Fractal Julia sets of polynomial Diffeomorphisms of $\mathbb{C}^2$

Christian Wolf

Abstract

We summarize the main results of the authors thesis [W] about the fractal structure of Julia sets of polynomial automorphisms of $\mathbb{C}^2$.

We consider polynomial automorphisms $g$, which are finite compositions of so called generalized Hénon mappings. Each generalized Hénon mapping $g_j$ has the form

$$g_j : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \quad (z, w) \mapsto (w, p_j(w) - a_j)$$

where $p_j$ is a monic polynomial with degree at least 2 and $a_j$ is a non-zero complex number. Friedland and Milnor [FM] proved that every polynomial automorphisms of $\mathbb{C}^2$ with non trivial dynamics (in the sense of possessing positive topological entropy) is affine conjugate to a finite composition of generalized Hénon mappings. Thus we consider the space of all polynomial automorphisms of $\mathbb{C}^2$ with complicated dynamics. Let $g$ be a finite composition of generalized Hénon mappings. Following [H] we define $K^+$ and $K^-$ as the set of points with bounded forward respectively backward orbits and $J^\pm := \partial K^\pm$, $K := K^+ \cap K^-$ and $J := J^+ \cap J^-$. The sets $K^\pm$ and $J^\pm$ are connected by [BS1]. According to [FM] $K$, and $J$ are $g$-invariant compact sets. The set $J$ is called the Julia set of $g$. We are interested in the fractal properties of the Julia set $J$ especially in the case when $g$ is hyperbolic, i.e. the Julia set $J$ is a hyperbolic set for $g$. It follows from a Theorem of Bedford and Smillie [BS1] that under that condition $g$ is an Axiom A diffeomorphism. First we give a definition of fractals.

**Definition 1** A set $A \subset \mathbb{R}^n$ is called fractal, if the Hausdorff dimension of $A$ exceeds the topological dimension of $A$, in notation: $\dim_{top} A < \dim_H A$.

There are other possibilities for the definition of fractals. Here our considerations are restricted to the analysis of the Hausdorff dimension and the topological dimension. We have the following theorem:

**Theorem 2** Let $g$ be a hyperbolic finite composition of generalized Hénon mappings. Assume that $\dim_{top} J^+ = 2$ or $\dim_{top} J^- = 2$ or both. Then the Julia set $J$ is fractal.
Theorem 2 follows easily from two other theorems, each of which is interesting by itself. We first discuss the assumptions of Theorem 2 and then present the theorems inducing Theorem 2.

It is easy to see that $\dim_{\text{top}} J^z \in \{2, 3\}$. Therefore Theorem 2 implies fractality of $J$ for all cases except $\dim_{\text{top}} J^+ = \dim_{\text{top}} J^- = 3$. It is not known whether this case can occur even without the assumption of hyperbolicity.

We will use the standard sets $V, V^+$ and $V^-$ for a finite composition of generalized Hénon mappings as in [BS1], chapter 2.

Next we give an upper bound of the Hausdorff dimension of $J^-$ in the volume decreasing case.

**Theorem 3** Let $g$ be a finite composition of generalized Hénon mappings with complex Jacobian determinant $a \in \mathbb{C}, 0 < |a| < 1$. Define

$$s := \lim_{n \to \infty} \frac{1}{n} \log^+ \left( \max\{| |Dg^{-n}(p)|| : p \in J^- \cap V \} \right).$$

Then

$$\dim_H J^- \leq 4 - \frac{2 \log |a^{-1}|}{s} < 4$$

Observe that Theorem 3 does not require hyperbolicity. The upper bound of the Hausdorff dimension of $J^-$ has the advantage that estimations for this bound can be derived.

Example:

Let us consider the family of generalized Hénon mappings

$$g_{a,c}(z, w) = (w, w^2 + c - az)$$

where $(a, c) \in D(0, 1) \setminus \{0\} \times \mathbb{C}$, then

$$\dim_H J^-_{a,c} \leq 4 - \frac{2 \log |a^{-1}|}{\log \left( |a^{-1}|r_{a,c} + \sqrt{|a^{-2}|r_{a,c}^2 + |a^{-1}|} \right)}$$

where $r_{a,c} := \frac{|a| + \sqrt{(|a| + 1)^2 + 4|c|}}{2}$

We get the following Corollary to Theorem 3.

**Corollary 4** Let $g$ be a finite composition of generalized Hénon mappings with complex Jacobian determinant $a \in \mathbb{C}, 0 < |a| < 1$ and let $s$ be as in Theorem 3, then

i) If $s < 2 \log |a^{-1}|$, then $2 = \dim_{\text{top}} J^- < \dim_H J^- < 3$

ii) Let $(p_c)_{c \in C}$ be a family of monic polynomials of degree $d \geq 2$ and $C$ compact. For $(a, c) \in \mathbb{C} \setminus \{0\} \times C$ define $g_{a,c} = (w, p_c(w) - az)$. Then

$$\lim_{t \downarrow 0} \sup \{ \dim_H J^-_{a,c} : |a| \leq t, c \in C \} = 2$$
Remarks:
The statement that the Hausdorff dimension of $J^-$ exceeds 2 in part i) of Corollary 4 is due to Fornæss and Sibony [FS]. Part ii) remains true if one considers a family of mappings $g_c = g_{c_n} \circ \ldots \circ g_{c_1}$.
Part i) of Corollary 4 provides a set of parameters of volume decreasing finite composition of generalized Hénon mappings with $\dim_{top} J^- = 2$. Thus all hyperbolic mappings belonging to this set have fractal Julia sets by Theorem 2.
We now present the results required for the proof of Theorem 2.
The next theorem shows that in the hyperbolic situation the slice of $J^+$ and $J^-$ is nice in sense of topological dimension.

**Theorem 5** Let $g$ be a hyperbolic finite composition of generalized Hénon mappings. Then

$$\dim_{top} J = \dim_{top} J^+ + \dim_{top} J^- - 4$$

Theorem 5 shows that under the assumptions of Theorem 2 it is sufficient to consider the cases: $\dim_{top} J \in \{0, 1\}$. It is wellknown that $\dim_H J > 0$, see [BS3]. Hence Theorem 2 follows, if $\dim_{top} J = 0$.
The next result gives lower bounds for the Hausdorff dimension of $J$.

**Theorem 6** Let $g$ be a hyperbolic finite composition of generalized Hénon mappings with degree $d \geq 2$. Define

$$s^+ := \lim_{n \to \infty} \frac{1}{n} \log^+ (\max \{||Dg^n(p)|| : p \in J\})$$

$$s^- := \lim_{n \to \infty} \frac{1}{n} \log^+ (\max \{||Dg^{-n}(p)|| : p \in J\})$$

Then we have

i) $$\dim_H J \geq \frac{\log d}{s^+} + \frac{\log d}{s^-} > 0$$

ii) If $\dim_{top} J \geq 1$ then there exists $s \in \{s^+, s^-\}$ such that

$$\dim_H J \geq 1 + \frac{\log d}{s} > 1$$

Remarks:
These lower bounds for the Hausdorff dimension of $J$ guarantee the fractality of $J$ in the case $\dim_{top} J \in \{0, 1\}$ and thus Theorem 2 follows from Theorem 5 and 6. The lower bound for $\dim_H J$ in Theorem 6 i) is quite similar to the lower bound for the Hausdorff dimension of the equilibrium measure $\mu_K$ in [BS3].
Examples can be constructed where the lower bound in i) is trivial, i.e lower than the topological dimension of $J$.
Until now we do not know if the topological dimension of $J$ can exceed 0. This is a consequence of the following Theorem.
Theorem 7 Let \( g \) be a finite composition of generalized Hénon mappings with an attracting periodic point. Then \( \dim_{\text{top}} I \geq 1 \).

There are examples of hyperbolic finite composition of generalized Hénon mappings with attracting periodic points, thus \( ii \) of Theorem 6 really occurs. Together with Theorem 5 and 6 we get the following Corollary to Theorem 2.

Corollary 8 Let \( d = (d_1, \ldots, d_l), d_j \geq 2 \) and let \( \mathcal{H}_d \) be the space consisting of all hyperbolic finite compositions of generalized Hénon mappings \( g = g_n \circ \ldots \circ g_1 \) with \( \deg(g_j) = d_j \). Let \( C \) be a connected component of \( \mathcal{H}_d \). Assume there exists \( g_c \in C \) with \( \dim_{\text{top}} J_c^+ = 2 \) or \( \dim_{\text{top}} J_c^- = 2 \) or both, then the Julia set of \( g \) is fractal for all \( g \in C \). This property is particularly satisfied if one of the following conditions holds.

\( i \) \( \mathcal{C} \) contains a 1-dimensional degenerate mapping in the sense of \( [BS3] \), chapter 6.

\( i i \) There exists \( g \in C \) with \( s < 2 \log |a^{-1}| \), where \( a \) is the complex Jacobian determinant of \( g \) with \( |a| < 1 \) and \( s \) is defined like in Theorem 3.

The last result is independent to the preceding Theorems. It gives an upper bound for the Hausdorff dimension of hyperbolic Julia sets.

Theorem 9 Let \( g \) be a finite composition of generalized Hénon mappings with complex Jacobian determinant \( a \in \mathbb{C}, 0 < |a| \leq 1 \) and degree \( d \geq 2 \). Let \( E^s \) and \( E^u \) be the stable and the unstable subbundle of the hyperbolic splitting of \( T_x \mathbb{C}^2 \). Define

\[
s := \lim_{n \to \infty} \frac{1}{n} \log \left( \min \{ \| Dg^n(p)_{E^u} \| : p \in J \} \right) > 0
\]

and \( t := \min \{ 2s, s - \log |a| \} \), then

\[
\dim_H J \leq \frac{2(t - s + \log d)}{t}
\]

We note that Theorem 9 can be generalized to any hyperbolic set of a holomorphic diffeomorphism of \( \mathbb{C}^n \).

References


