A Note on Rengel's Inequality
and the Crowding Phenomenon in Conformal Mapping

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Abstract. We show how Rengel's inequality for the conformal module of a quadrilateral can be applied in a simple way to estimate the crowding phenomenon for slender regions.

1. INTRODUCTION

Recently there has been some interest in the severe ill-conditioning of the conformal mapping problem known as the crowding phenomenon. Briefly, this ill-conditioning occurs whenever a "straight" section of a domain is mapped to a highly elongated section of a target domain, as is the case, for instance, with the map from the disk to a slender domain or the map from a strip to a strip with "long fingers." Since this phenomenon can make the numerical solution of the mapping problem difficult or impossible, some methods which compute the map between domains which do not exhibit crowding have been developed; see [3] and [5]. However, the general problem seems to be inherent, since for any convenient computational domain it would seem possible to find a domain for which the map crowds.

Estimates for the crowding can be given for some explicit examples; for instance, the case of the map from the unit disk to a rectangle of large aspect ratio. Some more general estimates have been given in [1] and [9]. The results in [9] are not entirely rigorous and the more general theory in [1] is not very accessible. It is the purpose of this short note to point out a simple inequality for the conformal module which can be applied, in particular, to estimate the crowding for the map from the unit disk to slender regions. More precise estimates of a similar nature are applied in [6] to a method for decomposing slender domains.

2. RENGEL'S INEQUALITY

Let \( D \) be a Jordan domain with four distinct points \( z_j; j = 1, 2, 3, 4 \) numbered in counterclockwise order on its boundary as shown below. Such a system is called a quadrilateral, denoted

\[
Q = \{D; z_1, z_2, z_3, z_4\}. \tag{2.1}
\]

Let \( R = \{x + iy : 0 \leq x \leq H, 0 \leq y \leq 1\} \) be a rectangle. For only one value of \( H \) can \( D \) be mapped conformally onto \( R \) so that the \( z_j \)'s map to the corners with \( z_1 \) corresponding to \( H + i \). In this case \( M(Q) = H \) is called the conformal module of \( Q \). Let \( A \) be the area of \( Q \), \( w \) the length of the shortest path in \( D \) from the side \( z_1z_2 \) to the side \( z_3z_4 \) of \( Q \), and \( l \) the length of the shortest path in \( D \) from the side \( z_2z_3 \) to the side \( z_4z_1 \). Then Rengel's inequality is

\[
M_l := l|2/A \leq M(Q) \leq A/w|2 =: M_u \tag{2.2}
\]

with equality when and only when \( Q \) is a rectangle; see [4, p. 22].

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3. Applications

3.1. For a slender quadrilateral \( Q \) with \( z_1, z_4 \) and \( z_2, z_3 \) at opposite ends and \( l \gg w \) as shown above, we have \( A \approx lw \). Therefore (2.2) gives the rough estimate

\[ M(Q) \approx l/w. \]

\( Q \) can thus be mapped to a rectangle \( R \) with sides of ratio approximately \( l/w \). If such a rectangle is mapped to the disk with the center going to the center, then, as a consequence of properties of the elliptic integrals which arise in the Schwarz-Christoffel map of \( R \) to the unit disk, the preimages on the unit circle of the corners corresponding to \( z_1 \) and \( z_4 \) (\( z_2 \) and \( z_3 \)) have arguments differing by \( \theta \approx 8e^{-\pi/2w} \). That is, the crowding of the preimages varies exponentially with the aspect ratio; see e.g. [5, pp. 351-352] or [3]. If we assume that the map from \( D \) to \( R \) exhibits no severe crowding (we have no theorem here, but the experiments in [3] provide much evidence that this is true), then the map from \( D \) to the unit disk given by the composition of the maps from \( D \) to \( R \) and \( R \) to the unit disk will exhibit crowding near the preimages of \( z_1 \) and \( z_4 \) (\( z_2 \) and \( z_3 \)) of roughly \( \theta \), as given above. A linear fractional transformation of the disk to itself may relieve the crowding near one pair of preimages, but not near both pairs. Maps between rectangles and certain quadrilaterals are studied in [5]. It is shown there that, even though the maps are well-conditioned, a numerical method that attempts to construct the map by mapping through the disk (or, equivalently, the upper half plane) will be ill-conditioned for slender regions and is therefore a bad choice.

3.2. Inequality (2.2) gives estimates for regions which are not slender. However, these estimates may be quite crude as can be seen by checking some of the examples in [7].

3.3. The estimate in [6, section 2.3] can be gotten from (2.2).

3.4. For an L-shaped polygon \( D \) with \( L > 1 \) and corners \( 0, L, L + i, 1 + i, 1 + iL, \) and \( iL \) let \( Q = \{D; L + i, 1 + iL, iL, L \} \). Then \( A = 2L - 1, I = 2L - 2, w = 1 \) and (2.2) shows that \( M(Q)/2L \rightarrow 1 \) as \( L \rightarrow \infty \).

3.5. For a slit rectangle \( D \) with corners \( 0, l, l + i, l/2 + i + (1 - a)i, l/2 + i, \) and \( i \) with \( 0 < a < 1 \), let \( Q = \{D; l + i, i, 0, l \} \). Then (2.2) gives \( l \leq M(Q) \leq l/(1 - a) \).

3.6. For a Z-shaped polygon \( D \) with \( L > 1 \) and corners \( 0, L - 1, L - 1 - i, 2L - 1 - i, 2L - 1, L, L + i, \) and \( i \), let \( Q = \{D; 2L - 1, i, 0, 2L - 1 - i \} \). Then \( A = 2L, I = 2L - 1, w = 1 \) and (2.2) shows that \( M(Q)/2L \rightarrow 1 \) as \( L \rightarrow \infty \).

3.7. The table below gives some sample calculations of \( M(Q) \) and \( \theta \) using the subroutine RESIST as in [8, section 5.3] and the regions above, with \( L = 5 \) in 3.4 and 3.6 and \( l = 5 \) and \( a = .5 \) in 3.5. \( \theta = 8e^{-\pi M}/2, \theta_M = 8e^{-\pi M(Q)/2}, \) and \( \theta_u = 8e^{-\pi M_1}/2 \).

<table>
<thead>
<tr>
<th>Region</th>
<th>( M_1 )</th>
<th>( M(Q) )</th>
<th>( M_u )</th>
<th>( \theta_l )</th>
<th>( \theta )</th>
<th>( \theta_M )</th>
<th>( \theta_u )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.4</td>
<td>7.1</td>
<td>8.5</td>
<td>9.</td>
<td>.0000058</td>
<td>.000012</td>
<td>.000013</td>
<td>.00011</td>
</tr>
<tr>
<td>3.5</td>
<td>5.</td>
<td>5.4</td>
<td>20.</td>
<td>1.8 \times 10^{-13}</td>
<td>.0016</td>
<td>.0017</td>
<td>.0031</td>
</tr>
<tr>
<td>3.6</td>
<td>8.1</td>
<td>9.2</td>
<td>10.</td>
<td>.0000012</td>
<td>.0000042</td>
<td>.0000042</td>
<td>.000024</td>
</tr>
</tbody>
</table>

3.8. A simple caricature of Saint Venant's Principle from the theory of elasticity [2] expressing the exponential decay of influence of boundary data can be given. Consider solving the Laplace equation \( \Delta u = 0 \) in the region \( D \) in the figure above with Dirichlet data \( u = 1 \) on side \( z_1 z_4 \) and \( u = 0 \) on the rest of the boundary. Let \( z = f(v) \) map the unit disk...
in the $v$-plane to $D$ such that $f(0)$ is a point in the middle of $D$ and $z_1 = f(e^{-i\theta/2}), z_4 = f(e^{-i\theta/2})$. Then $u(f(0))$ is harmonic in the disk and

$$u(f(0)) = \frac{1}{2\pi} \int_{-\theta/2}^{\theta/2} |\theta/2u(f(e^{i\theta}))\,d\theta = \frac{4}{\pi} e^{-\pi M(Q)/2} \approx \frac{4}{\pi} e^{-\pi l/2w}.$$ 

Thus the influence of the values of $u$ on $z_1 z_4$ on the value of $u$ near the middle of $D$ varies exponentially with the "length" of $D$.

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REFERENCES


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