Chapter 8

Two-way ANOVA: Mixed factor effects Equal sample sizes

8.1 Assumptions and model

- There are two factors, factors A and B, that are of interest.
- Factor A is studied at a fixed levels, and factor B at b random levels.
  All ab factor level combinations are included in the study.

Table 8.1: Format of (balanced) data set

		Factor B				
		level 1	...	level j	...	level b
	level 1	$Y_{111}, \ldots, Y_{11n}$		$Y_{1j1}, \ldots, Y_{1jn}$		$Y_{1b1}, \ldots, Y_{1bn}$
	$\vdots$					
Factor A	level i	$Y_{i11}, \ldots, Y_{i1n}$		$Y_{ij1}, \ldots, Y_{ijn}$		$Y_{ib1}, \ldots, Y_{ibn}$
	$\vdots$					
	level a	$Y_{a11}, \ldots, Y_{a1n}$		$Y_{aj1}, \ldots, Y_{ajn}$		$Y_{ab1}, \ldots, Y_{abn}$

8.1.1 Restricted mixed factor effects model

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}, \qquad i = 1, \dots, a, \quad j = 1, \dots, b, \quad k = 1, \dots, n.$$

$\mu_{\cdot\cdot}$ is a constant

$\alpha_1, \dots, \alpha_a$ are constants subject to $\sum_{i=1}^{a} \alpha_i = 0$

$\beta_1, \dots, \beta_b \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \sigma_\beta^2)$

$(\alpha\beta)_{ij},\ i = 1, \dots, a,\ j = 1, \dots, b, \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \frac{a-1}{a}\sigma_{\alpha\beta}^2)$ subject to the restrictions:

$$\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0, \qquad \text{for all } j$$

$$cov((\alpha\beta)_{ij}, (\alpha\beta)_{i'j}) = -\frac{1}{a}\sigma_{\alpha\beta}^2, \qquad i \neq i'$$

Error terms $\varepsilon_{ijk}, \quad i = 1, \dots, a,\ j = 1, \dots, b,\ k = 1, \dots, n, \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \sigma^2)$

$\beta_j, (\alpha\beta)_{ij}$ and $\varepsilon_{ijk}, \quad i = 1, \dots, a,\ j = 1, \dots, b,\ k = 1, \dots, n,$ are independent

- Features of model

1° The response variable Y_{ijk} is normally distributed with

$$\begin{aligned} EY_{ijk} &= \mu_{\cdot\cdot} + \alpha_i \\ Var(Y_{ijk}) &= \sigma_\beta^2 + \frac{a-1}{a}\sigma_{\alpha\beta}^2 + \sigma^2 \end{aligned}$$

2° (Covariance structure of observations) Unlike for the fixed factor levels model where all observations Y_{ijk} are independent, the Y_{ijk} for the random factor effects model are correlated when they are from the same random factor B level.

$$\begin{aligned} cov(Y_{ijk}, Y_{ijk'}) &= \sigma_\beta^2 + \tfrac{a-1}{a}\sigma_{\alpha\beta}^2, && k \neq k' \\ cov(Y_{ijk}, Y_{i'jk'}) &= \sigma_\beta^2 - \tfrac{1}{a}\sigma_{\alpha\beta}^2, && i \neq i' \quad \text{(Same levels of factor B)} \\ cov(Y_{ijk}, Y_{i'j'k'}) &= 0, && j \neq j' \quad \text{(Different levels of B)} \end{aligned}$$

8.1.2 Notations

Means, and Sums of squares are defined the same as the fixed factor levels case.

8.1.3 Questions of interest

1° Variance components $\sigma^2_\beta, \sigma^2_{\alpha\beta}$

Point estimation;

Interval estimation;

Test whether $\sigma^2_\beta = 0, \sigma^2_{\alpha\beta} = 0$.

2° About α.

Point estimation;

Interval estimation;

Test whether $\alpha_1 = \ldots = \alpha_a = 0$.

3° About $\mu_{\cdot\cdot}$.

Point estimation;

Interval estimation.

8.2 Inference on variance components

8.2.1 Tests

Source of Variation	df	Sum of Squares (SS)	Mean Square (MS)	F^*-value	Test for
Factor A	$a-1$	SSA	MSA=$\frac{SSA}{a-1}$	$F^* = \frac{MSA}{MSAB}$	$\mathcal{H}_0: \ \alpha_1 = \ldots = \alpha_a = 0$ (factor A main effects)
Factor B	$b-1$	SSB	MSB=$\frac{SSB}{b-1}$	$F^* = \frac{MSB}{MSE}$	$\mathcal{H}_0: \ \sigma^2_\beta = 0$ (factor B main effects)
AB interactions	$(a-1)(b-1)$	SSAB	MSAB=$\frac{SSAB}{(a-1)(b-1)}$	$F^* = \frac{MSAB}{MSE}$	$\mathcal{H}_0: \sigma^2_{\alpha\beta} = 0$ (interactions)
Error(Residuals)	$ab(n-1)$	SSE	MSE=$\frac{SSE}{ab(n-1)}$		
Total	$nab-1$	SSTO			

Question: Why the denominators of F statistics for testing factor A and B main effects of the mixed factor effects model are not the same?

Imitation pearls, p.1006
Factor A (number of coats) has 3 fixed factor levels, and factor B (batch) has 4 random factor levels.

```
> y <- read.table("CH24PR17.DAT")
> data <- y[,1]
> coat <- y[,2]
> batch <- y[,3]
> pearls.df <- data.frame(data=data,coat=factor(coat),batch=factor(batch))
```

Look at the data set graphically.

```
> plot.factor(coat,data)
> plot.factor(batch,data)
> plot.data.frame(pearls.df)
```

The ANOVA table is obtained by

```
> myfit <- anova(lm(data~coat+batch+coat*batch, pearls.df))
             Df   Sum Sq   Mean Sq  F value    Pr(>F)
  coat        2   150.388  75.194   15.591     1.327e-05
  batch       3   152.852  50.951   10.564     3.984e-05
  coat:batch  6   1.852    0.309    0.064      0.9988
  Residuals   36  173.625  4.823
```

Be careful: the F-value for `coat` is not `15.591`. The corresponding F-value and p-values are

```
> F.a <- myfit[1,3]/myfit[3,3]
> p.a <- 1-pf(F.a, myfit[1,1], myfit[3,1])
  [1] 1.806033e-06
```

8.2.2 Estimation of variance components

- Expected mean squares

$$
\begin{aligned}
E(MSA) &= \sigma^2 + nb\frac{\sum_{i=1}^{a} \alpha_i^2}{a-1} + n\sigma_{\alpha\beta}^2 \\
E(MSB) &= \sigma^2 + na\sigma_{\beta}^2 \\
E(MSAB) &= \sigma^2 + n\sigma_{\alpha\beta}^2 \\
E(MSE) &= \sigma^2
\end{aligned}
$$

- Unbiased estimators:

$$
\begin{aligned}
\hat{\sigma}^2 &= MSE \\
\hat{\sigma}_{\alpha\beta}^2 &= \frac{MSAB - MSE}{n} \\
\hat{\sigma}_{\beta}^2 &= \frac{MSB - MSE}{na}
\end{aligned}
$$

Occasionally, one of last three estimators may be negative.

- It is not possible to construct an exact CI for variance components.