Conformal mapping of simply and multiply connected domains - my work with Alan Elcrat and John Pfaltzgraff

Tom DeLillo

Wichita State U Math Dept

Aalto U., January 14, 2016

Outline

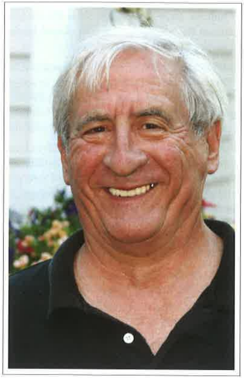
My mentors/coauthors

Alan Ross Elcrat, 1942–2013
PhD Indiana University 1967
Wichita State University 1967–2013
Full Professor 1977

John Andrew Pfaltzgraff, 1936–2014
PhD University of Kentucky 1963
KU 1963–65
Indiana U 1965–67
University of North Carolina 1967–2007
Full Professor 1974
Chair 1984–93

Alan R. Elcrat

January 13, 1942 – December 20, 2013



John Andrew Pfaltzgraff, November 1, 1936 (All Saints Day) - October 12, 2014

CMFT 2005 Joensuu, Finland

front row: Tom DeLillo, Alan Elcrat, Nick Trefethen, Ken Stephenson

back row: John Pfaltzgraff, Bengt Fornberg, Nick Papamichael, Ed Saff, Ted Suffridge, Roger Barnard, David Minda and his

“Competitors” and friends

Rudolf Wegmann on Fourier series methods based on Riemann-Hilbert problems; see his survey, R. Wegmann, *Methods for Numerical Conformal Mapping*, in Handbook of Complex Analysis: Geometric Function Theory, Vol. 2, R. Kühnau, ed., Elsevier, 2005, pp. 351–477.
Darren Crowdy on Schwarz-Christoffel mappings for multiply connected domains using the Schottky-Klein prime function; see Darren’s webpage and my 2006 CMFT J. paper relating our methods.

We learned a lot from both of them...and at least someone else usually read our papers!

Other general references for numerical conformal mapping

1. D. Gaier, *Konstruktive Methoden der konformen Abbildung*, Springer, 1964.
2. P. Henrici, *Applied and Computational Complex Analysis, Vol. 3*, Wiley, 1986.
3. T. A. Driscoll, and L. N. Trefethen, *Schwarz-Christoffel Mapping*, 2002, Cambridge.

Fourier series for maps $w = f(z)$ from disk

See tutorial on Fourier series methods on my webpage `http://www.math.wichita.edu/~delillo/TD_tutorial.pdf`.

Goals of conformal mapping in 1980s on (see Henrici ACCA, v. 3, 1986) for maps from the disk to the domain were to find and investigate:

fast, fft-based methods: Theodorsen, Timman, Friberg (linearly convergent) and Newton-like methods of Fornberg and Wegmann; see M. H. Gutknecht, JCAM special issue 1986.

Crowding (Menikoff and Zemach 1980, Gaier 1972) = severe ill-conditioning of the mapping problem

Conformal map $w = f(z)$ from disk to target domain

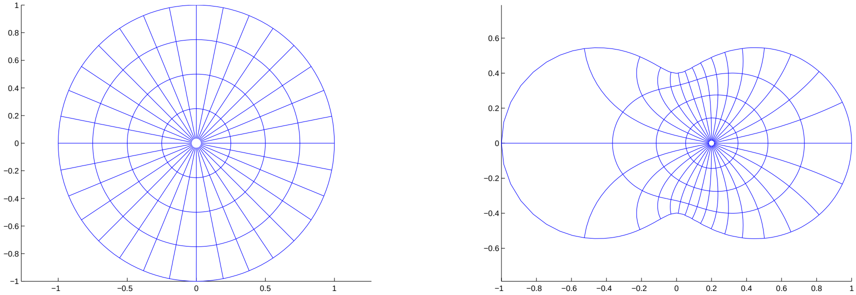


Figure: Fornberg (Fourier series) map from unit disk to interior of an inverted ellipse using N=64 Fourier points. $f'(z) \neq 0$, so locally $f(a+h) \approx f(a) + f'(a)h$ and f maps a small circle near $z = a$ to a circle near $f(a)$ magnified by $|f'(a)|$ and rotated by $\arg f'(a)$, therefore f is *angle-preserving* or *conformal*. Existence and uniqueness given by Riemann Mapping Theorem with $f(0)$ and $f(1)$ fixed.

Boundary correspondence

The boundary Γ of Ω is parametrized by S (e.g., arclength or polar angle), $\Gamma : \gamma(S), 0 \leq S \leq L, \gamma(0) = \gamma(L)$. If $S = S(\theta)$ or its inverse $\theta(S) = \arg f^{-1}(\gamma(S))$ is known, then the map is known for $z \in D$ or $w \in \Omega$ by the Cauchy Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\gamma(S(\theta))}{\zeta - z} d\zeta(\theta)$$

or

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{i\theta(S)}}{\gamma(S) - w} d\gamma(S).$$

Two classes of methods

1. Find $S = S(\theta)$ such that $f(e^{i\theta}) = \gamma(S(\theta))$. We will discuss this case. These methods solve a nonlinear integral equation for $S(\theta)$ by linearly convergent methods of successive approximation (Picard-like iteration) such as Theodorsen's method, or quadratically convergent Newton-like methods such as Fornberg's or Wegmann's methods. Cost: $O(N \log N)$ with FFTs.
2. Find $\theta = \theta(S)$ such that $f^{-1}(\gamma(S)) = e^{i\theta(S)}$. These methods solve linear integral equations arising from potential theory for $\theta(S)$ or $\theta'(S)$. Cost: $O(N^2)$, but can handle more highly distorted regions.

Taylor series = Fourier series

For $|z| < |\zeta| = 1, \zeta = e^{i\theta}, d\zeta = ie^{i\theta}d\theta$

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\gamma(S(\theta))}{\zeta - z}d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma(S(\theta))\left(1 + \frac{z}{\zeta} + \left(\frac{z}{\zeta}\right)^2 + \cdots\right)\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))(1 + ze^{-i\theta} + z^2e^{-2i\theta} + \cdots)d\theta \\
&= \sum_{k=0}^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))e^{-ik\theta}d\theta\right)z^k \\
&= \sum_{k=0}^{\infty}a_k z^k,
\end{aligned}
$$

Taylor coeff. = Fourier coeff. $a_k := \frac{1}{2\pi}\int_0^{2\pi}\gamma(S(\theta))e^{-ik\theta}d\theta.$

Applications:

Transplant boundary value problems for Laplace equation from complicated domain to circle domain or model domain and solve using (fast) Fourier/Laurent series or elementary methods.

(BVP for biharmonic equation can also be solved by transplanting the analytic functions of the Goursat representation.)

Advantages: *fast methods* and *spectral accuracy* for analytic data and boundaries.

Disadvantages: *Crowding phenomenon*–mapping problem can be *severely ill-conditioned* for distorted domains, e.g., an $L \times 1$ elongated domain has derivatives of order $exp(cL)$.

Simply-connected case: crowding=large distortions=Ill-conditioning

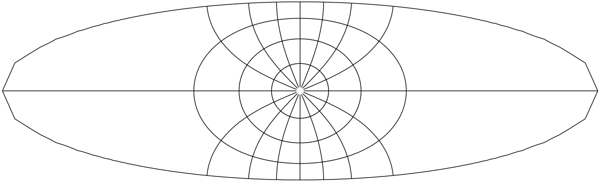


Figure: Fornberg (Fourier series) map from unit disk to interior of ellipse using N=1024 Fourier points.

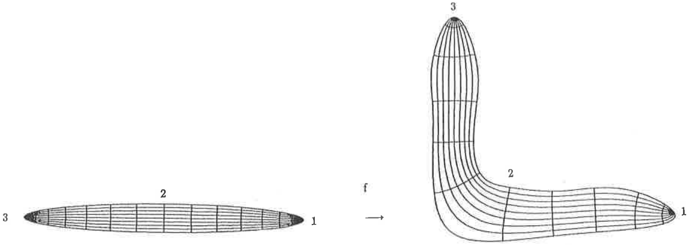


Figure: Map from the $\alpha = 0.1$ ellipse to sock with $N = 256$ Fourier points, D. and Elcrat (1993), D., Elcrat, amd Pfaltzgraff (1997).

Map from annulus–D. and Pfaltzgraff (1998)

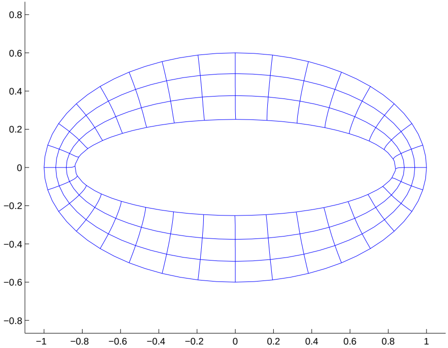


Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Exterior mult. conn. case, D., Horn, and Pfaltzgraff (1999), Benchama et al, (2007)

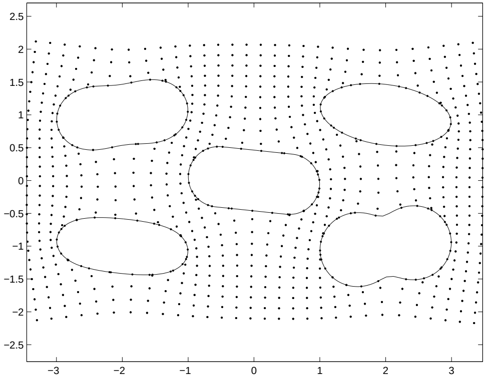


Figure: Fornberg map from exterior of five disks to the exterior of five smooth curves. Normalize at ∞ and find $m = 5$ boundary correspondences and centers and radii of circles (unique "conformal moduli") must be computed.

Interior mult. conn. case–Kropf's MS thesis (2009)

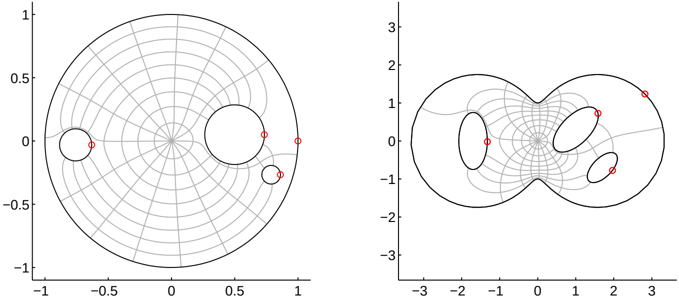


Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.

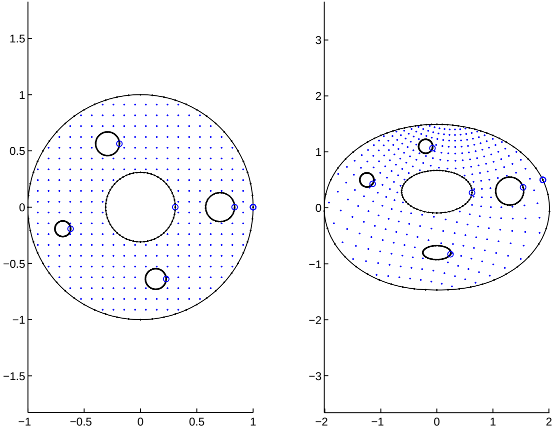


Figure: It's easier to count moduli for an annulus with holes.

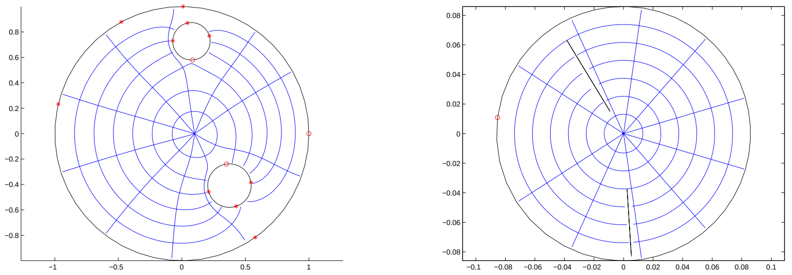


Figure: Infinite product map from circle domain to radial slit disk.

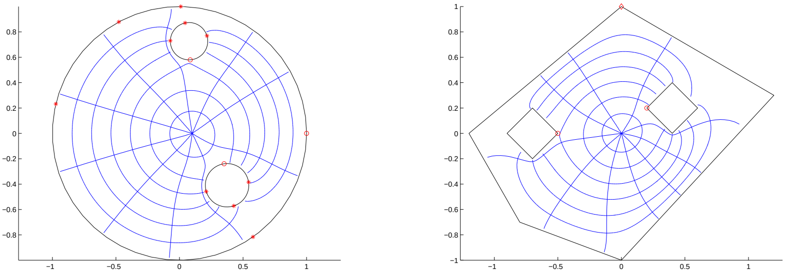
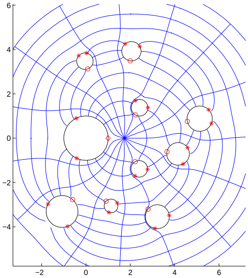


Figure: An orthogonal grid using level lines of map to radial slit disk, MCSC formulas by D., Elcrat, and Pfaltzgraff, numerics with Driscoll and Kropf



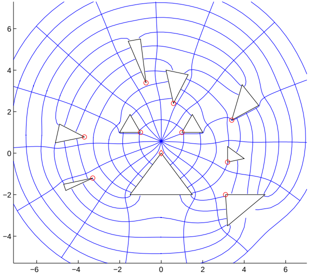


Figure: Schwarz-Christoffel map from the exterior of m=10 disks to the exterior of m=10 polygons using Laurent series centered at disks and least squares fit to BCs, above, D., Elcrat, Kropf, and Pfaltzgraff (2013).

Application to resistance calculation

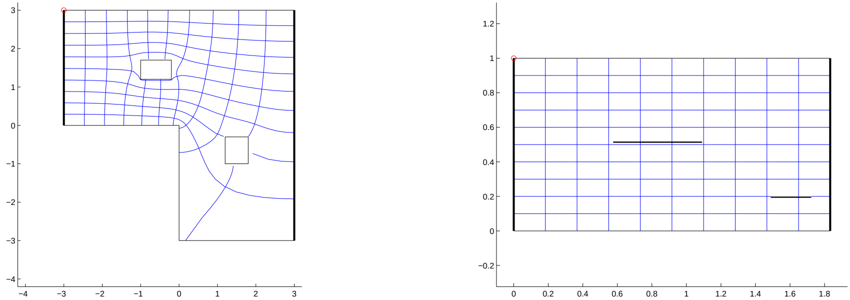


Figure: Map from the interior of unit disk minus $m - 1 = 2$ disks to interior of a bounded polygonal domain and to a rectangle with horizontal slits. Resistance = length/width of rectangle = 1.832838728. (D., Elcrat, and Kropf, CMFT J., 11 (2011), 725–745)

Application to resistance calculation

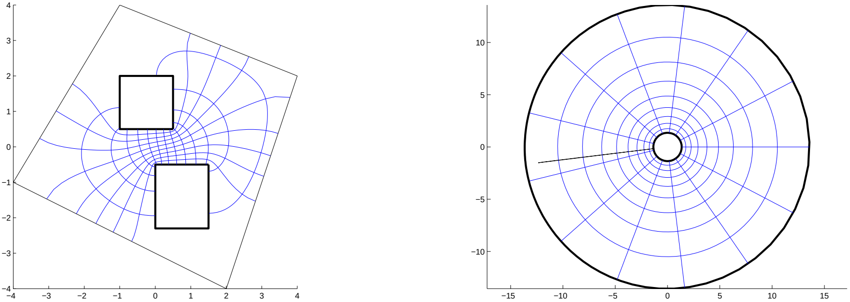


Figure: Map from the interior of unit disk minus $m-1=2$ disks to interior of a bounded polygonal domain and to an annulus with a radial slit. Resistance = $\frac{1}{2\pi}\log(\textit{outer radius}/\textit{inner radius}) = 0.3671$. (D., Elcrat, and Kropf, CMFT J. 11 (2011), 725–745))

Outline

Simply-connected case: crowding=large distortions=Ill-conditioning

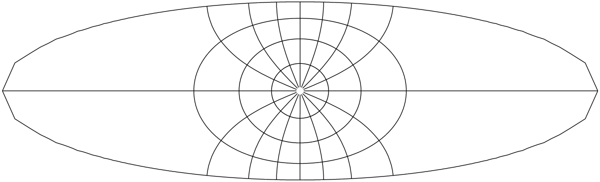
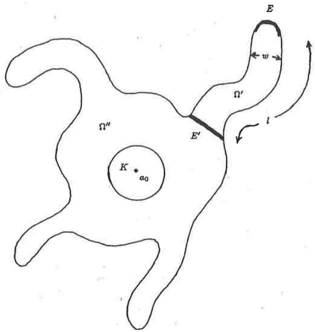


Figure: Fornberg (Fourier series) map from unit disk to interior of ellipse using N=1024 Fourier points.



D. and Pfaltzgraff, JCAM 46 (1993), 103–113 and Ch. Pommerenke, Boundary Behavior of Conformal Maps, 1992. Estimate of crowding by conformal invariants, harmonic measure $\omega(a_0, E, \Omega)$ and extremal distance $\lambda(K, E, \Omega) \approx length^2/area$, for map to amoeba

$$\omega(a_0, E, \Omega) \leq Ce^{-\pi\lambda(K,E,\Omega')} \approx Ce^{-\pi l/w}.$$

Rudolf Wegmann on crowding 2005 review article:
"The behavior of conformal mapping depends on the local property of smoothness— and the global property of shape.
On small scales a conformal mapping maps disks to disks, but on large scales a disk can be mapped to any simply-connected bounded region, however elongated and distorted it may be. But it takes some effort for a mapping which has such a strong tendency to map disks to disks, to map a disk to an elongated region. The mapping suffers, lying on a Procrustean bed, and the numerical conformal mapper must share the pains."

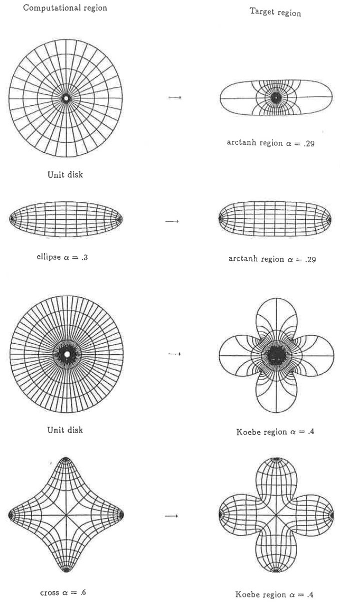


Fig. 1. Top two maps: arctanh region $\alpha = 0.29$ with disk map with $N = 1024$ and ellipse map with $\alpha_{\text{ellipse}} = 0.3, N = 64$. Bottom two maps: 4-leaf Koebe region $\alpha = 0.4$ with disk map with $N = 1024$ and cross map with $\alpha_{\text{cross}} = 0.6, N = 256$.

Maps based on Faber series

D. and Elcrat, JCAM 46 (1993) 49–64, D., Elcrat, and Pfaltzgraff, JCAM 83 (1997) 205–236.

Represent map from a simply-connected domain $E : z = z(\theta)$ to target domain $\Omega : \gamma = \gamma(S)$ with normalizatiion $f(0) = 0, f(z(0)) = \gamma(0)$

$$f(z) = A_0 + \sum_{k=1}^{\infty} A_k F_k(z)$$

where $F_k(z)$ is the kth-degree Faber polynomial of E.
For E = unit disk, $F_k(z) = z^k$.
For E = ellipse, $F_k(z)$ = kth-degree Chebyshev polynomial.

Newton method following Fornberg and Wegmann

Fornberg, 1 SISSC (1980) 386-400, also Wegmann, Numer. Math., (1978,1984).

Find boundary correspondence $S = S(\theta)$ such that $f(z(\theta)) = \gamma(S(\theta))$.

Linearization:

$$\begin{aligned} f(z(\theta)) &= \gamma(S^{(k)}(\theta) + U^{(k)}(\theta)) \\ &\approx \gamma(S^{(k)}(\theta)) + \gamma'(S^{(k)}(\theta))U^{(k)}(\theta)) \end{aligned}$$

Conditions for analytic extension to interior of $E : z(\theta)$ give linear system for $U = |\gamma'|U^{(k)}(\theta)$ of form

$$AU = r$$

where A = identity + compact operator (as Widlund 1980, Wegmann 1986 proved for disk), spd, and can be solved by conjugate gradient with superlinear convergence in $O(N \log N)$.

Newton update: $S^{(k+1)}(\theta)) = S^{(k)}(\theta) + U^{(k)}(\theta))$.

Conditions for analytic extension into E

Theorem

f Hölder continuous on E extends analytically to interior if and only if

$$\int_E f(z)F_k(z)dz = 0, k \geq 0.$$

For unit disk, conditions are just $a_{-k} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{ik\theta}d\theta = 0$, $k \geq 1$.

Conditions for extension into ellipse

$z(\theta) = \psi(\zeta) = \zeta + 1/\zeta,\ \zeta = \rho e^{i\theta}$, $F_k(\psi(\zeta)) = \zeta^k + \zeta^{-k}$

$$
\begin{aligned}
0 &= \int_E f(z) F_k(z) dz = \int f(\psi(\zeta))(\zeta^k + \zeta^{-k})(\zeta - \zeta^{-1}) d\zeta/\zeta \\
&= i \int f(\psi(\zeta))(\zeta^{k+1} - \zeta^{-k-1} - \zeta^{k-1} - \zeta^{-k+1}) d\theta \\
&= i \int_0^{2\pi} f(\psi(\rho e^{i\theta})[(\rho^{k+1} e^{i(k+1)\theta} - \rho^{-(k+1)} e^{-i(k+1)\theta}) \\
&\quad -(\rho^{k-1} e^{i(k-1)\theta} - \rho^{-(k-1)} e^{-i(k-1)\theta})] d\theta.
\end{aligned}
$$

This gives, for the Fourier coefficients a_k of $f(\psi(\rho e^{i\theta}))$,

$$
\rho^{k+1} a_{-(k+1)} - \rho^{-(k+1)} a_{k+1} = \rho^{k-1} a_{-(k-1)} - \rho^{-(k-1)} a_{k-1}.
$$

Conditions for extension into ellipse...cont.

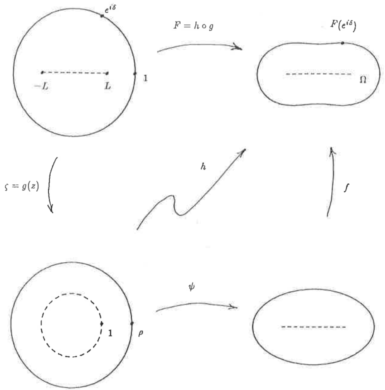
Alan's solution is

$$a_{-k} = \rho^{-2k} a_k, k \geq 0.$$

Identity + compact structure preserved since multiplication by ρ^{-2k} can be represented by convolution.

Similar condition for cross $\psi(\zeta) = \sqrt{\zeta^2 + 1/\zeta^2}$.

Wegmann solves finds updates $U^{(k)}$ as solutions to Riemann-Hilbert problems; see his survey in R. Kühnau, ed., Handbook of Complex Analysis, v. 2, (2005).


$e^{i\delta}$
$F = h \circ g$
$F(e^{i\delta})$
$-L$
L
1
Ω
$\zeta = g(z)$
h
f
ψ
1
ρ

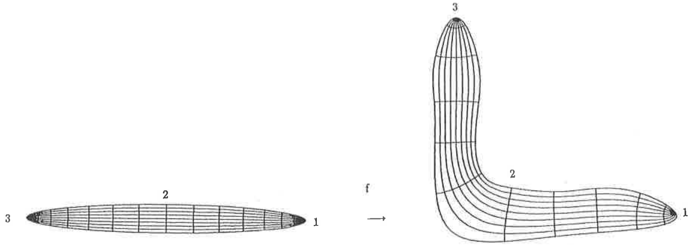


Figure: Map from the $\alpha = 0.1$ ellipse to sock with $N = 256$ Fourier points.

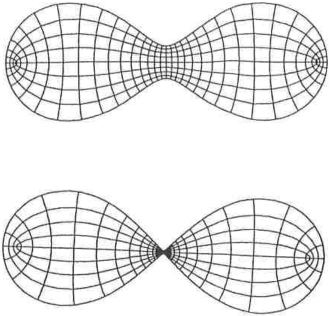
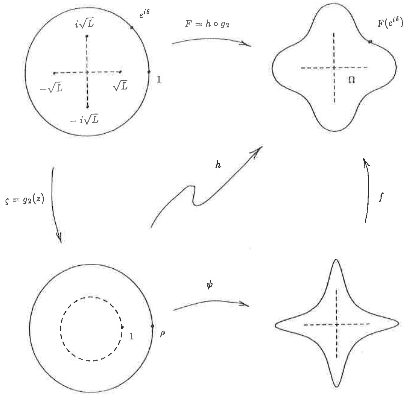


Figure: Map from $\alpha = 0.3, 0.2$ ellipses to $\alpha = 0.1, 0.01$ Cassini ovals with $N = 128, 256$, resp.


$e^{i\delta}$
$i\sqrt{L}$
$-\sqrt{L}$
$\sqrt{L}$
$-i\sqrt{L}$
1
$F = h \circ g_2$
$F(e^{i\delta})$
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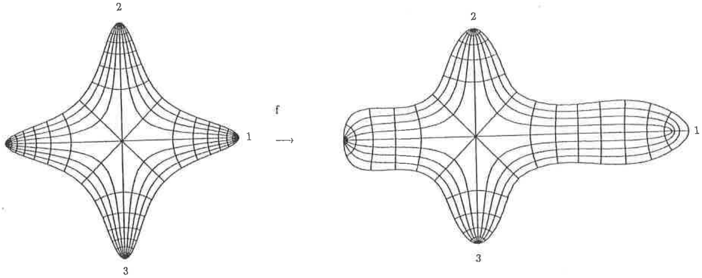

Figure: Map from cross to symmetric amoeba with N=256 Fourier points.

Outline

Map from annulus–D. and Pfaltzgraff (1998)

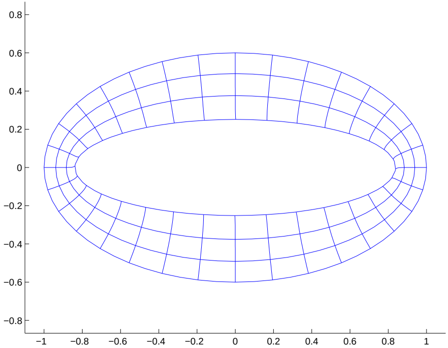

Figure: Doubly connected Fornberg maps annulus $\rho < |z| < 1$ to domain between two ellipses $\alpha = .3, .6$ with $N = 64$. Normalization fixes one boundary point $f(1)$ to fix rotation of annulus. The inner and outer boundary correspondences $S = S_1(\theta)$ and $S = S_2(\theta)$ along with the unique ρ(=1/conformal modulus) must be computed numerically.

Analyticty conditions

A function f given on boundary of annulus extends analytically to interior $\rho < |z| < 1$ if and only if

$$\int_{|z|=1} f(z)z^k dz = \int_{|z|=\rho} f(z)z^k dz, \quad k \in \mathbf{Z}.$$

If we let

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \qquad f(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} b_k e^{ik\theta} = \sum_{k=-\infty}^{\infty} a_k \rho^k e^{ik\theta}$$

then $\rho^k a_k = b_k, \quad k \in \mathbf{Z}$ or (to prevent overflow)

$$\rho^k a_k = b_k, a_{-k} = \rho^k b_{-k}, k = 0, 1, 2, \ldots.$$

Mapping problem

Target region Ω bounded by two smooth curves $\Gamma_1 : \gamma_1(S_1)$ and $\Gamma_2 : \gamma_2(S_2)$.

Problem: Find the *boundary correspondences* $S_1(\theta)$ and $S_2(\theta)$ and the *conformal modulus* ρ such that $f(z)$ is analytic in the annulus $\rho < |z| < 1$ and $f(e^{i\theta}) = \gamma_1(S_1(\theta))$ and $f(\rho e^{i\theta}) = \gamma_2(S_2(\theta))$.

Linearization for Newton-like method

At each Newton step we want to compute corrections $U_1(\theta),\ U_2(\theta)$, and $\delta\rho$ to $S_1(\theta),\ S_2(\theta)$, and ρ. With S_j arclength, $\beta_j(\theta) := \arg \gamma_j'(S_j(\theta)),\ \xi_j(\theta) := \gamma_j(S_j(\theta)),\ j = 1,2,\ \zeta(\theta) := f'(\rho e^{i\theta})e^{i\theta} = -ie^{i\beta_2(\theta)}dS_2(\theta)/d\theta/\rho$, as in [LM] we linearize about S_1, S_2, and ρ,

$$\begin{aligned} \gamma_j(S_j(\theta) + U_j(\theta)) &\approx \gamma_j(S_j(\theta)) + \gamma_j'(S_j(\theta))U_j(\theta)),\ j = 1,2, \\ f((\rho + \delta\rho)e^{i\theta}) &\approx f(\rho e^{i\theta}) + f'(\rho e^{i\theta})\delta\rho e^{i\theta} \end{aligned}$$

giving

$$\begin{aligned} f(e^{i\theta}) &\approx \xi_1(\theta) + e^{i\beta_1(\theta)}U_1(\theta) \\ f(\rho e^{i\theta}) &\approx \xi_2(\theta) + e^{i\beta_2(\theta)}U_2(\theta) - \zeta(\theta)\delta\rho. \end{aligned}$$

We find $U_1, U_2, \delta\rho$ to force these BVs to satisfy the analyticity conditions for the annulus.

Linear system and Newton update

Applying above conditions and fixing $f(1)$ leads to linear system

$$AU = r$$

for $U = [U_1^T, U_2^T, \delta\rho]^T$ Where $A = I + Compact$, eigenvalues cluster around 1, and CG with fft converges superlinearly.
Newton update:

$$\underline{S}_1^{(k+1)} = \underline{S}_1^{(k)} + \underline{U}_1^{(k)}$$
$$\underline{S}_2^{(k+1)} = \underline{S}_2^{(k)} + \underline{U}_2^{(k)}$$
$$\rho^{(k+1)} = \rho^{(k)} + \delta\rho^{(k)}.$$

See my tutorial or D. and Pfaltzgraff (1998) for details.

Outline

Exterior mult. conn. case, D., Horn, and Pfaltzgraff (1999), Benchama et al, (2007)

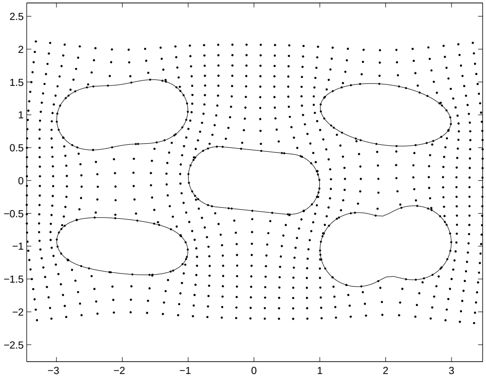

Figure: Fornberg map from exterior of five disks to the exterior of five smooth curves. Normalize at ∞ and find $m = 5$ boundary correspondences and centers and radii of circles (unique "conformal moduli") must be computed.

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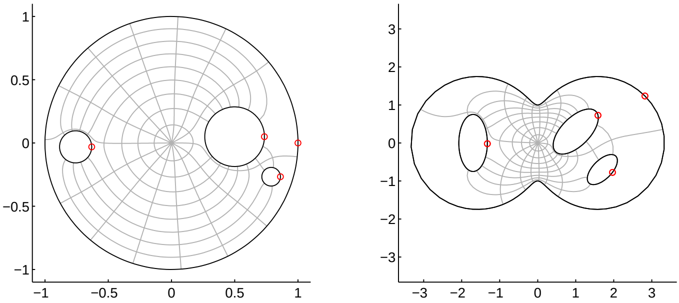


Figure: Outer circle is unit circle. Map normalization fixes $f(0)$ and $f(1)$. $m = 4$ boundary correspondences and centers and radii of inner circles (unique "conformal moduli") must be computed.

Form of the Map (bounded case)

The conformal map has the series representation

$$f(z) = \sum_{j=0}^{\infty} a_{1,j} z^j + \sum_{\nu=2}^{m} \sum_{j=1}^{\infty} a_{\nu,-j} \left(\frac{\rho_\nu}{z - c_\nu} \right)^j ,$$

where for $1 \leq \nu \leq m$ and $j > 0$ the Fourier coefficients $a_{\nu,j}$ are defined

$$a_{\nu,j} := \frac{1}{2\pi} \int_0^{2\pi} f(c_\nu + \rho_\nu e^{i\theta}) e^{-ij\theta} \, d\theta.$$

Linearization and (complicated) conditions on $a_{\nu,j}$'s for analytic extension again leads to system of form "identity + compact" for Newton updates to centers, radii, and boundary correspondences solved in $O(N^2)$ by cg; see my tutorial and Wegmann's R-H formulation.

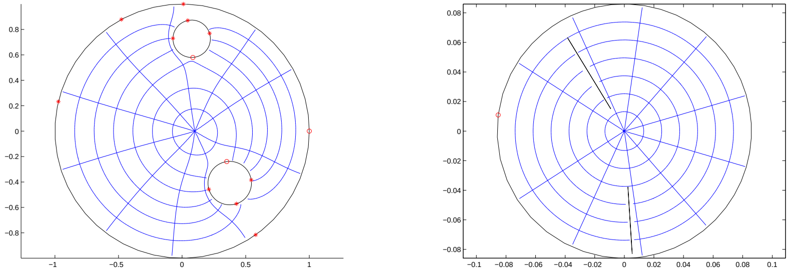
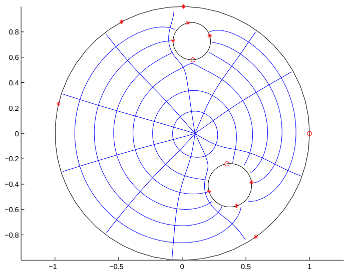


Figure: Infinite product map from circle domain to radial slit disk.



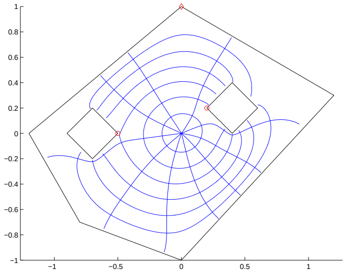


Figure: An orthogonal grid using level lines of map to radial slit disk.

Outline

Re-derivation of Schwarz-Christoffel map of annulus

D., Elcrat, and Pfaltzgraff, SIREV 43 (2001) 469–477. Motivated by the derivation of the formula for the disk in Nehari's book, John had the idea that the known formula for the annulus $\mu < |z| < 1$ could be derived by reflection arguments and that this could probably be extended to the multiply connected case...

$$f(z) = A \int^{z} \prod_{k=1}^{m} \left[\Theta \left(\frac{\zeta}{\mu z_{0,k}} \right) \right]^{-\beta_{0,k}} \prod_{k=1}^{n} \left[\Theta \left(\frac{\mu \zeta}{z_{1,k}} \right) \right]^{\beta_{1,k}} d\zeta + B$$

where $z_{\nu,k}$ are the prevertices and $\beta_{\nu,k}$ are the turning parameters and

$$\Theta(w) = \prod_{\nu=0}^{\infty} \left(1 - \mu^{2\nu+1} w\right) \left(1 - \mu^{2\nu+1}/w\right).$$

The $\mu^k z_{0,k}$'s and $\mu^k z_{1,k}$'s are reflections of the prevertices on the outer and inner circles, resp., through concentric reflected circles.

Outline

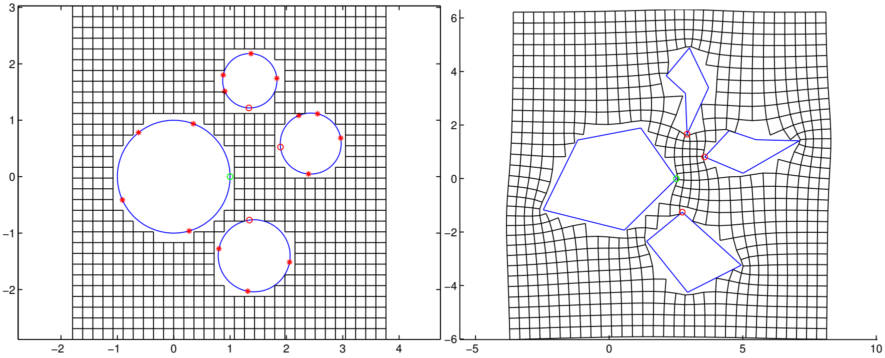


Figure: Exterior MCSC with Cartesian grid for m=4 polygons.

MCSC for unbounded case- D., Elcrat, and Pfaltzgraff (2004)

Given polygons and fixing normalization of $w = f(z)$ at infinity the circles (and the map) will be uniquely determined.
$z_{k,\nu i}, s_{\nu i}$ = reflections of prevertices, centers, $\sum_{k=1}^{K_i} \beta_{k,i} = 2, i = 1, \ldots, m$

$$f(z) = A \int^{z} \prod_{i=1}^{m} \prod_{k=1}^{K_i} \left[\prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left(\frac{\zeta - z_{k,\nu i}}{\zeta - s_{\nu i}} \right) \right]^{\beta_{k,i}} d\zeta + B$$

Numerics: "parameter problem": Find A, B, prevertices $z_{k,i} = c_i + r_i e^{i\theta_{k,i}}$'s, circle centers c_i, and radii r_i, such that sidelengths, positions, and orientations of polygons are correct.
Important: Transform $\theta_{k,i}$'s, r_i's to unconstrained parameters.

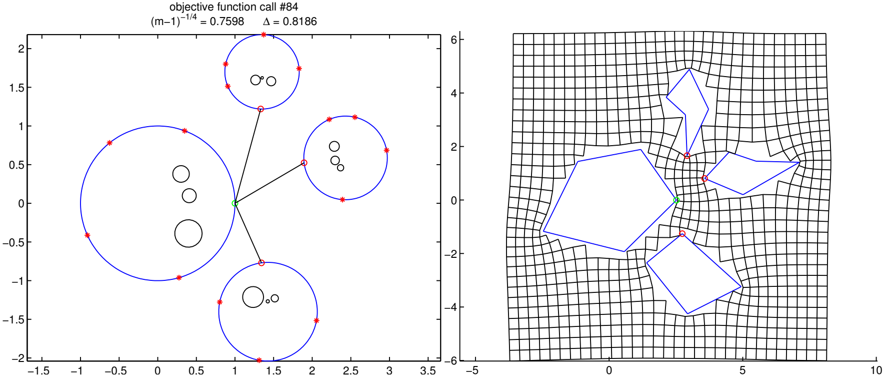


Figure: Objective function integration paths for m=4 polygons.

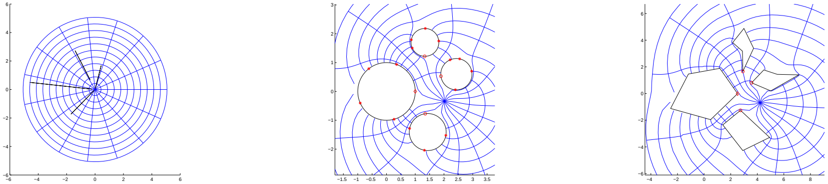


Figure: The (inverted) map from the exterior of $m = 4$ disks to the exterior of $m = 4$ slits gives a better orthogonal grid.

Our first attempt at numerics was D., T. A. Driscoll, A. R. Elcrat, and J. A. Pfaltzgraff, *Computation of multiply connected Schwarz-Christoffel maps for exterior domains*, CMFT J., 6, (2006), 301–315.

Most of the MCSC computations here are from Everett Kropf's PhD dissertation and from D. and E. H. Kropf, *Numerical computation of the Schwarz-Christoffel transformation for multiply connected domains*, SISC, 33 (2011), 1369–1394.

Some remarks on numerics

Most effective nonlinear solver is the continuation algorithm CONTUP, Program 3 from Allgower and Georg's book.

Some MFILES from Driscoll's SC Toolbox were used, such as a function for automatically calculating the turning angles from input polygon vertices.

Evaluation of the Schwarz-Christoffel integrals is done using Gauss-Jacobi quadrature, GAUSSJ from SC Toolbox. A fixed number of Gauss-Jacobi points (typically 30) is generally sufficient for each integral.

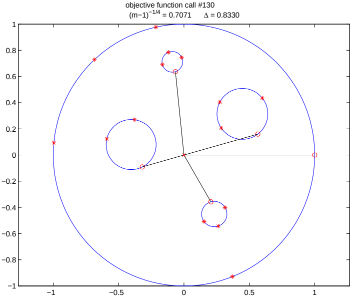
Integration paths may cross circles. This causes no trouble unless they come close to interior singularities.

MCSC formula for the bounded case

$$f'(z) = A \prod_{k=1}^{K_1} (z - z_{k,1})^{\beta_{k,1}} \prod_{i=2}^{m} \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left(\prod_{k=1}^{K_1} \left(z - z_{k,\nu i 1}\right)^{\beta_{k,1}} \prod_{k=1}^{K_i} \left(z - z_{k,\nu i}\right)^{\beta_{k,i}} \right)$$

Here C_1 is the (outer) unit disk and the other C_i are in the unit disk. Also, $\sum_{k=1}^{K_1} \beta_{k,1} = -2$.

Note: The reflections $z_{k,\nu 1}$ of the prevertices on outer circle replace the reflections $s_{\nu i}$ of the centers in the formula for the unbounded case.



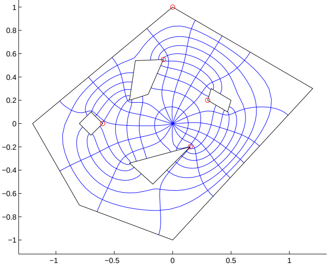


Figure: Schwarz-Christoffel map from the interior of unit disk with minus $m - 1 = 4$ disks to a bounded polygonal domain with origin and one boundary point fixed. Integration paths between circles plotted.

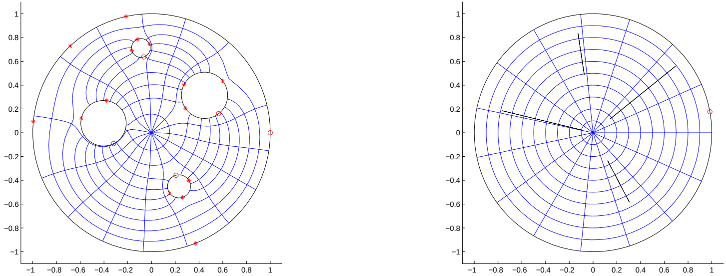


Figure: Interior map to radial slit disk with $m = 5$.

Outline

Theorem

If the unbounded m-connected circular domain satisfies the separation property $\Delta < (m-1)^{-1/4}$ for $m > 1$, then the SC map to the polygonal domain $\mathbb{P}$ is

$$f(z) = A \int^{z} \prod_{i=i}^{m} \prod_{k=1}^{K_i} \left[\prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left(\frac{\zeta - z_{k,\nu i}}{\zeta - s_{\nu i}} \right) \right]^{\beta_{k,i}} d\zeta + B, \qquad (1)$$

where $-1 < \beta_{k,i} \leq 1$ *and* $\sum_{k=1}^{m} \beta_{k,i} = 2$ *and the separation parameter*

$$\Delta := \max_{i,j;i \neq j} \frac{r_i + r_j}{|c_i - c_j|} < 1, \quad 1 \leq i,j \leq m.$$

Separation parameter Δ

Here

$$\Delta := \max_{i,j;i\neq j} \frac{r_i + r_j}{|c_i - c_j|} < 1, \;\; 1 \leq i,j \leq m \tag{2}$$

is the *separation parameter* of the circle domain. Let $\widetilde{C}_j$ denote the circle with center c_j and radius r_j/Δ. Then geometrically, $1/\Delta$ is the smallest magnification of the m radii such that at least two $\widetilde{C}_j$'s just touch.

Note: $\Delta < (m-1)^{-1/4}$ is a sufficient condition for convergence. It is not necessary and is violated in many of our numerical examples.

Remark on extension of $w = f(z)$ by reflection.

(More details below): We use Schwarz reflection of z across circles and $w = f(z)$ across sides of polygons to extend $f(z)$ analytically to a (multivalued) function of the entire plane. ν is a multi-index labeling reflections.

The extended $f(z)$ will have singularities at the reflections of the $z_{k,i}$'s.

$f(z) \approx Az + B$ for $z \approx \infty$, and so f has a simple pole at ∞.

(For unbounded multiply connected case, reflection across polygonal sides keeps ∞ fixed. Therefore, the reflections $s_{\nu i}$ of the centers c_i are simple poles of the extended f.)

The preSchwarzian $f''(z)/f'(z)$

Extend f (to global multi-valued function) by Schwarz reflection.
The (global) preSchwarzian of f, $f''(z)/f'(z)$, is invariant under affine maps $w \longmapsto aw + b$ and, hence, single-valued, i.e.,

$$\frac{(af(z)+b)''}{(af(z)+b)'} = \frac{f''(z)}{f'(z)}.$$

Behavior near corner $f(z_{k,i})$ is represented by an $\alpha_{k,i}$ root,

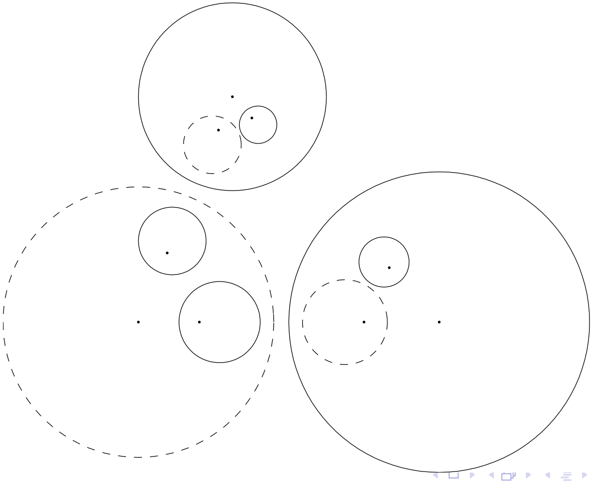
$$f(z) - f(z_{k,i}) = \left(z - z_{k,i}\right)^{\alpha_{k,i}} h_{k,i}(z)$$

where $h_{k,i}(z)$ is analytic and nonvanishing near z_k. Local expansion:

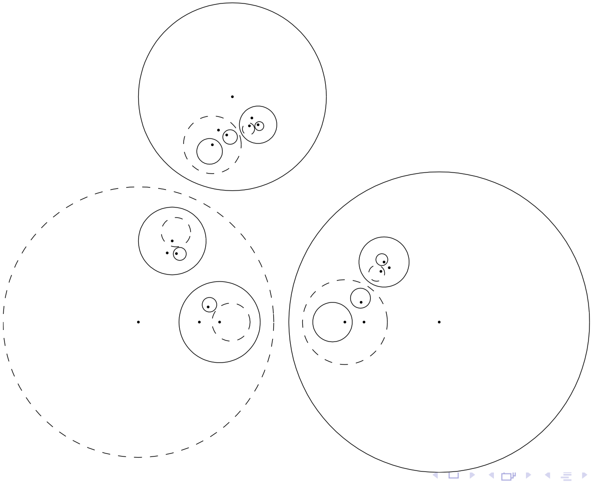
$$f''(z)/f'(z) = \beta_{k,i}/\left(z - z_{k,i}\right) + H_{k,i}(z), \ \ \beta_{k,i} = \alpha_{k,i} - 1$$

where $H_{k,i}(z)$ is analytic in a neighborhood of $z_{k,i}$,

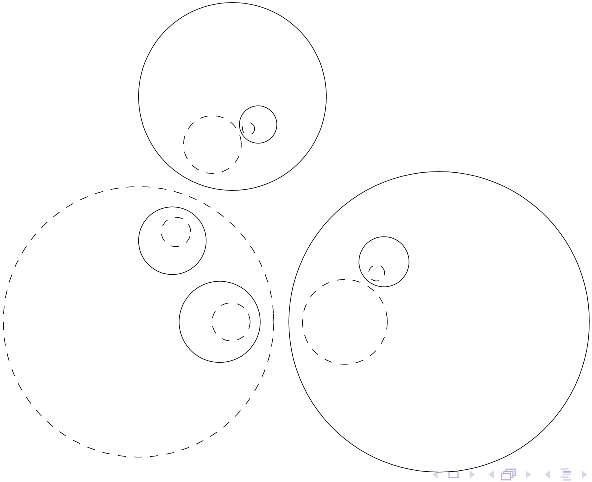
$N = 1$ levels of mutual reflections of circles and centers for unbounded case with connectivity $m = 3$.



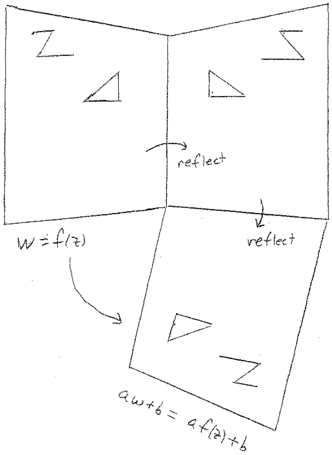
$N = 2$ levels of mutual reflections of circles and centers for unbounded case with connectivity $m = 3$.



$N = 1$ levels of mutual reflections of circles for bounded case with connectivity $m = 3$.



Reflections through sides of polygons



Singularity function $S(z)$ for unbounded mc case

Single-valued (!) and constructed by reflecting poles $\frac{\beta_{k,i}}{z-z_{k,i}}$ and $\frac{-2}{z-c_i}$ (cf: *method of images*)

$$S(z) = \sum_{j=0}^{\infty} \sum_{i=1}^{m} \sum_{\nu \in \sigma_j(i)} \left[\sum_{k=1}^{K_i} \frac{\beta_{k,i}}{z - z_{k,\nu i}} - \frac{2}{z - s_{\nu i}} \right] \quad \text{(nonconvergent form)}$$

$$= \frac{f''(z)}{f'(z)} \quad \text{(for proof see [DEP](2004).}$$

BCs for SC maps and convergence

Tangent angle $\psi(t) = \arg\{ie^{it}f'(e^{it})\}$ is constant each arc between the prevertices and hence (for disk)

$$\psi'(t) = 1 + \mathrm{Re}\left\{e^{it}\frac{f''(e^{it})}{f'(e^{it})}\right\} = 0.$$

To prove convergence of MCSC infinite product formula, show singularity functions truncated to N levels of reflection converge to $S(z)$ and $S(z) = f''(z)/f'(z)$ and satisfies BCs:

$$\mathrm{Re}\{(z - c_k)S(z)\} = -1 \quad z = c_k + r_k e^{it}.$$

(Proved for $\Delta < (m-1)^{1/4}$ for connectivity $m > 2$.)

A useful lemma (Henrici, ACCA, v.3, p.505)

To prove convergence, we need to estimate how fast the reflected circles shrink.

Lemma

$$\sum_{\nu \in \sigma_{n+1}} r_\nu^2 \leq \Delta^{4n} \sum_{i=1}^{m} r_i^2 .$$

This shows decrease in total area of circles at nth level of reflection.

Truncated $S(z)$

$$S_N(z) = \sum_{j=0}^{N} A_j(z), \quad S(z) := \lim_{N\to\infty} S_N(z)$$

where

$$\begin{aligned}
A_j(z) &:= \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} \left[\sum_{k=1}^{K_i} \frac{\beta_{k,i}}{z - z_{k,\nu i}} - \frac{2}{z - s_{\nu i}} \right] \\
&= \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} \left[\sum_{k=1}^{K_i} \left(\frac{\beta_{k,i}}{z - z_{k,\nu i}} - \frac{\beta_{k,i}}{z - s_{\nu i}} \right) \right] \\
&= \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} \left[\sum_{k=1}^{K_i} \frac{\beta_{k,i}(z_{k,\nu i} - s_{\nu i})}{(z - z_{k,\nu i})(z - s_{\nu i})} \right]
\end{aligned}$$

Convergence of $S_N(z)$

The lemmas and the Cauchy-Schwarz inequality for z bded away from $z_{k,i}$'s give

$$
\begin{aligned}
|A_j(z)| &\le \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} \sum_{k=1}^{K_i} \frac{|\beta_{k,i}||z_{k,\nu i} - s_{\nu i}|}{|z - z_{k,\nu i}||z - s_{\nu i}|} \\
&\le \frac{2}{\delta^2} \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} \sum_{k=1}^{K_i} r_{\nu i} \le \frac{2K_{\max}}{\delta^2} \sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} r_{\nu i} \\
&\le \frac{2K_{\max}}{\delta^2} \left(\sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} r_{\nu i}^2 \right)^{1/2} \left(\sum_{i=1}^{m} \sum_{\nu\in\sigma_j(i)} 1 \right)^{1/2} \\
&\le \frac{2K_{\max}}{\delta^2} \Delta^{2j} \left(\sum_{i=1}^{m} r_i^2 \right)^{1/2} \sqrt{m}(m-1)^{j/2} \le C\Delta^{2j}(m-1)^{j/2}
\end{aligned}
$$

Therefore the series converges if $\Delta^2\sqrt{m-1} < 1$.

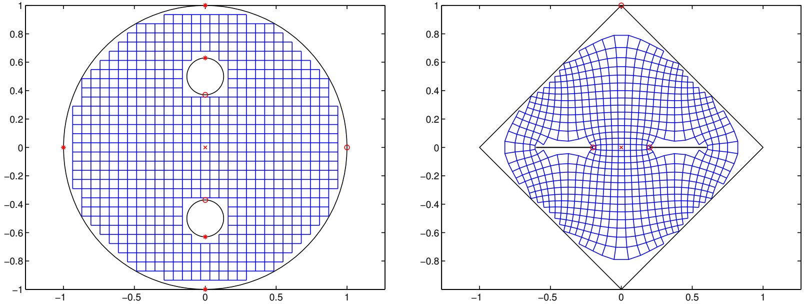
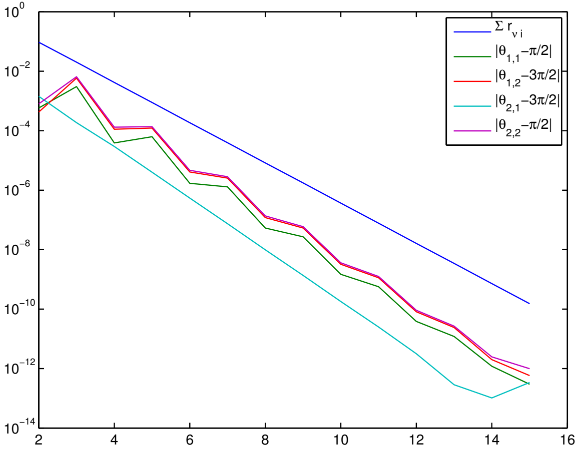


Figure: Geometry used to test numerical accuracy. For orientation note that C_1 is the lower, inner circle, and C_2 is the upper.

log of error vs. levels of reflection N



Outline

Truncated infinite products

We truncate the infinite Schwarz-Christoffel product for $f'(z)$ after N levels of reflection and denote them in the unbounded case as

$$p_u(z) = \exp\left(\int S_N(z)dz\right) = \prod_{i=1}^{m}\prod_{k=1}^{K_i}\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N}\left(\frac{z-z_{k,\nu i}}{z-s_{\nu i}}\right)^{\beta_{k,i}},$$

and in the bounded case as

$$p_b(z) = \prod_{k=1}^{K_1}(z-z_{k,1})^{\beta_{k,1}}\prod_{i=2}^{m}\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N}\left(\prod_{k=1}^{K_1}(z-z_{k,\nu i1})^{\beta_{k,1}}\prod_{k=1}^{K_i}(z-z_{k,\nu i})^{\beta_{k,i}}\right).$$

Parameter count-unbounded case

Parametrize the prevertices by $z_{k,i} = c_i + r_i e^{i\theta_{k,i}}$ for $k = 1, \ldots, K_i$ with

$$\theta_{1,i} < \theta_{2,i} < \ldots < \theta_{K_i,i}.$$

The unknown c_i's, r_i's, and $\theta_{k,i}$'s amount to a total of

$$K_1 + \cdots + K_m + 3m.$$

real parameters.

Parameter count continued

Relax normalization $f(z) = z + O(1/z), z \approx \infty$ to

$$f(z) = Az + B + O(1/z)$$

and A and B are determined implicitly by setting $c_1 = 0$, $r_1 = 1$, and $\theta_{1,1} = 0$. We then write

$$f(z) = C \int_{z_{1,1}}^{z} p_u(\zeta)\, d\zeta + D$$

with $D = w_{1,1} = f(z_{1,1})$ and define

$$C = \frac{w_{2,1} - w_{1,1}}{\int_{z_{1,1}}^{z_{2,1}} p_u(\zeta)\, d\zeta}.$$

This normalization takes care of 4 of the real parameters, leaving

$$(K_1 - 1) + K_2 + \cdots + K_m + (3m - 3) = K_1 + \cdots + K_m + 3m - 4$$

real parameters to be determined.

Nonlinear equations-unbounded case

Side-length conditions

$$|f(z_{k+1,i}) - f(z_{k,i})| = |w_{k+1,i} - w_{k,i}|$$

for $i = 1, \ldots, m$ and $k = 1, \ldots, K_i$ gives $K_1 + \ldots + K_m - 1$ equations, since C above fixes the first side-length of the first polygon.

Positions of polygons Γ_2 through Γ_m relative to Γ_1

$$f(z_{1,i}) - f(z_{1,1}) = w_{1,i} - w_{1,1}$$

for $i = 2, \ldots, m$ gives $2(m-1)$ real equations.

Orientation of polygons Γ_2 through Γ_m is determined by

$$\arg(f(z_{2,i}) - f(z_{1,i})) = \arg(w_{2,i} - w_{1,i})$$

for $i = 2, \ldots, m$ gives $(m-1)$ real equations (position and orientation of Γ_1 determined by the normalization). (These equations can be combined with the side-length conditions for $k = 1$ to obtain

$$f(z_{2,i}) - f(z_{1,i}) = w_{2,i} - w_{1,i}, \qquad i = 2, \ldots, m.)$$

Total: $K_1 + \cdots + K_m + 3m - 4$ equations, as needed

Transformation to unconstrained parameters

Key improvement leading to very robust method.

We use the unconstrained variables, $\mathrm{Re}\{c_i\}, \mathrm{Im}\{c_i\}$, and $\log r_i$.

$\sum_{k=1}^{K}(\theta_{k+1} - \theta_k) = 2\pi$, with $\theta_{K+1} := \theta_1 + 2\pi$.

Denote $\phi_k := \theta_{k+1} - \theta_k, k = 1, \dots, K$.

Unconstrained variables $\psi_{k,i} = \psi_k := \log \frac{\phi_{k+1}}{\phi_1}$ for $k = 1, \dots, K-1$.

Inverse, given θ_1, is $\theta_k = \theta_1 + 2\pi \frac{1+\sum_{j=1}^{k-2} e^{\psi_j}}{1+\sum_{j=1}^{K-1} e^{\psi_j}}$.

Unconstrained angle variables: $\psi_{1,1}, \psi_{2,1}, \dots, \psi_{K_1-1,1}$ and $\theta_{1,i}, \psi_{1,i}, \psi_{2,i}, \dots, \psi_{K_i-1,i}, i = 2, \dots, m$.

Numerical continuation

The equations above can be expressed as a non-linear system,

$$F(x) = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, where $n = K_1 + \cdots + K_m + 3m - 4$. As in [DDEP06], we use the continuation algorithm CONTUP, Program 3 from book by Allgower and Georg. We give a brief discription of this algorithm. It is assumed that F is smooth enough – that is F has enough derivatives to facilitate the required analysis. Let G be a trivial map $G : \mathbb{R}^n \to \mathbb{R}^n$ with known zeros. Let $x_0, x_1 \in \mathbb{R}^n$ be such that $G(x_0) = 0$ and $F(x_1) = 0$. Define the homotopy function

$$H(x, \lambda) = \lambda F(x) + (1 - \lambda)G(x)$$

for $\lambda \in [0, 1]$.

Some remarks on numerics

Other nonlinear solvers from MATLAB's Optimization Toolbox were also tried, but the above continuation method was best.

Some MFILES from Driscoll's SC Toolbox were used, such as a function for automatically calculating the turning angles from input polygon vertices.

Evaluation of the Schwarz-Christoffel integrals is done using Gauss-Jacobi quadrature, GAUSSJ from SC Toolbox. A fixed number of Gauss-Jacobi points (typically 30) is generally sufficient for each integral.

Integration paths may cross circles. This causes no trouble unless they come close to interior singularities.

Outline

MCSC for unbounded case [DEP04](from exterior of disks):

$$f'(z) = A \prod_{i=1}^{m} \prod_{k=1}^{K_i} \prod_{\substack{j=0 \\ \nu\in\sigma_j(i)}}^{\infty} \left(\frac{z - z_{k,\nu i}}{z - s_{\nu i}} \right)^{\beta_{k,i}}$$

$s_{\nu i}$ = refls. of ctrs, $\sum_{k=1}^{K_i} \beta_{k,i} = 2, i = 1, \ldots, m$

Crowdy's formula (from interior of C_1=unit disk):

$$f'(z) = AS_{\infty}(z) \prod_{k=1}^{K_1} \left[\omega(z, z_{k,1})\right]^{\beta_{k,1}} \prod_{i=2}^{m} \prod_{k=1}^{K_i} \left[\omega(z, z_{k,i})\right]^{\beta_{k,i}}$$

ω = Schottky-Klein prime function.

Crowdy's formula uses the **Schottky-Klein (SK) prime functions**;

$$\omega(z,a) := (z-a)\prod_{\theta_i\in\Theta''}\frac{(\theta_i(z)-a)(\theta_i(a)-z)}{(\theta_i(z)-z)(\theta_i(a)-a)},$$

where $\theta_i \in \Theta''$ involve all compositions of the "forward" maps $\theta_j = \rho_j\rho_1, j \neq 1$ giving "half" of the Schottky group Θ, and Θ'' does not include any θ_i^{-1} or the identity map, *id*.
See Chap. 12 of H. F. Baker, *Abelian Functions: Abel's Theorem and the Allied Theory including the Theory of the Theta Functions*, Cambridge U. Press (1897) reissued 1995; for characterization as BVP free of infinite product, see Crowdy and Marshall (2007). D. (2006) relates reflection and SK formulas for bounded case.

Crowdy's additional factor:

$$S_\infty(z) = \frac{S_B(z)}{\omega(z, z_\infty)^2 \omega(z, \overline{z}_\infty^{\,-1})^2},$$

where $f(z_\infty) = \infty$ and the factor arising from the bounded case is used:

$$S_B(z) := \frac{\omega_z(z, a)\omega(z, \overline{a}^{\,-1}) - \omega_z(z, \overline{\alpha}^{\,-1})\omega(z, \alpha)}{\prod_{j=2}^{m} \omega(z, \gamma_1^j)\omega(z, \gamma_2^j)}.$$

The map

$$\eta_a(z) = \frac{\omega(z, a)}{|a|\omega(z, \overline{a}^{\,-1})}, \quad \eta_a'(\gamma_1^j) = \eta_a'(\gamma_2^j) = 0.$$

taking the circle domain to a circular slit disk with $\eta(a) = 0$ is used.

Alternate representation [De] of ratios of Schottky-Klein prime functions using the full Schottky group Θ:

$$\frac{\omega(z,a)}{\omega(z,b)} = C(a,b) \prod_{\theta_i \in \Theta} \frac{z-\theta_i(a)}{z-\theta_i(b)},$$

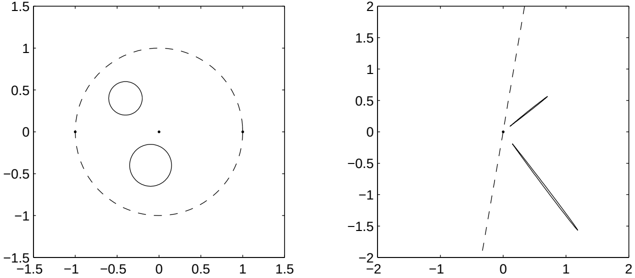
where $C(a,b)$ is a ratio of integration constants.
Therefore for $a_i, b_i \in C_i$

$$\frac{\omega(z,a_i)}{\omega(z,b_i)} = C \prod_{\nu} \frac{z-\rho_\nu(a_i)}{z-\rho_\nu(b_i)}$$

is a **slit map** and so

$$\arg \frac{\omega(z,a_i)}{\omega(z,b_i)} = \text{constant}$$

for $z \in C_j, j = 0, \ldots, m$; see [Cr].



Radial slit map: Extend f by reflection. Choose $a_i \in C_i$. $f(z_{k,i}) = 0$ implies $f(\rho_\nu(z_{k,i})) = 0$ (simple zeros). $f(a_i) = \infty$ implies $f(\rho_\nu(a_i)) = \infty$ (simple poles). f has the simple form

$$f(z) = f_{z_{k,i},a_i}(z) := C \prod_\nu \frac{z - \rho_\nu(z_{k,i})}{z - \rho_\nu(a_i)} = \frac{\omega(z, z_{k,i})}{\omega(z, a_i)}.$$

Remark on Crowdy's approach

Our derivations are based on the invariance of the preSchwarzian $f''(z)/f'(z)$ under reflections. They give "geometric" in solutions of boundary conditions guaranteeing "straight" sides by a method of images.

The Schottky-Klein (SK) prime functions in Darren Crowdy's similar formulas are the generalization of the elliptic Θ functions to finitely connected domains bounded by circles and invariance properites under the group of reflections in circles. This approach is more "algebraic" based on the fact that certain ratios of prime functions are constant on the circles.

Outline

DEP=DT≈Cr factorizations of $f'(z)$ for unbounded case

$$f'(z) = A\prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty}\left(\frac{z-z_{k,\nu i}}{z-s_{\nu i}}\right)\right]^{\beta_{k,i}}$$

$$= A\prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[f_{z_{k,i}}(z)\right]^{\beta_{k,i}} \qquad \text{finite product}$$

$$= Af_a(z)\prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[f_{z_{k,i},a_i}(z)\right]^{\beta_{k,i}} \qquad \text{product of radial slit maps}$$

$$= Af_a(z)\prod_{i=1}^{m}\prod_{k=1}^{K_i}\left(\frac{\omega(z,z_{k,i})}{\omega(z,a_i)}\right)^{\beta_{k,i}}$$

where

$$f_a(z) := \left(\prod_{i=1}^{m} f_{a_i}(z) \right)^2 , \quad \text{with} \quad f_{a_i}(z) := \prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left(\frac{z - a_{\nu i}}{z - s_{\nu i}} \right) .$$

Which factorization is “best” may be determined by numerics.

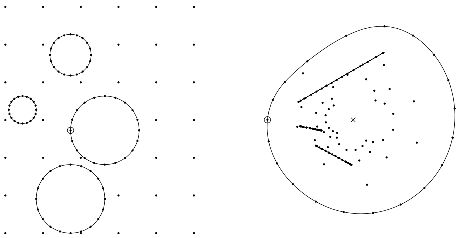


Figure: MCSC factor, $w = f_{z_{k,i}}(z)$ with $f_{z_{k,i}}(z_{k,i}) = 0$, from the exterior of four disks to the interior of a starlike curve $f(C_i)$ through 0 with three radial slits removed, satisfying BCs: $\arg f_{z_{k,i}}(z) =$const. on $C_j, j \neq i$, $\partial \arg f_{z_{k,i}}(z)/\partial\theta = -1/2, z = c_i + r_i e^{i\theta} \in C_i$.

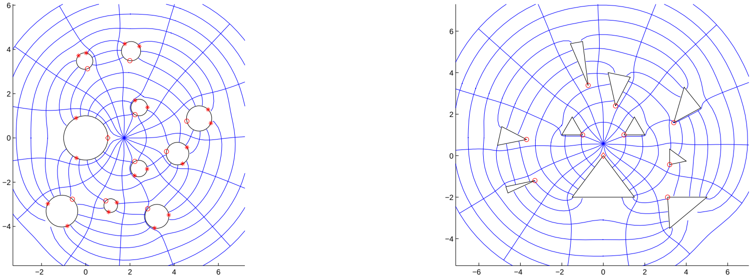


Figure: Schwarz-Christoffel map from the exterior of m=10 disks to the exterior of m=10 polygons using Laurent series centered at disks and least squares fit to BCs, above.

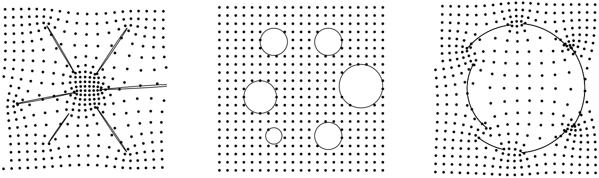
Comment

Boundary value (RH?) problems seem to form a common theoretical basis for most of the methods; see also V. Mityushev, CMFT 12 (2012), 449-463, on RH problem for $S(z)$. (Kropf tried to solve this directly.)

Outline

Maps to radial and circular slit domains

D., Driscoll, Elcrat, and Pfaltzgraff, Proc. R. Soc. A, 464 (2008), 1719–1737



Here the pre-preSchwarzian $f'(z)/f(z)$ is invariant under reflections.

$$f(z) = (z - a) \prod_{k=1}^{m} \prod_{\substack{j=0 \\ \nu \in \sigma_j(k)}}^{\infty} \frac{z - \rho_\nu(a_k)}{z - \rho_\nu(c_k)},$$

where the $\rho_\nu(a_k)$'s and $\rho_\nu(c_k)$'s are the reflections of a and ∞ across the circles.

Our boundary conditions for map to radial slits are

$$\mathrm{Re}\{(z-c_k)f'(z)/f(z)\} = 0, z \in C_k.$$

i.e., for $z = c_k + r_k e^{i\theta} \in C_k$, we have arg $f(z) =$const. Therefore,

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta} \arg f(z) \\ &= \frac{\partial}{\partial\theta} \mathrm{Im} \log f(c_k + r_k e^{i\theta}) = \mathrm{Im}\ i r_k e^{i\theta} \frac{f'}{f} = \mathrm{Re}\ r_k e^{i\theta} \frac{f'}{f}(c_k + r_k e^{i\theta}). \end{aligned}$$

Least squares for more efficient numerics

E.g. for radial slit map for more efficient numerics following Driscoll, Trefethen,..., Finn et al. (2003)...

$$f(z) = (z - a)e^{g(z)}$$

Let

$$g(z) \approx \sum_{k=1}^{m} \sum_{j=1}^{J} \frac{a_{k,j}}{(z - c_k)^j}$$

Solve linear least squares problem for $a_{k,j}$'s to satisfy BCs.

For circular slits concentric wrt origin, boundary conditions are

$$\mathrm{Im}\{(z - c_k) f'(z)/f(z)\} = 0, z \in C_k.$$

Then Ω is mapped conformally onto circular slits with $f(a) = 0$ and $f(\infty) = \infty$ if

$$f(z) = (z - a) \prod_{i=1}^{m} \prod_{\substack{j=0 \\ \nu_e, \nu_o \in \sigma_j(i)}}^{\infty} \frac{(z - \rho_{\nu_o}(a_i))(z - \rho_{\nu_e}(c_i))}{(z - \rho_{\nu_e}(a_i))(z - \rho_{\nu_o}(c_i))}$$

Outline

Theodorsen's method

See D. and Elcrat, SISC, 12 (1991), 399–422 for comparisons.
Requires that the boundary Γ be *starlike with respect to the origin*, i.e.,

$$\Gamma : \gamma(\phi) = \rho(\phi)e^{i\phi}, 0 < \rho(\phi), 0 \le \phi \le 2\pi.$$

The method finds the boundary correspondence $\phi = \phi(\theta)$ by successive conjugation such that $f(e^{i\theta}) = \rho(\phi(\theta))e^{i\phi(\theta)}$
Start with auxiliary function $h(z) := \log f(z)/z$.
Normalization $f(0) = 0$ and $f'(0) > 0$.
Note that $h(0) = \log f'(0)$ is real and $h(z)$ is analytic in $|z| < 1$ and so

$$h(e^{i\theta}) = \log \frac{\rho(\phi(\theta))e^{i\phi(\theta)}}{e^{i\theta}} = \log \rho(\phi(\theta)) + i(\phi(\theta) - \theta)$$

Theodorsen iteration

Apply conjugation operator K to the real and imaginary parts of

$$h(e^{i\theta}) = \log \rho(\phi(\theta)) + i(\phi(\theta) - \theta)$$

giving *Theordorsen's equation*

$$\phi(\theta) - \theta = K[\log \rho(\phi(\theta))].$$

The iteration

$$\phi^{(n+1)}(\theta) - \theta = K[\log \rho(\phi^{(n)}(\theta))].$$

converges linearly for nearly circular regions.

K as singular integral operator

$$Kh(\theta) = \frac{1}{2\pi} PV \int_0^{2\pi} h(\tau) \cot\left(\frac{\theta - \tau}{2}\right) d\tau,$$

Timman's (James') method for exterior domains

Here $f(z) = cz + a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots, \quad c = \tau e^{i\delta}$ for map to exterior of a general smooth boundary, parametrized by, e.g., $\sigma =$ arclength,

$$\Gamma : \gamma = \gamma(\sigma), 0 \leq \sigma \leq L.$$

The method finds the boundary correspondence $\sigma = \sigma(\theta)$ by successive conjugation.
Now auxiliary function $\qquad h(z) := \log f'(z).$
Conjugation for exterior of disk, $\mathrm{Re} h(e^{i\theta}) - \mathrm{Re} h(\infty) = K \mathrm{Im} h(e^{i\theta})$ and $f'(e^{i\theta}) = -\gamma'(\sigma(\theta))\sigma'(\theta) i e^{-i\theta}$, we have

$$\log \sigma'(\theta) + \log |\gamma'(\sigma(\theta))| - \log \tau = K \left[\arg \gamma'(\sigma(\theta)) - \theta - \pi/2\right].$$

Timman iteration (linear convergence)

With normalization $f(1) = \gamma(\sigma_0)$, we have

$$\sigma^{(n+1)}(\theta) = \tau_n \int_0^\theta exp\left(K\left[\arg\gamma'(\sigma^{(n)}(\theta)) - \theta\right]\right) / |\gamma'(\sigma^{(n)}(\theta))| d\theta$$

where $L\tau_n^{-1}$ is given by the integral over $[0, 2\pi]$.

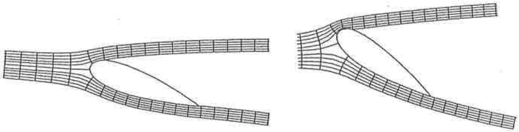


Fig. 9. H–K flow past an inclined airfoil with fixed separation points.

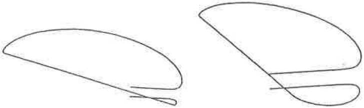


Fig. 10. R-J flow past a flat plate.

T. K. DeLillo, A. Elcrat, and C. Hu, *Computation of the Helmholtz-Kirchhoff and reentrant jet flows using Fourier series,* Applied Mathematics and Computation, 163 (2005), pp. 397–422, related to Timman's method

Outline

"Schwarz-Christoffel" (SC) for curvilinear polygons

Ives, Davis, Noble, and others; see D., JCAM, 19 (1987), 363-377. Express map derrivative as

$$f'(z) = g(z)e^{h(z)}.$$

If $h(z) = 0$ and $g(z) = \prod_k (z - z_k)^{-\beta_k}$, we have standard SC

$$f'(z) = \prod_k (z - z_k)^{-\beta_k}.$$

$h(z)$ can be used to include effects of curvature.

Timman for exterior domains with with corners

D. and Elcrat, JCP 1993, for exterior of curvilinear domains with corners at $f(z_k) = \gamma(\sigma(\theta_k))$ of angle $\alpha_k\pi$ with $\beta_k := \alpha_k - 1$ used auxiliary function

$$h(z) = i\log((f'(z)/f'(\infty))\prod_k (1 - z_k/z)^{-\beta_k}$$

Iteration with underrelaxation was used. The results only gave a couple digits accuracy for $\alpha_k\pi > \pi$ and failed to converge for $\alpha_k\pi < \pi$.

Inverse Timman with corners

D. and Elcrat, JCP 1993: Integrating the Hilbert transform by parts a la Menikoff and Zemach and others and solving for the inverse boundary correspondence $\theta = \theta(\sigma)$ worked somewhat better:
Denote $\lambda(\sigma) := \arg \gamma'(\sigma)$ = tangent angle with jumps at corners. Using auxiliary function $h(z) = \log f'(z)$ we have

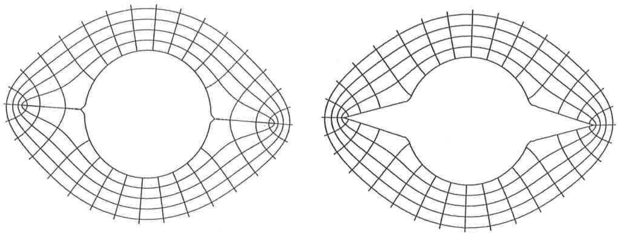
$$\begin{aligned} log \frac{d\sigma}{d\theta} &= \log \tau + K\left[\lambda(\sigma(\theta) - \theta\right] \\ &= \log \tau + \frac{1}{2\pi} PV \int \cot\left(\frac{\theta - \tilde{\theta}}{2}\right)(\lambda(\sigma(\tilde{\theta})) - \tilde{\theta}) d\tilde{\theta} \\ &= \log \tau + \frac{1}{\pi} \int \log \left|\sin \frac{\theta - \tilde{\theta}}{2}\right| d(\lambda(\sigma(\tilde{\theta})) - \tilde{\theta}) \end{aligned}$$

Inverse Timman with corners...cont.

Interchanging dependent and independent variables σ and θ and letting $\kappa(\sigma) = \lambda'(\sigma)$= curvature of smooth sections, we get and an equation for $\theta = \theta(\sigma)$

$$\begin{aligned} \frac{d\theta(\sigma)}{d\sigma} &= \frac{1}{4\tau} \prod_k \left| \sin \frac{\theta(\sigma) - \theta_k}{2} \right|^{-\beta_k} \\ &= \times \exp\left(-\frac{1}{\pi} \int_0^L \log \left| \sin \frac{\theta(\sigma) - \theta(\tilde{\sigma}}{2} \right| \kappa(\tilde{\sigma}) d\tilde{\sigma} \right) \end{aligned}$$

which (with smoothing of singularities and numerical integration) can be solved by successive approximation. σ_k at corners are now mesh points.



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5. D. Crowdy, *The Schottky-Klein prime function on the Schottky double of planar domains*, Comput. Methods Funct. Theory, 10, 2 (2010), pp. 501–517.
6. ...plus various overview papers on Darren's webpage...