15] under certain hypotheses, and the suggestion has been made that it is superior to all known methods for a large class of problems [10, p. 422]. Theodorsen's method is the oldest and most thoroughly studied of all these methods; we include it for completeness and as a bridge to earlier work. Friberg's method may be thought of as a hybrid of that of Timman and Theodorsen. The methods of Theodorsen, Timman, and Friberg converge linearly.

Theodorsen's method requires a star-shaped curve, but the others can be formulated for a general curve with arbitrary parameterizations. In many of the previous studies arclength has been used; we use both arclength and other parameters and in some of our examples this has made a significant difference in the observed convergence behavior. All of the methods considered here may be thought of as approximating the Fourier series of the desired mapping, and they make use of Fast Fourier Transforms (FFTs) to perform conjugation of a function defined in the unit circle. For Timman and Friberg the equation being solved iteratively is an integrodifferential equation and each step requires integration of a function defined on the circle. Henrici [10] has suggested using Fourier series to do this integration, and we have used this idea. To our knowledge this has not been done in earlier computational studies.

The part of our work which compares Timman and Theodorsen expands on Halsey's work [9], and is made in light of the crowding phenomenon [16]. For all of the methods we have given a more careful study of the discretization error than has been presented before. For Timman and Friberg, Kaiser [13], [7] has given a theoretical analysis which implies convergence for curves near to a circle. Our experiments confirm these results. Our general conclusion is that Wegmann's method compares favorably

* Received by the editors August 7, 1989; accepted for publication (in revised form) January 16, 1990.
† Department of Mathematics and Statistics, Wichita State University, Wichita, Kansas 67208–1595.
‡‡ The research of this author was partially supported by U.S. Air Force grant AFOSR-86-0274.
with the others in many situations. We have not included relaxation techniques in our study. It is possible that they could significantly improve the range of performance of both Theodorsen [6] and Timman/James [12, p. 118].

An outline of the paper is as follows. Section 2 gives a derivation of the methods based on function conjugation, as in [7], and specifies the iterations. Section 3 treats six examples.

2. Methods for exterior regions. Suppose that \( \Gamma \) is a smooth, simple, closed curve in the \( z \)-plane with the origin in its interior, and that \( z = z(\sigma), 0 \leq \sigma \leq \Gamma \) is a smooth parameterization of \( \Gamma \) with \( z'(\sigma) \neq 0 \). We seek a conformal mapping \( g \) from the exterior of the unit disk in the \( w \)-plane to the exterior of \( \Gamma \) in the \( z \)-plane. \( g \) is normalized by \( g(\infty) = \infty \), and either \( g'(\infty) > 0 \) or \( g(1) = z(\sigma_0) \) for some fixed \( \sigma_0 \in [0, \Gamma] \). (The second condition makes sense by virtue of the Osgood–Carathéodory theorem [10] and the smoothness of \( \Gamma \).) We note that \( g \) has the Laurent expansion

\[
g(w) = cw + a_0 + a_1 w^{-1} + \cdots
\]

with

\[
c = \gamma e^{i\alpha},
\]
where the positive number \( \gamma \) is the "capacity" of the curve \( \Gamma \) [10, Chap. 16]. We may write

\[
g(e^{i\theta}) = z(\sigma(\theta))
\]
and, since \( g \) is determined by its boundary values, our problem reduces to finding \( \sigma(\theta) \).

The unit disk in the \( w \)-plane is denoted by \( D \), the exterior by \( D^c \), and the circle by \( S \).

Each of our methods is associated with an auxiliary function \( h \) of \( g \) defined on \( D^c \) [3], [7], [10]. We need the following proposition.

**Proposition.** For \( h \in H^1(D^c) \):

(a) \( \text{Im } h(e^{i\theta}) - \text{Im } h(\infty) = -K [\text{Re } h(e^{i\theta})] \),

(b) \( \text{Re } h(e^{i\theta}) - \text{Re } h(\infty) = K [\text{Im } h(e^{i\theta})] \).

Here \( K \) is the conjugation operator,

\[
K = F^{-1} \hat{K} F
\]
where \( F \) takes a function to its Fourier coefficients, \( a_n \) and

\[
\hat{K} : a_n \rightarrow -i \text{ sgn } (n) a_n.
\]

The discrete version of \( K \) is

\[
K_N = F^{-1} \hat{K} F_N
\]
where \( F_N \) is the discrete Fourier transform, which can be realized in \( O(N \log N) \) multiplications using the FFT algorithm (see [5]). \( H^1(D^c) \) is the Hardy space associated with \( L^1(S) \). Note that \( K \) maps constant functions to zero.

The methods to be discussed are now derived from the above proposition using the appropriate auxiliary function.

(i) Theodorsen [3], [6], [7], [9]. We assume here that \( \Gamma \) is given by

\[
z(\sigma) = \rho(\sigma) e^{i\sigma}, \quad 0 \leq \sigma < 2\pi
\]
for a smooth positive function \( \rho \). Then

\[
h(w) = \log \left( \frac{g(w)}{w} \right)
\]
and

\[ h(e^{i\theta}) = \log |z(\sigma(\theta))| + i(\arg z(\sigma(\theta)) - \theta) = \log \rho(\sigma(\theta)) + i(\sigma(\theta) - \theta). \]

Using the normalization \( g'(\infty) > 0 \) (\( \alpha = 0 \)), Proposition (a) implies

\[ \sigma(\theta) - \theta = -K[\log \rho(\sigma(\theta))]. \]

For Theodorsen’s method

\[ \sigma^{n+1}(\theta) - \theta = -K[\log \rho(\sigma^n(\theta))]. \]

It is known that the “(e-condition”

\[ \epsilon = \max \left| \frac{\rho'}{\rho} \right| < 1 \]

implies that \( \sigma^n(\theta) \) converges linearly to \( \sigma(\theta) \) in \( L^2 \); there is a corresponding theory for the discrete problem and a bridge between the discrete and continuous.

(ii) Timman [3], [7], [9], [10], [13]. We define

\[ h(w) = \log g'(w). \]

Since \( g'(e^{i\theta}) = -z'(\sigma(\theta))\sigma'(\theta) e^{-i\theta} \), we have

\[ h(e^{i\theta}) = \log \sigma'(\theta) + \log |z'(\sigma(\theta))| + i \left( \arg z'(\sigma(\theta)) - \theta - \frac{\pi}{2} \right). \]

Proposition (b) implies

\[ \log \sigma'(\theta) + \log |z'(\sigma(\theta))| - \log \gamma = K \left[ \arg z'(\sigma(\theta)) - \theta - \frac{\pi}{2} \right]. \]

Suppose we use \( g(1) = z(\sigma_0) \) as our normalization. Then

\[ \alpha = \text{Im} h(\infty) = \frac{1}{2\pi} \int_0^{2\pi} (\arg z'(\sigma(\theta)) - \theta) \, d\theta - \frac{\pi}{2} \]

since \( \text{Im} h(w) \) is harmonic. The iteration is

\[ \sigma^{n+1}(\theta) = \gamma_n \int_0^\theta \exp \left( K[\arg z'(\sigma^n(\theta)) - \theta] / |z'(\sigma^n(\theta))| \right) \, d\theta \]

where \( L\gamma_n^{-1} \) is given by the integral over \([0, 2\pi]\). We perform the integration by integrating the Fourier series of the integrand term by term as suggested by Henrici [10]. Since

\[ h(w) = \log (c - O(w^{-2})) = O(w^{-2}) + \log c \]

at infinity, we set the first two Fourier coefficients of \( K[\arg z'(\sigma^n(\theta)) - \theta] \) equal to zero [10], [1].

Kaiser [13] has shown that \( \sigma^n \) converges linearly to \( \sigma \) if \( \Gamma \) satisfies certain hypotheses. In particular, if \( \sigma \) is arclength and \( \Gamma \) is close to a circle the convergence factor is .5.

(iii) Friberg [3], [7], [13]. Here

\[ h(w) = \log g'(w) - \log \frac{g(w)}{w}. \]
Note that \( h(w) = O(w^{-1}) \) at infinity and that \( h(\infty) = 0 \). Proposition (b) implies
\[
\log \sigma'(\theta) = K[\arg z'(\sigma(\theta)) - \arg z(\sigma(\theta))] + \log |z(\sigma(\theta))| - \log |z'(\sigma(\theta))|;
\]
The iteration is
\[
\hat{\sigma}^{n+1}(\theta) = \int_{0}^{\theta} |z'(\sigma^n(\theta))|^{-1} |z(\sigma^n(\theta))| \exp \left( K[\arg z'(\sigma^n(\theta)) - \arg z(\sigma^n(\theta))] \right) d\theta
\]
and \( \sigma^{n+1}(\theta) = \hat{\sigma}^{n+1}(\theta) L/\hat{\sigma}^{n+1}(2\pi) \). We note that \( \alpha, \gamma \) can be found as a part of this iteration using the Timman auxiliary function.

Kaiser [13] also proved a convergence theorem for this method; in this case the convergence factor for a curve close to a circle is nearly zero.

(iv) Wegmann [10], [15]. This method is described by giving a correction to a current approximation to \( \sigma(\theta) \). We want to compute \( \tau(\theta) \) so that
\[
g(e^{i\theta}) = z(\sigma(\theta) + \tau(\theta));
\]
linearizing
\[
z(\sigma(\theta)) + z'(\sigma(\theta))\tau(\theta) = g^-(e^{i\theta})
\]
where \( g^-(e^{i\theta}) \) are the boundary values of a map to a nearby curve. We normalize by assuming \( g'(\infty) \) is real. Since \( \tau(\theta) \) is real \( g^-(e^{i\theta}) \) must be a solution of the Riemann-Hilbert problem
\[
\text{Re} (i\bar{z}'(s)g^-(s)) = \text{Re} (i\bar{z}'(s)z(s))
\]
where \( s = e^{i\theta}, z(s) = z(\sigma(\theta)), z'(s) = z'(\sigma(\theta)) \). It can be shown that
\[
g^-(w) = [e^{-i\delta}k_i(w) - \hat{\lambda}/\sin \hat{\delta}]w e^{i\theta}
\]
where (note the correction of a misprint in [10] in the first of the following and the use of a general parameterization)
\[
l(w) = -\frac{1}{\pi} \int_{|s|=1} \frac{\arg z'(s) - \theta}{s - w} ds,
\]
\[
\lambda(s) = \exp (-K[\arg z'(s) - \theta]) \text{Im} (z'(s)\bar{z}(s))|z'(s)|^{-1},
\]
\[
k_i(w) = \frac{1}{\pi} \int_{|s|=1} \frac{\lambda(s)}{s - w} ds,
\]
\[
\hat{\lambda} = \frac{1}{2\pi} \int_{0}^{2\pi} \lambda(e^{i\theta}) d\theta
\]
\[
\hat{\delta} = -\frac{1}{2\pi} \int_{0}^{2\pi} [\arg (z'(\sigma(\theta)) - \theta)] d\theta.
\]
Then
\[
g^-(s) = -\exp (i\arg z'(s) + K[\arg z'(s) - \theta])\{K[\lambda(s)] + \hat{\lambda} \cot \hat{\delta} + i\lambda(s)\}
\]
\[
= z(\sigma(\theta)) + z'(\sigma(\theta))\tau(\theta).
\]
The iteration is given by
\[
\tau^n(\theta) = \frac{g^-(e^{i\theta})}{z'(\sigma^n(\theta))} - \frac{z(\sigma^n(\theta))}{z'(\sigma^n(\theta))}
\]
and
\[
\sigma^{n+1}(\theta) = \sigma^n(\theta) + \tau^n(\theta).
\]
Wegmann [15] has shown that for smooth curves and a good enough initial guess, \( \sigma^n \) converges to \( \sigma \) quadratically in \( W^{1,2}(S) \).

3. Numerical experiments. We have done numerical experiments with maps to the exterior of the following regions: (a) ellipse, (b) sports ground, (c) inverted ellipse, (d) cosine airfoil, (e) a perturbed circle, and (f) a general spline curve. Our implementation uses the radix-2 \( N \) point FFT routine listed in [2, p. 416] where \( N = 2^M \).

There are two errors associated with the iterations:

\[
\max_{1 \leq j \leq N} |\sigma^{(n)}(\theta_j) - \sigma^{(n-1)}(\theta_j)|
\]

and

\[
\max_{1 \leq j \leq N} |\sigma(\theta_j) - \sigma^{(n)}(\theta_j)|
\]

where \( \theta_j = 2\pi(j-1)/N \) are the Fourier points, \( \sigma \) is the exact boundary correspondence, and \( \sigma^{(n)}(\theta_j) \) are the values at the \( n \)th iterate. The first error can generally be iterated to zero in machine precision. Exceptions to this occur for Wegmann’s method, where it may cease to decrease much below the level of the discretization error and often exhibits a convergence/divergence behavior, to be discussed below. The second error is our computational measure of the discretization error. We have \( \sigma(\theta) \) only in the

\[(K - K(N)) (\log(\rho)) \text{ for ellipse}\]

\[\text{for ellipse.}\]
case (a), so in the other examples we estimate the discretization error by computing an approximation to

$$\max_{0 \leq \theta \leq 2\pi} \text{dist} \left( g^{(n)}(e^{i\theta}), \Gamma \right),$$

the Hausdorff distance between $g^{(n)}(S)$ and $\Gamma$. Here $g^{(n)}$ is the approximation to the Taylor series of $g$ found using $\sigma^{(n)}(\theta)$ and $N' > N$ Fourier points; the values of $g^{(n)}(e^{i\theta})$ at these $N'$ points are found using the $N'$-point FFT and the Taylor series of $g^{(n)}$ zero padded from $N$ to $N'$ points. (For example, (b) dist $(g^{(n)}(e^{i\theta}), \Gamma)$ can be found exactly and it is approximated by

$$|g^{(n)}(e^{i\theta}) - z(\arg g^{(n)}(e^{i\theta}))|$$

for (c) and (d).) We do not compute the discretization error for (e) and (f).

The curves (a), (b), (c), (d) are all given by piecewise analytic expressions. We also consider periodic cubic splines parameterized by the chordal distance [11] fitted to points on these curves. The chordal distance approximates the arclength so $|z'(\sigma)| = 1$. Usually we use $NN = 1,000$ knots evenly distributed in the original parameterization. In this case the spline approximates the analytic curves to within about $10^{-7}$ to $10^{-10}$ in most of our examples, so the discretization error can never go below that. However, the spline is needed in example (d) where a parameterization is needed for the near circular region generated by the Karman–Trefftz map. (To treat this case with Theodorsen’s method we would have to use a spline parameterized by polar angle;
we have chosen to omit this case.) In practice, we usually have a finite number of points to define a boundary. This can be done conveniently with a spline as illustrated in example (f).

The initial guess for the examples below is always $\sigma^{(0)}(\theta) = L\theta/2\pi$, where $\theta \in [0, 2\pi]$.

The numbers near the data points on the figures are rough timings of CPU second (IBM 3081, FORTVS compiler, double precision arithmetic).

(a) Ellipse [10, p. 412]. $\Gamma$ is given by

$$z(\theta) = \cos \theta + i\alpha \sin \theta, \quad 0 \leq \theta \leq 2\pi.$$  

$\sigma$ is a “thinness” parameter, $0 < \alpha \leq 1$. The exact solution, if $g'(\infty) > 0$ or $g(1) = 1$ is imposed, is

$$g(w) = \frac{1}{2}(1 + \alpha)w + (1 - \alpha)w^{-1}.$$  

We can also represent $\Gamma$ in polar coordinates, $z(\sigma) = \rho(\sigma)e^{i\sigma}$, $0 \leq \sigma \leq 2\pi$, with

$$\rho(\sigma) = \alpha/\sqrt{1 - (1 - \alpha^2) \cos^2 \sigma}.$$  

In this representation the exact boundary correspondence is

$$\sigma(\theta) = \arctan(\alpha \tan \theta).$$
We note that the Laurent series of $g$ converges (in fact has only two terms) for all $w \neq 0$ but that $g'(w) = 0$ at

$$w = \pm \sqrt{\frac{1 - \alpha}{1 + \alpha}}.$$  

From the expression for $g(w)$ we derive the Fourier coefficients of the auxiliary function for Theodorsen's method:

$$h(e^{i\theta}) = \log \left( \frac{g(e^{i\theta})}{e^{i\theta}} \right)$$

$$= \log \rho(\sigma(\theta)) + i(\sigma(\theta) - \theta)$$

$$= \log \left( \frac{1 + \alpha}{2} \right) + \beta \cos(2\theta) - \frac{\beta^2}{2} \cos(4\theta) + \cdots$$

$$+ i \left( -\beta \sin(2\theta) + \frac{\beta^2}{2} \sin(4\theta) - \cdots \right)$$

where $\beta = (1 - \alpha)/(1 + \alpha)$. The $N$-point FFT gives an approximation to the first $N/2$ terms of the sin/cos series (see [5]). Thus for the above series we have

$$\log \rho(\sigma(\theta)) = \log \left( \frac{1 + \alpha}{2} \right) + \beta \cos(2\theta) - \frac{\beta^2}{2} \cos(4\theta) + \cdots \pm \frac{\beta^{N/4}}{N/4} \cos \frac{N}{2} \theta.$$
Fig. 3(a). Ellipse $\alpha = .8$ with Theodorsen $N = 64$.

Fig. 3(b). Ellipse $\alpha = .4$ with Friberg $N = 256$. 
We see that the error committed in truncating the series is

\[(K - K_N)\left[\log \rho(\sigma(\theta))\right] = O\left(\frac{\beta^{N/N}}{N}\right)\]

in agreement with Gaier [4]. This is also in agreement with Wegmann [15, Ex. b, § 9]; there, bounds of order \(R^{-N/2}\) are exhibited for the interior problem where \(R > 1\) is the modulus of the nearest singularity or zero of the derivative. Here \(R^2 = \beta < 1\).

Similarly for Timman's method,

\[h(e^{i\theta}) = \log (g'(e^{i\theta}))\]

\[= \log \left(\frac{1 + \alpha}{2}\right) - \left(\beta \cos \theta + \frac{\beta^2}{2} \cos 4\theta + \cdots\right)\]

\[+ i \left(\beta \sin \theta + \frac{\beta^2}{2} \sin 4\theta + \cdots\right).\]
As observed previously,
\[
\log (\sigma'(\theta)) + \log |z'(\sigma(\theta))| = \text{Re } h(e^{i\theta})
\]
and
\[
\arg (z'(\sigma(\theta))) - \theta - \frac{\pi}{2} = \text{Im } h(e^{i\theta}),
\]
so that the errors here exhibit the same order as above.

We also recall that the \(\varepsilon\)-condition in this example can be computed exactly, and \(\varepsilon < 1\) if and only if \(\alpha > \sqrt{2} - 1 \approx 0.4142\).

Our computations of the discretization error are given in Fig. 1. Gaier [4] shows that the aliasing error of the conjugated trigonometric interpolant is of the same order as the error in truncating the Fourier series of the conjugate function, as exhibited above. The errors shown here follow the trend predicted by the theory. The discretization error for Theodorsen follows Fig. 1 for \(\alpha > 0.41\) \ldots. The discretization errors for Timman, Friberg, and Wegmann for the ellipse are given in Figs. 2(a), 2(b), and 2(c). The data there are for analytic parameterization for \(\alpha = 0.8, 0.4\), and for \(\alpha = 0.2\) a spline was used. Often the spline gave a smaller discretization error. Here for \(\alpha = 0.2\), larger \(N\) was required for convergence in the analytic case. These errors follow Fig. 1 for
larger \( \alpha \), but degrade for smaller \( \alpha \) (say \( \alpha < .4 \)). Some computed regions are shown in Fig. 3.

(b) Sports ground. This region of length two is bounded by two semicircles of radius \( \alpha \) connected by horizontal line segments. See Fig. 4. We use arclength as a parameter; \( z'(\sigma) \) is not differentiable at the points connecting the semicircles and lines. Here \( |z'(\sigma)| = 1 \) for all \( \sigma \), in contrast to example (a) where \( |z'(\sigma)| \) varies between \( \alpha \) and 1.

We have observed a convergence/divergence phenomenon (further addressed in [16]) in which errors decrease up to a certain point and then increase, for Timman, Friberg, and Wegmann. This occurs only in certain situations. For Timman it never occurs for a spline, never for the sports ground, and only for small \( \alpha \) for an ellipse with analytic parameterization. For Friberg it occurs for an ellipse with small \( \alpha \) for both the spline and the analytic parameterizations. For Wegmann it occurs to some extent in all cases.

These results indicate that the division by \( |z'(\sigma(\theta))| \) in Timman or Wegmann, and also the multiplication by \( |z'(\sigma(\theta))| \) in Friberg has some effect on the convergence behavior, and may trigger this phenomenon. It may also be associated with discretization error. It does not occur for Theodorsen where the discrete equations can be solved to machine accuracy for \( \varepsilon < 1 \). Wegmann [15, p. 457] noticed the start of this behavior for the interior problem and attributed it to roundoff error. Examples of
CONVERGENCE/DIVERGENCE FOR WEGMANN FOR .6 ELLIPSE, N = 32

Fig. 5(c). Convergence/divergence for Wegmann for .6 ellipse N = 32.

convergence/divergence are shown in Fig. 5 for the ellipse; Figs. 5(a) and 5(b) give discretization error versus number of iterations for Timman and Friberg, respectively (error between successive iterates is roughly the same); Fig. 5(c) shows discretization error and that between successive iterates for Wegmann. It is interesting to observe that convergence rates are independent of N.

(c) Inverted ellipse. If we write $z(\sigma) = \rho(\sigma)e^{i\epsilon}$, then

$$\rho(\sigma) = \sqrt{1 - (1 - \alpha^2)\cos^2(\sigma)}.$$ 

The $\epsilon$-condition is the same as in (a). (See Figs. 7(a)–(c).)

The discretization error for Timman’s method with this parameterization is given in Fig. 6(a). Friberg yields almost indistinguishable results. Using a spline gives results which are often an order of magnitude more accurate. The timings given in Fig. 6(a) clearly indicate that Friberg is faster than Timman for larger $\alpha$. This confirms Kaiser’s analysis [13], [7, p. 62], which says that the linear convergence rate for Friberg’s method is nearly zero for nearly circular regions. We did not succeed in getting converged solutions for $\alpha$ smaller than $.5$ with Friberg or Timman.

Figure 6(b) gives the discretization error for Wegmann’s method. The results for Theodorsen’s method were nearly the same, except that Theodorsen’s method failed to converge for $\alpha < \sqrt{2} - 1 \approx .41$ ($\epsilon > 1$). The use of a spline fit produced nearly identical results down to the level of accuracy of the spline. Some timings are again indicated as above.
Wegmann's method converged for \( N = 2,048, \alpha = .3 \), also (see Fig. 7(c)). It is clearly seen to be the best of the four methods here both in timing and in range of \( \alpha \). Since it converges quadratically Wegmann's method usually only required three or four iterations to reach minimum discretization error. For \( \alpha = .6 \), \( N = 512 \), for instance, Theodorsen required 49 iterations. See [14] for similar observations.

Note that the exterior problem for the inverted ellipse is, like the interior problem for the ellipse, a highly ill-conditioned problem as \( \alpha \downarrow 0 \). (The inverse map exhibits severe crowding; see [6] and [17].) We may expect such difficulty for any method which has an elongation or fingering of a portion of the boundary between the canonical and the target regions. Examples (a) and (b) are easier in this sense, i.e., maps to the interior of slender regions are harder than maps to the exterior of slender regions.

(d) cosine airfoil. We use the following curve to illustrate the map to the exterior of a curve with a corner:

\[
z(\tau) = -\cos(k\tau) e^{i\pi}, \quad \frac{-\pi}{2k} \leq \tau \leq \frac{\pi}{2k}.
\]

The exterior angle \( 2\pi - \lambda \) is removed by the Karman–Trefftz transformation

\[
\frac{\xi - \xi_1}{\xi - \xi_2} = \left( \frac{z - z_1}{z - z_2} \right)^{\delta}
\]
where $z_1, z_2$ map to $\zeta_1, \zeta_2$, and

$$\delta = \frac{\pi}{2\pi - \lambda}.$$  

We map $NN = 1,000$ points to the $\zeta$-plane and interpolate with a periodic cubic spline; this yields a smooth curve which is more nearly circular for a good choice of $z_2$. (This curve is not parameterized by argument, so we omitted Theodorsen’s method.) Timman, Friberg, and Wegmann give the map to the region in the $\zeta$-plane. This is composed with the inverted Karman–Trefftz map. Below $z_1 = \zeta_1 = 0$, $z_2 = \zeta_2 < 0$ for Timman and Wegmann. (Since Friberg requires the curve to surround the origin we translated the spline to the right by $-\zeta_2/2$ and then back by $-g(1)$ after the iteration.) Plots of the image in the $z$ and $\zeta$ planes are shown in Fig. 8.

Figures 8(a)–8(d) show the maps for two cosine curves and their images under the Karman–Trefftz transformation. Figure 9 shows the discretization error in the plane of the cosine curve plotted against $M$ for Wegmann with $z_2 = -.9$ and $k = 2$ (Figs. 8(a) and (b)). An error of $10^{-9}$ was reached in successive iterates of the map to the spline curve. The discretization errors for Timman and Friberg were similar but somewhat erratic; however, the errors in the successive iterates could be decreased to $10^{-15}$. Timings for $N = 128$ were: Wegmann, 1.3 sec; Friberg, 1.8 sec; and Timman, 4.0 sec.
(e) Perturbed circle. Here $z(\sigma) = (1 + 0.03 \sin(16\sigma)) e^{i\sigma}$. The methods of Theodorsen and Friberg worked best here converging to machine precision in only two iterations (see Fig. 10(a)). For $N = 128$ Wegmann reached an error of $10^{-11}$ in the successive iterates. For $N = 64$ Timman’s method converged to machine precision to a map with loops exhibited in Fig. 10(b). In this case the approximate map must have zeros of its derivative just outside the unit circle. For $N = 256$ Timman converged to the correct map. Similar loops are shown in [6] for Theodorsen’s method for maps to the interior of a thin ellipse and to regions with corners.

(f) Spline curves. Here our boundary curves were generated by selecting a few points in the plane and interpolating them with our periodic cubic spline routine. For the region in Fig. 11(a), Timman, Friberg, and Wegmann all converged. Machine precision was achieved in the successive iteration error for Timman and Friberg and most $N$ tried. Wegmann achieved the successive iteration errors given in Fig. 11(c) in a few iterations and then diverged. This convergence/divergence behavior may be somewhat disadvantageous in practice without a good procedure to stop the iterations. However, Wegmann again is more robust. It converged for Fig. 11(b) where Timman and Friberg failed. The effect of the curve geometry is evident again: as in the comparison of inverted ellipse and the ellipse, the fingering in Fig. 11(b) causes more inaccuracy than the slenderness in Fig. 11(a). Note also that the errors in Fig. 11(c) are roughly $O(N^{-3})$ as expected with cubic spline boundary, $z(\sigma) \in W^{3,\infty}$. Sample timings for Fig. 11(b) are 3.7 sec for $N = 1024$ and 7.5 sec for $N = 2048$. Thus
Wegmann's method seems quite promising in this practical case. In future work we hope to report on computations where the spline curve is generated by explicit maps, such as the osculation maps in [10].

4. Conclusion. In comparing the methods of Wegmann, Timman, Friberg, and Theodorsen our general conclusion is that Wegmann is the most efficient and most robust. Friberg is superior to Timman for nearly circular domains as the theory suggests; for extreme regions Timman is preferable if the arclength is used as the parameter.

We have observed a curious convergence/divergence phenomenon. For Timman it can be correlated with the use of a general parameterization; it did not occur for Timman when the arclength was used. It may also be associated with the relation between the discrete and continuous problems; it does not occur for Theodorsen with $\varepsilon < 1$ where the discrete equation can be solved to machine precision.

A careful analysis has been made of discretization error; for extreme regions the error is worse than the theory indicates for Timman, Friberg, or Wegmann. If a remedy for this could be found, the range of usefulness of Fourier methods might be extended significantly.

Acknowledgments. We thank Professor Mike Papadakis of our Aeronautical Engineering Department for assistance with the graphics and one of the referees for suggesting some corrections and improvements.
FIG. 7(c). Inverted ellipse $a = .3$ with Wegmann $N = 2,048$.

FIG. 8(a). Cosine curve $k = 2$ with Wegmann $N = 128$. 
FIG. 8(b). Image of Fig. 8(a) under Karman-Trefftz with $z_2 = -0.9$.

FIG. 8(c). Cosine curve $k = 4$ with Timman $N = 512$. 
FIG. 8(d). Image of Fig. 8(c) under Karman–Trefftz with $z_2 = -0.99$.

**WEGMANN COSINE CURVE, $Z_2 = -0.9, K = 2$**

![Graph showing discretization error for Wegmann for cosine curve, $z_2 = -0.9, k = 2$.]

FIG. 9. Discretization error for Wegmann for cosine curve, $z_2 = -0.9, k = 2$. 
Fig. 10(a). Perturbed circle with Theodorsen $N = 64$.

Fig. 10(b). Loops produced by perturbed circle with Timman $N = 64$. 
FIG. 11(a). Spline curve with Friberg $N = 128$.

FIG. 11(b). Spline curve with Wegmann $N = 1,024$. 
**REFERENCES**


