**Stable Sets, Hyperbolicity and Dimension**

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**Abstract.** In this note we derive an upper bound for the Hausdorff dimension of the stable set of a hyperbolic set $\Lambda$ of a $C^2$ diffeomorphisms on an $n$-dimensional manifold. As a consequence we obtain that $\dim_H W^s(\Lambda) = n$ is equivalent to the existence of a SRB-measure. We also discuss related results in the case of expanding maps.

1. Introduction

Let $M$ be an $n$-dimensional smooth Riemannian manifold, $f : M \to M$ a diffeomorphism, and $\Lambda \subset M$ a locally maximal hyperbolic set of $f$. We define the stable set of $\Lambda$ by

$$W^s(\Lambda) = \{x \in M : \text{dist}(f^k(x), \Lambda) \to 0 \text{ for } k \to \infty\}. \tag{1}$$

In this paper we investigate the complexity of $W^s(\Lambda)$ in terms of its Hausdorff dimension $\dim_H W^s(\Lambda)$. It is a classical result of Bowen [Bo1] that there exist examples of $C^1$ horseshoes $\Lambda$ with positive Lebesgue measure, in particular $\dim_H W^s(\Lambda) = n$. On the other hand, by a result of Bowen and Ruelle [BR], if $f$ is a $C^2$-diffeomorphism and $\Lambda$ is not an attractor then $W^s(\Lambda)$ has zero Lebesgue measure. We extend the latter result by showing that the Hausdorff dimension of $W^s(\Lambda)$ is strictly smaller than $n$. More precisely, we derive an upper bound for the Hausdorff dimension of $W^s(\Lambda)$ which is given in terms of the exponential expansion rate of the tangent map defined by

$$s = \lim_{k \to \infty} \frac{1}{k} \log \left( \max \{|Df^k(x)| : x \in \Lambda\} \right), \tag{2}$$

and the topological pressure of the unstable Jacobian. If $\Lambda$ is not an attractor this bound is strictly smaller than $n$.

**Theorem 1.** Let $f : M \to M$ be a $C^2$-diffeomorphism, and $\Lambda$ a locally maximal hyperbolic set of $f$, which is not a periodic orbit, such that $f|\Lambda$ is topologically mixing. Define $\phi^u = -\log |\det Df|E^u|$. Then

$$\dim_H W^s(\Lambda) \leq n + \frac{P(\phi^u)}{s}. \tag{2}$$

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Here \( P(\phi^n) \) denotes the topological pressure of \( \phi^n \), see Section 2 for details. We note that our result holds for manifolds of arbitrary dimension, in particular, we do not require that \( f \) is conformal on \( \Lambda \). Since \( f \) is a diffeomorphism, the analogous result also holds for the unstable set \( W^u(\Lambda) \) of \( \Lambda \).

It follows from the proof of Theorem 1 that the upper bound in (2) also provides an upper bound for the upper box dimension of certain subsets of \( W^s(\Lambda) \). In particular, the following holds.

**Corollary 2.** Suppose that \( f \) and \( \Lambda \) are as in Theorem 1, and assume that \( \Lambda \) has empty interior. Then the upper box dimension \( \dim B \Lambda \) of \( \Lambda \) is strictly smaller than \( n \).

We note that the case \( \Lambda \) having non-empty interior actually occurs. For example, if \( f \) is an Anosov diffeomorphism, then the entire manifold \( M \) is a hyperbolic set. The result of Corollary 2 is known in some special cases. In particular, if \( M \) is a surface, that is, \( n = 2 \), then the classical result of McCluskey and Manning [MM] states that the Hausdorff dimension of \( \Lambda \) coincides with the box dimension of \( \Lambda \), and that its value is strictly smaller than 2. Recently, significant progress has also been made towards the estimation of the Hausdorff and box dimension of hyperbolic sets of higher dimensional manifolds, see for instance [Ba], [F], and the references in [P]. These estimates are typically given in terms of the topological pressure of functions related to the expansion/contraction rates of the tangent map on the hyperbolic splitting. However, it is often difficult to calculate these upper bounds in particular examples. Corollary 2, on the other hand, states without further calculations that the dimension of a hyperbolic set is strictly smaller than \( n \). This has not been known before in this generality.

Another consequence of Theorem 1 is a new characterization for \( f \) to have an invariant probability measure \( \mu \) supported on \( \Lambda \) whose conditional measures on the unstable manifolds are absolutely continuous with respect to the Lebesgue measure. Such a measure \( \mu \) is called a SRB measure of the diffeomorphism \( f \).

**Theorem 3.** Suppose that \( f \) and \( \Lambda \) are as in Theorem 1. Then the following are equivalent.

(i) \( \dim_H W^s(\Lambda) = n \);
(ii) \( f \) admits a SRB measure on \( \Lambda \);
(iii) \( \Lambda \) is an attractor of \( f \);
(iv) \( W^s(\Lambda) \) has positive Lebesgue measure.

The novelty of Theorem 3 is the implication (i) implies (ii), while the other implications are well-known, and are not consequences of our results.

This paper is organized as follows. In Section 2 we consider hyperbolic diffeomorphisms and derive an upper bound of the box dimension of particular sets associated with a hyperbolic set. Furthermore, we apply this bound to establish the results stated in the introduction. Finally, in Section
3 we study repellers of expanding maps, and derive related results as in the case of diffeomorphisms.

2. Stable sets for diffeomorphisms

Let \( f : M \to M \) be a \( C^2 \) diffeomorphism on an \( n \)-dimensional Riemannian manifold, and \( \Lambda \subset M \) a hyperbolic set. This means that \( \Lambda \) is a compact \( f \)-invariant set, and that there exist a continuous splitting of the tangent bundle \( T_\Lambda M = E^u \oplus E^s \), and constants \( c > 0 \) and \( \lambda \in (0, 1) \) such that for each \( x \in \Lambda \):

1. \( Df(x)(E^u_x) = E^u_{f(x)} \) and \( Df(x)(E^s_x) = E^s_{f(x)} \);
2. \( \|Df^{-k}(x)v\| \leq c \lambda^k \|v\| \) whenever \( v \in E^u_x \) and \( k \in \mathbb{N} \);
3. \( \|Df^k(x)v\| \leq c \lambda^k \|v\| \) whenever \( v \in E^s_x \) and \( k \in \mathbb{N} \).

We say that \( \Lambda \) is locally maximal if there exists an open neighborhood \( U \) of \( \Lambda \) such that \( \Lambda = \bigcap_{k \in \mathbb{Z}} f^k U \). We shall always assume that \( \Lambda \) is a locally maximal hyperbolic set, which is not a periodic orbit, and that \( f|\Lambda \) is topologically mixing.

We say that \( \Lambda \) is an attractor of \( f \) if there are arbitrarily small neighborhoods \( U \) of \( \Lambda \) such that \( f(U) \subset U \). This includes the case of Anosov diffeomorphisms for which the entire manifold \( M \) is a hyperbolic set. We say that \( \Lambda \) is a repeller of \( f \) if it is an attractor of \( f^{-1} \). We note that there are other definitions of an attractor and repeller. Here we follow the notation of [Bo2] which suits well our purposes.

Following Bowen [Bo2], for \( x \in \Lambda \), \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) we define the sets

\[
B(x, \varepsilon, k) = \{y \in M : |f^i(x) - f^i(y)| < \varepsilon, i = 0, \ldots, k - 1\},
\]

where \( |\cdot| \) denotes the distance induced by the Riemannian metric. Define

\[
B(\Lambda, \varepsilon, k) = \bigcup_{x \in \Lambda} B(x, \varepsilon, k).
\]

(4)

Furthermore, we define the local stable set of \( \Lambda \) by

\[
W^s_\varepsilon(\Lambda) = \{x \in M : \text{dist}(f^k(x), \Lambda) < \varepsilon \text{ for all } k \in \mathbb{N}\}.
\]

(5)

Analogously, we define the local unstable set \( W^u_\varepsilon(\Lambda) \) of \( \Lambda \) by considering \( f^{-1} \). It is an immediate consequence of the shadowing lemma that if \( \varepsilon \) is sufficiently small, then there exists \( \varepsilon > 0 \) such that

\[
W^s_\varepsilon(\Lambda) \subset \bigcup_{x \in \Lambda} W^u_\varepsilon(x),
\]

(6)

where \( W^u_\varepsilon(x) \) denotes the local stable manifold of size \( \varepsilon \) of \( x \in \Lambda \). Furthermore, by choice of \( \varepsilon \), the number \( \varepsilon \) can be chosen arbitrarily small. We will also need the topological pressure \( P(\varphi) \) of a continuous function \( \varphi : \Lambda \to \mathbb{R} \) (see [Wa] for the definition and details). We are interested in the particular function \( \phi^u : \Lambda \to \mathbb{R} \) defined by \( \phi^u(x) = -\log |\det Df(x)|E^u_x| \), where \( \det Df(x)|E^u_x| \) denotes the Jacobian of the linear map \( Df(x)|E^u_x| \). We will need the following result which is due to Bowen.
Proposition 4 ([Bo2]). If \( \varepsilon > 0 \) is small enough then:

(i) \( \lim_{k \to \infty} \frac{1}{k} \log \left( \text{vol}(B(\Lambda, \varepsilon, k)) \right) = P(\phi^u) \leq 0 \);

(ii) \( P(\phi^u) = 0 \) if and only if \( W^u_\varepsilon(\Lambda) \) has nonempty interior, in which case \( \Lambda \) is an attractor.

We note that the right-hand side inequality in (i) also follows as an application of the Margulis-Ruelle inequality and the variational principle. The following is the main result of this paper.

Theorem 5. Let \( f : M \to M \) be a \( C^2 \) diffeomorphism on an \( n \)-dimensional Riemannian manifold, and let \( \Lambda \subset M \) be a locally maximal hyperbolic set, which is not a periodic orbit, such that \( f|\Lambda \) is topologically mixing. Then for sufficiently small \( \varepsilon > 0 \) we have

\[
\dim_B W^s_\varepsilon(\Lambda) \leq n + \frac{P(\phi^u)}{s},
\]

(7)

where \( s \) is defined as in (1).

Remark. It follows from Proposition 4 that if \( \Lambda \) is not an attractor, then \( P(\phi^u) < 0 \), and therefore inequality (7) provides a non-trivial estimate.

Proof of Theorem 5. First observe that since the operator norm is submultiplicative, the limit defining \( s \) exists (see e.g. [Wa]). Furthermore, since \( \Lambda \) is not an attracting cycle, \( s > 0 \). If \( \Lambda \) is an attractor, then by Proposition 4, \( P(\phi^u) = 0 \), and inequality (7) trivially holds. Thus, we may assume that \( \Lambda \) is not an attractor, in which case \( P(\phi^u) < 0 \).

Let \( \delta > 0 \). It follows from a simple continuity argument that there exist \( \varepsilon > 0 \) and \( k_\delta \in \mathbb{N} \) such that for all \( x \in W^s_\varepsilon(\Lambda) \), \( |x - y| < \varepsilon \) we have

\[
\|Df^{k_\delta}(x)\| < \exp(k_\delta(s + \delta)).
\]

(8)

From now on we consider the map \( g = f^{k_\delta} \). Note that \( \Lambda \) is also a hyperbolic set of \( g \). Evidently \( W^s_\varepsilon(\Lambda) \) is forward invariant under \( g \). It follows from the variational principle that \( P_g(\phi^u_g) = k_\delta P_f(\phi^u_f) \); moreover \( s_g = k_\delta s_f \). Thus it is sufficient to prove inequality (7) for \( g \). We continue to use the notation \( s, \phi^u, P(\phi^u), \) etc. for \( g \) instead of \( f \). Let \( x \in \Lambda \) and \( k \in \mathbb{N} \). It follows from Proposition 4 that for \( \varepsilon \) sufficiently small,

\[
P(\phi^u) = \lim_{k \to \infty} \frac{1}{k} \log(\text{vol}(B(\Lambda, 2\varepsilon, k))),
\]

(9)

where \( B(\Lambda, 2\varepsilon, k) = \bigcup_{x \in \Lambda} B(x, 2\varepsilon, k) \) (see (3)). From this we obtain that if \( k \) is sufficiently large, then

\[
\text{vol}(B(\Lambda, 2\varepsilon, k)) < \exp(k(P(\phi^u) + \delta)).
\]

(10)

For all \( k \in \mathbb{N} \) we define real numbers

\[
r_k = \frac{\varepsilon}{\exp(s + \delta)^k}
\]
and neighborhoods $B_k = B(W^s_\varepsilon(\Lambda), r_k)$ of $W^s_\varepsilon(\Lambda)$. Let $y \in B_k$. Then there exists $x \in W^s_\varepsilon(\Lambda)$ with $|x - y| < r_k$. An elementary induction argument in combination with the mean-value theorem implies $|g^i(x) - g^i(y)| < \varepsilon$ for all $i \in \{0, \ldots, k-1\}$. Using (6) and making $\varepsilon$ smaller if necessary, we can assure that $x$ is contained in the local stable manifold of size $\varepsilon$ of a point in $\Lambda$. It follows that $y \in B(\Lambda, 2\varepsilon, k)$. Hence $B_k \subset B(\Lambda, 2\varepsilon, k)$. Therefore, (10) implies that

$$\text{vol}(B_k) < \exp(k(\phi^u + \delta))$$

for sufficiently large $k$. Let us recall that for $t \in [0, n]$ the $t$-dimensional upper Minkowski content of a relatively compact set $A \subset M$ is defined by

$$M^s_t(A) = \limsup_{\rho \to 0} \frac{\text{vol}(A_\rho)}{(2\rho)^{n-t}},$$

where $A_\rho = \{ p \in M : \exists q \in A : |p - q| \leq \rho \}$, and vol denotes the volume induced by the Riemannian metric on $M$. Let $t \in [0, n]$ and $\rho_k = \frac{\varepsilon}{2^k}$ for all $k \in \mathbb{N}$. Then we have

$$M^s_t(W^s_\varepsilon(\Lambda)) = \limsup_{\rho \to 0} \frac{\text{vol}(W^s_\varepsilon(\Lambda)_\rho)}{(2\rho)^{n-t}} \leq \limsup_{k \to \infty} \frac{\text{vol}(W^s_\varepsilon(\Lambda)_{\rho_k})}{(2\rho_k)^{n-t}} \leq \limsup_{k \to \infty} \frac{\text{vol}(B_k)}{(r_{k+1})^{n-t}} \leq \frac{\exp(s + \delta)^{n-t}}{\varepsilon^{n-t}} \lim_{k \to \infty} \left( \exp(s + \delta)^{n-t} \exp(\phi^u + \delta) \right)^k.\ (12)$$

Let $t > n + \frac{P(\phi^u) + \delta}{s + \delta}$. Then $\exp(s + \delta)^{n-t} \exp(\phi^u + \delta) < 1$. This implies $M^s_t(W^s_\varepsilon(\Lambda)) = 0$, in particular, $t \geq \dim_B W^s_\varepsilon(\Lambda)$. Since $\delta$ can be chosen arbitrarily small, the result follows. \qed

Remark. We note that the idea of estimating the dimension of an invariant set by calculating the volume of neighborhoods of the set was introduced by the second author of this paper in [Wo] for estimating the dimension of an invariant set of a $C^1$ diffeomorphism. For a related result in the case of polynomial automorphisms of $\mathbb{C}^n$ see [SW].

Applying Theorem 1 to $f^{-1}$ we obtain the analogous result for the local unstable set

$$W^u_\varepsilon(\Lambda) = \{ x \in M : \text{dist}(f^{-k}(x), \Lambda) < \varepsilon \text{ for all } k \in \mathbb{N} \}.$$

**Theorem 5a.** Let $f : M \to M$ be a $C^2$ diffeomorphism on an $n$-dimensional Riemannian manifold, $\Lambda \subset M$ a locally maximal hyperbolic set, which is not a periodic orbit, such that $f|\Lambda$ is topologically mixing. Let $\phi^s = \log |\det Df|E^s|$. Then for sufficiently small $\varepsilon > 0$ we have

$$\dim_B W^u_\varepsilon(\Lambda) \leq n + \frac{P(\phi^s)}{s}.\ (13)$$
where \( s \) is defined as in (1) for the map \( f^{-1} \).

Note that if \( M \) is a surface, then the dimension of \( W^s_{ε}(Λ) \) can be expressed using Bowen’s formula. Namely,

\[
t^{u/s} := \dim_H W^{u/s}(x) \cap Λ = \overline{\dim}_B W^{u/s}(x) \cap Λ
\]

are independent of \( x \in Λ \) (see [MM]), and it is not too hard to see that

\[
\dim_H W^{s/u}(Λ) = \overline{\dim}_B W^{s/u}(Λ) = t^{u/s} + 1.
\]  \( (14) \)

The right equality in (14) can be shown using the fact that when \( n = 2 \) the holonomies are lipschitz continuous (see for instance [KH]). On the other hand, the following example shows that the dimension of \( W^s_{ε}(Λ) \) can be arbitrarily close to \( n \).

**Example 1.** Let \( B \subset \mathbb{R}^2 \) be a unit square, and let \( f : B \to \mathbb{R}^2 \) be a linear horseshoe map with the expansion rate \( λ^u > 2 \) and the contraction rate \( λ^s < \frac{1}{2} \), see Figure 1. It is well-known that \( Λ = \{ x \in B : f^k(x) \in B \text{ for all } k \in \mathbb{Z} \} \) is a locally maximal hyperbolic set of \( f \), and that \( f|Λ \) is topologically mixing. Moreover, we have

\[
t^u = \frac{\log 2}{\log λ^u} \quad \text{and} \quad t^s = -\frac{\log 2}{\log λ^s}.
\]

![Figure 1. A Linear Horseshoe Map](image)

As it was mentioned above, we also have \( \dim_H W^{s/u}_{ε}(Λ) = \overline{\dim}_B W^{s/u}_{ε}(Λ) = t^{u/s} + 1 \). Therefore, by choosing \( λ^u \) close to 2 (respectively \( λ^s \) close to \( \frac{1}{2} \)) we obtain the following.

**Corollary 6.** For each \( ε > 0 \) there exists a linear horseshoe map of \( \mathbb{R}^2 \) such that \( \dim_H W^{s/u}_{ε}(Λ) = \overline{\dim}_B W^{s/u}_{ε}(Λ) > 2 - ε \).
We now provide the proofs of the results stated in the introduction.

**Proof of Theorem 1.** It follows from Theorem 5 that $\dim H W^s(\Lambda) \leq n + \frac{P(\phi^u)}{s}$. Obviously, $W^s(\Lambda) \supset W^s(x)$ for all $x \in \Lambda$, hence

$$\dim H \bigcup_{x \in \Lambda} W^s(x) \leq n + \frac{P(\phi^u)}{s}. \quad (15)$$

It is a consequence of the shadowing lemma that $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$. \hspace{1cm} (16)

On the other hand, we have $W^s(x) = \bigcup_{k \in \mathbb{N}} f^{-k}(W^s(f^k(x)))$. Together we obtain

$$W^s(\Lambda) = \bigcup_{k \in \mathbb{N}} f^{-k} \left( \bigcup_{x \in \Lambda} W^s(x) \right). \quad (17)$$

The theorem follows now from equations (15), (17) and the fact that the Hausdorff dimension is stable with respect to countable unions. \hspace{1cm} □

**Remark.** Since we have taken countable unions of sets whose box dimensions are uniformly bounded above by $n + \frac{P(\phi^u)}{s}$, we obtain the same upper bound for the packing dimension of $W^s(\Lambda)$.

**Proof of Corollary 2.** If $\Lambda$ is an attractor, then $\Lambda$ cannot be a repeller. Otherwise, we would have $W^u(\Lambda) \supset W^u(x)$ for all $x \in \Lambda$, in which case, since $\Lambda$ has a local product structure, $\Lambda$ would have non-empty interior. Therefore, Proposition 4 (or the analogous proposition for $f^{-1}$) implies that either $P(\phi^u) < 0$ or $P(\phi^s) < 0$. The result now follows from Theorems 5 and 5a. \hspace{1cm} □

**Proof of Theorem 3.** (i)⇒(ii) Assume $\dim H W^s(\Lambda) = n$. Then it follows from Theorem 1 and Proposition 4 (i) that $P(\phi^u) = 0$. Since $\Lambda$ is a locally maximal hyperbolic set on which $f$ is topologically mixing, there exists a unique equilibrium measure $\mu_{\phi^u}$ of the potential $\phi^u$. This means that

$$0 = P(\phi^u) = h_{\mu_{\phi^u}}(f) + \int \phi^u d\mu_{\phi^u}, \quad (18)$$

where $h_{\mu_{\phi^u}}(f)$ denotes the measure-theoretic entropy of $f$ with respect to $\mu_{\phi^u}$. Moreover, $\mu_{\phi^u}$ is ergodic. Let $\lambda_1(x) < \cdots < \lambda_l(x)$ be the Lyapunov exponents of $x$ with respect to $f$ with multiplicities $m_1(x), \cdots, m_l(x)$. The fact that $\mu_{\phi^u}$ is ergodic implies that the Lyapunov exponents and the multiplicities are constant $\mu_{\phi^u}$-almost everywhere. We denote the corresponding values by $\lambda_i(\mu_{\phi^u})$ and $m_i(\mu_{\phi^u})$. It follows from the fact that $\Lambda$ is a hyperbolic set that

$$-\int \phi^u d\mu_{\phi^u} = \sum_{\lambda_i(\mu_{\phi^u}) > 0} \lambda_i(\mu_{\phi^u}) m_i(\mu_{\phi^u}). \quad (19)$$
Therefore, equations (18) and (19) imply that
\[ h_{\mu_{\phi^u}}(f) = \sum_{\lambda_i(\mu_{\phi^u}) > 0} \lambda_i(\mu_{\phi^u}) m_i(\mu_{\phi^u}) = \int \sum_{\lambda_i(x) > 0} \lambda_i(x) m_i(x) \, d\mu_{\phi^u}. \] (20)

Hence \( \mu_{\phi^u} \) satisfies Pesin’s entropy formula. Finally, from the work of Ledrappier, Strelcyn and Young in [LS] and [LY] we conclude that \( \mu_{\phi^u} \) is a SRB measure.

(ii)\( \Rightarrow \) (iii) follows from [Bo2] and [LY]. Finally, (iii)\( \Rightarrow \) (iv) and (iv)\( \Rightarrow \) (i) are trivial.

\[ \Box \]

3. Expanding maps

In this section we consider a \( C^2 \) map \( f \) on an \( n \)-dimensional smooth Riemannian manifold \( M \) to itself. The map \( f \) is not assumed to be invertible. Let \( \Lambda \subset M \) be a compact invariant set of \( f \). We say that \( f \) is expanding on \( \Lambda \) if there exist \( c > 0 \) and \( \lambda \in (1, \infty) \) such that for each \( x \in \Lambda \):

\[ \|Df^k(x)v\| \geq c\lambda^k\|v\| \quad \text{whenever} \quad v \in T_xM \text{ and } k \in \mathbb{N}. \]

Furthermore, we say that \( \Lambda \) is locally maximal if there exists an open neighborhood \( U \subset M \) of \( \Lambda \) such that \( \Lambda = \bigcap_{k \in \mathbb{N}} f^{-k}(U) \). If \( f \) is expanding on a locally maximal set \( \Lambda \) we say that \( \Lambda \) is a repeller of \( f \). We shall always assume that \( f \) is expanding on \( \Lambda \), \( \Lambda \) is locally maximal, and \( f|\Lambda \) is topologically mixing. We define the function \( \phi: \Lambda \to \mathbb{R} \) by

\[ \phi(x) = -\log |\operatorname{det} Df(x)|. \]

We start by proving the version of Proposition 4 for expanding maps.

**Proposition 7.** If \( \varepsilon > 0 \) is small enough then

\[ \lim_{k \to \infty} \frac{1}{k} \log (\text{vol}(B(\Lambda, \varepsilon, k))) \leq P(\phi) \leq 0 \] (21)

**Proof.** We first show the right-hand side inequality of (21). Consider an ergodic invariant probability measure \( \mu \) supported on \( \Lambda \), and let \( 0 < \lambda_1(\mu) < \cdots < \lambda_l(\mu) \) be the Lyapunov exponents of \( \mu \). Denote by \( m_i(\mu) \) the multiplicity of the Lyapunov exponent \( \lambda_i(\mu) \). It follows from the Margulis-Ruelle inequality that

\[ h_{\mu}(f) \leq \sum_i \lambda_i(\mu) m_i(\mu) = -\int \phi \, d\mu. \] (22)

On the other hand, the variational principle states that

\[ P(\phi) = \sup_{\mu} \left( h_{\mu}(f) + \int \phi \, d\mu \right), \] (23)

where the supremum is taken over all (ergodic) invariant probability measures on \( \Lambda \). Combining (22) and (23) yields \( P(\phi) \leq 0 \).

We now show the left-hand side inequality in (21). Fix some \( \delta \leq \varepsilon \). We say that a set \( E \subset M \) is \((k, \delta)\)-separated if for all \( y, z \in E \), \( y \neq z \), there exists \( i \in \{0, \ldots, k-1\} \) with \( |f^i(y) - f^i(z)| \geq \delta \). Let \( E_k(\delta) \) be a maximal \((k, \delta)\)-separated subset of \( \Lambda \). Let \( x \in \Lambda \); then \( x \in B(y, \delta, k) \) for some \( y \in E_k(\delta) \),
because otherwise $E_k(\delta) \cup \{x\}$ would be $(k, \delta)$-separated. Here $B(y, \delta, k)$ is defined as in (3). We conclude that $B(x, \varepsilon, k) \subset B(y, \delta + \varepsilon, k)$, and

$$B(\Lambda, \varepsilon, k) \subset \bigcup_{y \in E_k(\delta)} B(y, \delta + \varepsilon, k), \quad (24)$$

see (4) for the definition. For $x \in \Lambda$ and $k \in \mathbb{N}$ we define

$$S_k \phi(x) = \sum_{i=0}^{k-1} \phi(f^i(x)).$$

Analogously as in the case of diffeomorphisms (see [Bo2], [BR]), there exists $C_{\delta+\varepsilon} > 1$ such that if $\varepsilon$ is small enough then

$$\text{vol}(B(x, \delta + \varepsilon, k)) \leq C_{\delta+\varepsilon} \cdot \exp S_k \phi(x) \quad (25)$$

for all $x \in \Lambda$ and $k \in \mathbb{N}$. Therefore (24) and (25) imply that

$$\text{vol}(B(\Lambda, \varepsilon, k)) \leq C_{\delta+\varepsilon} \cdot \sum_{y \in E_k(\delta)} \exp S_k \phi(y). \quad (26)$$

The map $f|\Lambda$ is expansive; therefore for $\delta$ small enough we have

$$P(\phi) = \limsup_{k \to \infty} \frac{1}{k} \log \left( \sum_{y \in E_k(\delta)} \exp(S_k \phi(y)) \right). \quad (27)$$

The result follows now by taking the logarithm, dividing by $k$ and taking the upper limit in (26) and applying (27). □

We would like to point out that the proof of Proposition 7 is based on ideas of the corresponding proof of Bowen [Bo2] for diffeomorphisms.

Quan and Zhu classified in [QZ] the invariant measures $\mu$ of a $C^2$ endomorphism $f$ for which Pesin’s entropy formula holds; they showed that $\mu$ has this property if and only if $\mu$ satisfies the SRB property. This property is a generalized condition of an SRB measure for diffeomorphisms defined for the corresponding measure on the inverse limit map, see [QZ] for details. In particular, if $f$ is an expanding map and $\mu$ is absolutely continuous with respect to Lebesgue, then Pesin’s entropy formula holds, see [H].

**Proposition 8.** $P(\phi) = 0$ if and only if $f$ admits an invariant measure $\mu$ satisfying the SRB-property. In particular, this is the case when $f$ has an invariant measure which is absolutely continuous with respect to Lebesgue.

**Proof.** If $P(\phi) = 0$ then the same arguments as in the proof of Theorem 3 imply that there exists an invariant measure $\mu$ satisfying Pesin’s formula; hence, $\mu$ has the SRB property. On the other hand, if $\mu$ has the SRB property then $\mu$ satisfies Pesin’s entropy formula. It now follows from the variational principle and a similar argument as in the proof of Theorem 3 that $P(\phi) = 0$. □
Remark. It is easy to see that $P(\phi) = 0$ actually occurs. For example, if $f : S^1 \to S^1, f(z) = z^2$, then the entire manifold $S^1$ is a repeller. In this case the measure of maximal entropy (given by the distribution of the periodic points of $f$) is absolutely continuous with respect to Lebesgue.

We now present our main results for expanding maps.

**Theorem 9.** Let $f$ be a $C^2$ self map on an $n$-dimensional smooth Riemannian manifold $M$, and let $\Lambda$ be a locally maximal repeller of $f$ such that $f|\Lambda$ is topologically mixing. Then

$$\text{dim}_B \Lambda \leq n + \frac{P(\phi)}{s},$$

where $s$ is defined as in (1).

**Proof.** The proof of the theorem is analogous to the proof of Theorem 5 just by replacing $W^s_\varepsilon(\Lambda)$ by $\Lambda$ and applying Proposition 7 instead of Proposition 4. \hfill \Box

**Theorem 10.** Let $f$ be a $C^2$ self map of an $n$-dimensional smooth Riemannian manifold $M$, and let $\Lambda$ be a repeller of $f$ such that $f|\Lambda$ is topologically mixing. Then the following are equivalent.

(i) $\text{dim}_B \Lambda = n$;

(ii) $f$ admits an invariant measure $\mu$ satisfying the SRB property.

**Proof.** If $\text{dim}_B \Lambda = n$ then, by Theorem 9 and Proposition 7, $P(\phi) = 0$. Therefore, Proposition 8 implies that $f$ admits an invariant measure $\mu$ satisfying the SRB property. On the other hand, if $\mu$ is an invariant measure satisfying the SRB property, then Pesin’s entropy formula holds for $\mu$, see [QZ]. It now follows from Theorem B of [H] that $\text{vol}(\Lambda) > 0$, in particular, $\text{dim}_B \Lambda = n$. \hfill \Box

**References**


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