Math 511

$\boxed{Q's}$ $\begin{bmatrix} \boxed{\begin{matrix}4\\2\\2\\1\end{matrix}} & \begin{matrix}\downarrow\\2\\0\\0\\2\end{matrix} & \begin{matrix}1\\1\\-1\\1\end{matrix}\end{bmatrix}$ $Q = \begin{bmatrix} 4/5 & 1/5 \\ 2/5 & -2/5 \\ 2/5 & -2/5 \\ 1/5 & 4/5 \end{bmatrix}$ $R = \begin{bmatrix} 5 & 2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$

$q_1 = \frac{1}{5}[4\ 2\ 2\ 1]^T = [4/5\ \ 2/5\ \ 2/5\ \ 1/5]^T$

q_2 is x_2 remove parts $\boxed{\text{not } \perp}$ to q_1

P of x_2 onto q_1 (projection)

Remember project onto orthonormal basis..

$$v = \langle v, q_1\rangle q_1 + \langle v, q_2\rangle q_2 + \cdots + \langle v, q_n\rangle q_n$$

$$P = \underline{\langle x_2, q_1\rangle}\, q_1 = \boxed{2}\left(\overline{[4/5\ \ 2/5\ \ 2/5\ \ 1/5]^T}\right)$$

$$P = [8/5\ \ 4/5\ \ 4/5\ \ 2/5]^T \qquad x_2 = [2\ 0\ 0\ 2]^T$$

want the $\perp$ so $x_2 - P = [2/5\ \ -4/5\ \ -4/5\ \ 8/5]^T$

$$q_2 = \frac{1}{\boxed{2}}[2/5\ \ -4/5\ \ -4/5\ \ 8/5]^T$$

$$q_2 = [1/5\ \ -2/5\ \ -2/5\ \ 4/5]^T$$

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q_3 is x_3 but we remove the $\boxed{\text{non-}\perp}$ to q_1, q_2

P of x_2 onto q_1, q_2

$$\mathbb{P} = \langle \mathbb{X}_3, \mathbb{q}_1 \rangle \mathbb{q}_1 + \langle \mathbb{X}_3, \mathbb{q}_2 \rangle \mathbb{q}_2$$

where $$X = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 2/5 & -2/5 & 1/\sqrt{2} \\ 2/5 & -2/5 & -1/\sqrt{2} \\ 1/5 & 4/5 & 0 \end{bmatrix} \qquad R = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

so $\mathbb{P} = \textcircled{1} \cdot [4/5 \;\; 2/5 \;\; 2/5 \;\; 1/5]^T + \textcircled{1} \cdot [1/5 \;\; -2/5 \;\; -2/5 \;\; 4/5]$

$\mathbb{P} = [1 \;\; 0 \;\; 0 \;\; 1]^T$ (non $\perp$ to $\mathbb{q}_1, \mathbb{q}_2$)

→ to get $\perp$ part $\mathbb{X}_3 - \mathbb{P} = \underline{[0 \;\; 1 \;\; -1 \;\; 0]^T}$

$$\mathbb{q}_3 = \frac{1}{\boxed{\sqrt{2}}} [0 \;\; 1 \;\; -1 \;\; 0]^T =$$

check $\boxed{X = QR}$

$\boxed{6.3}$ $\quad A\mathbb{V} = \lambda \mathbb{V}$ eigen value / vector

→ find the eigen value / vector

$\boxed{\lambda_1, \mathbb{X}_1}$ $\quad \boxed{\lambda_2, \mathbb{X}_2}$ $\quad \cdots \quad \boxed{\lambda_K = \mathbb{X}_K}$

Mean:
$$A\mathbb{X}_1 = \lambda_1 \mathbb{X}_1$$
$$A\mathbb{X}_2 = \lambda_2 \mathbb{X}_2$$
$$\vdots$$
$$A\mathbb{X}_K = \lambda_K \mathbb{X}_K$$

left sides: $\quad A[\mathbb{X}_1 \;\; \mathbb{X}_2 \;\; \cdots \;\; \mathbb{X}_K]$

right sides: $A[\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_k]$

$$= [A\mathbf{x}_1\ A\mathbf{x}_2 \cdots A\mathbf{x}_k]$$

$$= [\lambda_1\mathbf{x}_1\ \lambda_2\mathbf{x}_2 \cdots \lambda_k\mathbf{x}_k]$$

$$= [\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_k]\begin{bmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & \ddots & \lambda_k\end{bmatrix}$$

so we can write this as..

$$A\underbrace{[\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_k]}_{X} = \underbrace{[\mathbf{x}_1\ \mathbf{x}_2 \cdots \mathbf{x}_k]}_{X}\overbrace{\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_k\end{bmatrix}}^{D}$$

so $\boxed{AX = XD}$

or $X^{-1}AX = D$ ← diagonalization of A

or $A = (X)(D)(X^{-1})$ ← a factorization of A

Def A is diagonalizable if there is a non-singular matrix X and diagonal matrix D such that $X^{-1}AX = D$

(we say X diagonalizes A)

$A \rightarrow \boxed{AX = XD}$ (solve eigen value/vector prob for A)

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ eigen values

$X = [\mathbf{x}_1 \ \mathbf{x}_2 \cdots \mathbf{x}_n]$

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when does X^{-1} exist so we can go to

$$A = XDX^{-1} \quad \text{or} \quad X^{-1}AX = D$$

(need is n-linearly ind. $\mathbf{x}_i$)

Thm $\lambda_1, \ldots, \lambda_k$ eigen values and they are distinct. $\rightarrow$ $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly ind.

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda_1 = 1 \rightarrow \mathbf{x}_1 = ?$

$\lambda_2 = -1 \rightarrow \mathbf{x}_2 = ?$

$\lambda_3 = 2 \rightarrow \mathbf{x}_3 = ?$

$$A = XDX^{-1} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]^{-1}$$

how do find $\mathbf{x}_i$?

$\lambda_1 = 1$ $\left[\begin{array}{ccc|c} 0 & \boxed{2} & 3 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$

$x_1 = \alpha$ $\quad x_3 = 0$

$\mathbf{x}_1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$

$$\left[\begin{array}{ccc|c} \boxed{2} & \textcircled{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 \end{array}\right] \qquad x_1 = -\alpha$$

$\uparrow$

$x_2 = \alpha \qquad x_3 = 0$

$$\mathbb{X}_2 = \begin{bmatrix} -\alpha \\ \alpha \\ 0 \end{bmatrix}$$

$$\mathbb{X}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$\lambda_3 = 2$

$$\left[\begin{array}{ccc|c} \boxed{-1} & 2 & 3 & 0 \\ 0 & \boxed{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x_1 = 3\alpha$$

$\uparrow$

$x_2 = 0 \qquad x_3 = \alpha$

$$\mathbb{X}_3 = \begin{bmatrix} 3\alpha \\ 0 \\ \alpha \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Note: b/c $\mathbb{X}_i$ are $\boxed{\text{not}}$ uniq

→ $\mathbb{X}$ is $\boxed{\text{not}}$ uniq

D is not uniq

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

What if λ_i are not distinct?

→ check that X_i are linearly ind.

ex

$$A = \begin{bmatrix}1&0&1\\0&2&0\\0&0&1\end{bmatrix}$$

$\lambda_1 = 1$

$$\left[\begin{array}{ccc|c}0&0&\textcircled{1}&0\\0&\textcircled{1}&0&0\\0&0&0&0\end{array}\right]$$

$\uparrow$

$x_1=\alpha$ $x_2=0$ $x_3=0$

$$X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$\lambda_2 = 2$

$$\left[\begin{array}{ccc|c}-1&0&1&0\\0&0&0&0\\0&0&-1&0\end{array}\right]$$

$\uparrow$

$x_1=0$ $x_2=\alpha$ $x_3=0$

$$X_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$A\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

X^{-1} does not exist.

$$B = \begin{bmatrix}1&1&-1\\0&2&-1\\0&0&1\end{bmatrix}$$

$\boxed{\lambda_1 = 1}$ $\lambda_2 = 2$

$$\left[\begin{array}{ccc|c}0&\boxed{1}&-1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

$\uparrow$

$x_1=\alpha$ | $x_3=\beta$

$x_2=\beta$

✓ ✓

$\lambda_2 = 2$

$X_2 =$

$$\begin{bmatrix}\alpha\\\beta\\\beta\end{bmatrix} = \alpha\begin{bmatrix}1\\0\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$\lambda_1 = 1$ $X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\lambda_3 = 1$ $X_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$.

for this X^{-1} exists

$$\boxed{\text{if } X^{-1} \text{ does not exist we call } A \text{ defective}}$$

Why? $AX = XD$

or $\boxed{A = XDX^{-1}}$ or $X^{-1}AX = D$

ex) Markov Chain

$$v_0,\ v_1 = \boxed{Av_0},\ v_2 = A\boxed{v_1},\ \dots,\ v_k = Av_{k-1}$$

$$v_2 = A^2 v_0 \qquad \boxed{v_k = A^k v_0}$$

$v_0, v_1, \dots v_n$: state vectors
Markov Chain

$v_k = A^k v_0$: Markov process

A : transition matrix

if $A = XDX^{-1}$

$$A^k = (XDX^{-1})^k$$

$$= (XD\underbrace{X^{-1})(X}_{I}D\underbrace{X^{-1})\cdots(X}_{I}D\underbrace{X^{-1})}_{I}$$

$$= XD^kX^{-1}$$

$$A^k = X\begin{pmatrix}\lambda_1^k & & 0\\ & \lambda_2^k & \\ 0 & & \lambda_n^k\end{pmatrix}X^{-1}$$

$\mathbb{V}_0 \quad X D X^{-1}$

$$A^k \mathbb{X}_0 = A \cdot A \cdot \ldots \cdot A \cdot \mathbb{V}_0 = \mathbb{V}_k$$

with $\quad X D X^{-1}$

$$\mathbb{V}_k = \underset{\substack{\uparrow \\ \text{eigen} \\ \text{vectors}}}{X} \underset{\substack{\uparrow \\ (\text{eigenvalues})^k}}{\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}} \underset{\substack{\uparrow \\ \text{eigen} \\ \text{vectors}}}{X^{-1}} \mathbb{V}_0$$

call $\quad X^{-1}\mathbb{V}_0 = \mathbb{W} = [w_1 \; w_2 \cdots w_n]$

$$[X \mid \mathbb{V}_0] \xrightarrow[\text{ops}]{\text{row}} [I \mid \mathbb{W}] \; \checkmark$$

$$\mathbb{V}_k = X \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

$$\mathbb{V}_k = [\mathbb{X}_1 \; \mathbb{X}_2 \cdots \mathbb{X}_n] \begin{bmatrix} \lambda_1^k w_1 \\ \lambda_2^k w_2 \\ \vdots \\ \lambda_n^k w_n \end{bmatrix}$$

$$\mathbb{V}_k = \lambda_1^{k} w_1 \mathbb{X}_1 + \underline{\lambda_2^{k} w_2} \mathbb{X}_2 + \cdots + \lambda_n^{k} w_n \mathbb{X}_n$$

$\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2, \ldots$