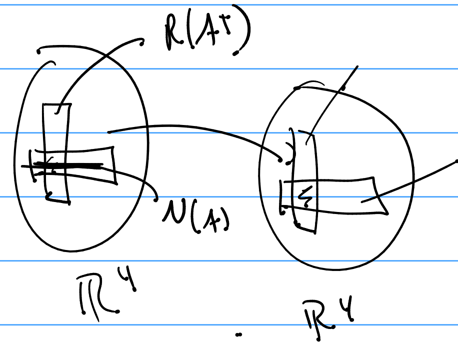
Math 511

Col. space of A^T
row space of A as col.

~~Q's~~ 5.2 #1d

$N(A)$, $R(A^T)$
$N(A^T)$, $R(A)$



$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

$N(A)$ $\quad Ax = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to find $N(A)$ is to solve $Ax = 0$
is to solve $Ux = 0$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = \alpha$
$x_3 = -\alpha$
$x_2 = 0$
$x_1 = 0$

all $x = \alpha \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

$$N(A) = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$R(A^T) = \text{span}\left(\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix}, \begin{matrix} 0 \\ 0 \\ 1 \\ 1 \end{matrix} \right)$$

5.2 #3

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \end{bmatrix}$$

$N(A) \to$ all $\mathbf{v}$ such that $A\mathbf{v} = 0$

$$N(A) = \{ \mathbf{v} \mid A\mathbf{v} = 0 \}$$

so $$A\mathbf{v} = \begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \end{bmatrix} \mathbf{v} = \begin{bmatrix} \mathbf{x}^T\mathbf{v} \\ \mathbf{y}^T\mathbf{v} \end{bmatrix}$$

$$N(A) = \left\{ \mathbf{v} \;\middle|\; \begin{bmatrix} \mathbf{x}^T\mathbf{v} \\ \mathbf{y}^T\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$N(A) = \{ \mathbf{v} \mid \underbrace{\mathbf{x}^T\mathbf{v} = 0}_{\mathbf{x} \perp \mathbf{v}} \text{ and } \underbrace{\mathbf{y}^T\mathbf{v} = 0}_{\mathbf{y} \perp \mathbf{v}} \}$$

b/c $S = \text{span}(\mathbf{x}, \mathbf{y})$

and $N(A)$ consisted of all $\mathbf{v}$ so $\mathbf{v} \perp \mathbf{x}$, $\mathbf{v} \perp \mathbf{y}$

$\to$ Def of $\boxed{S^\perp = N(A)}$

5.3 #5 points $(-1,0)$, $(0,1)$, $(1,3)$, $(2,9)$

(x,y)

approx by $y = ax + b$ (find $a = ?$, $b = ?$)

pt #1 $\to$ $0 = a(-1) + b$

pt #2 $\to$ $1 = a(0) + b$

pt#3 $\rightarrow$ $3 = a(1) + b$

pt#4 $\rightarrow$ $4 = a(2) + b$

$$\rightarrow \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{c} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}}_{y}$$

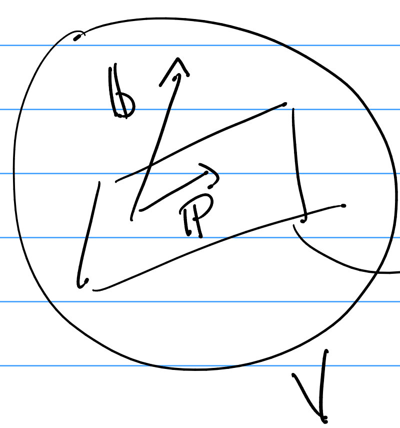
overdet. $\rightarrow$ so No soln

Find least squares soln. to $Ac = y$

$\Rightarrow$ to solve $\boxed{A^T A c = A^T y}$

V is an inner prod. space

5.5



S, a subspace of V

$\rightarrow$ S has an orthonormal basis $u_1, u_2, \ldots, u_n$

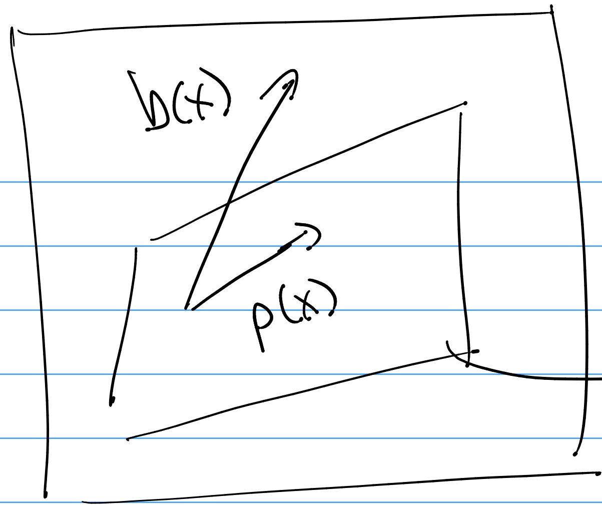
$$\boxed{P = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n}$$

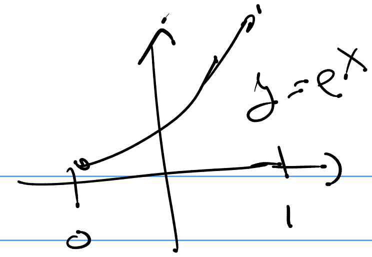
is the least squares projection of b onto S

$$\boxed{c_i = \langle b, u_i \rangle}$$

P is called the least squares approx. of b onto S, a subspace of V.

e^x





$S = \text{span}\left(f_1 = 1,\ f_2 = \sqrt{12}\left(x - \frac{1}{2}\right)\right)$

$C[0,1]$

$\langle f, g \rangle = \int_0^1 fg\, dx$

$$b(x) \approx p(x) = c_1 f_1 + c_2 f_2$$

$$c_1 = \langle b, f_1 \rangle = \int_0^1 b(1)\, dx$$

$$c_2 = \langle b, f_2 \rangle = \int_0^1 b\left(\sqrt{12}\left(x - \tfrac{1}{2}\right)\right) dx$$

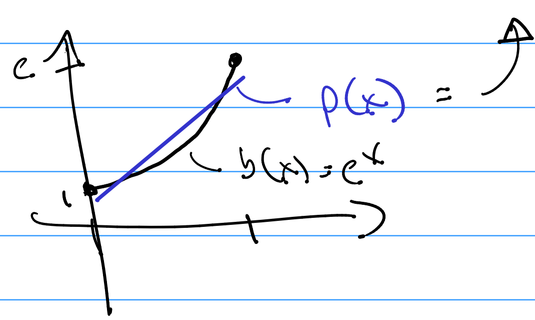
Ex: $b(x) = e^x$

$$e^x \approx p(x) = c_1(1) + c_2\left(\sqrt{12}\left(x - 1/2\right)\right)$$

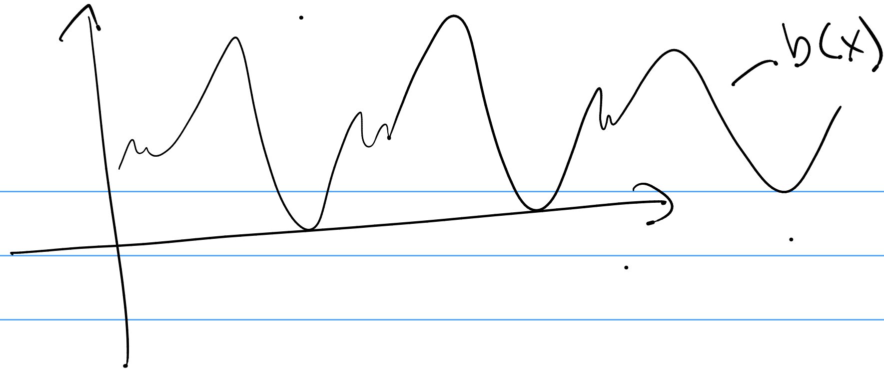
$$c_1 = \int_0^1 e^x\, dx = (e - 1)$$

$$c_2 = \int_0^1 e^x\left(\sqrt{12}\left(x - 1/2\right)\right) dx = \sqrt{3}(3 - e)$$

$$e^x \approx (e-1)\cdot 1 + \left(\sqrt{3}(3-e)\right)\sqrt{12}\left(x - 1/2\right)$$



ex



$b(x) \approx p(x)$ onto a subspace with orthonormal basis

$\to C[-\pi,\pi]$ $\quad \langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

Subspace = span$\left(\frac{1}{\sqrt{2}}, \underbrace{\cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos kx, \sin kx}\right)$

$\cos nx, \sin nx \quad n = 1,2,\ldots,k$

$f(x) \approx p(x) = c_0\left(\frac{1}{\sqrt{2}}\right) + \sum_{a=1}^{n} c_{1,a}\sin ax + \sum_{b=1}^{n} c_{2,b}\cos bx$

Fourier Transform

$p(x) = \langle f,1\rangle \frac{1}{2} + \langle f,\cos x\rangle \cos x + \langle f,\sin x\rangle \sin x$

$+ \cdots + \langle f,\cos kx\rangle \cos kx + \langle f,\sin kx\rangle \sin kx$

$\langle f,1\rangle = a_0$

$\langle f,\cos kx\rangle = a_k \qquad \langle f,\sin kx\rangle = b_k$

$$p(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)$$

is complex numbers $\quad c_k = \frac{1}{2}(a_k - i b_k)$

$$c_{-k} = \frac{1}{2}(a_k + i b_k)$$

$$p(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

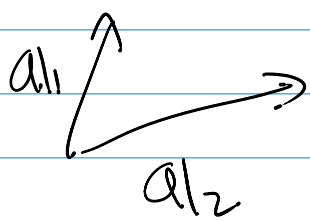
What if you get a set $a_1, a_2, \dots, a_n$ that is $\boxed{\text{not}}$ orthonormal?

(1) Not orthogonal?

(2) Not unit length?

$\boxed{5.6}$ Gram-Schmidt Orthogonalization Process

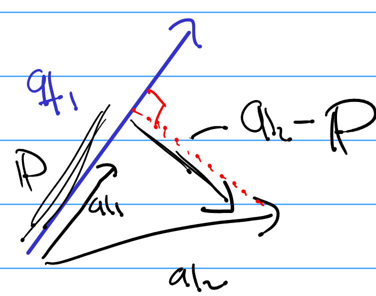
$\boxed{\text{Idea}}$



$$a_1, a_2 \to \underbrace{q_1, q_2}_{\text{orthonormal}}$$

Base Step: $a_1 \to q_1$

$$q_1 = \frac{1}{\|a_1\|} a_1$$



Induction:

q_k = a_k part that is $\perp$ to known $q_1, q_2, \dots, q_{k-1}$

$q_k = a_k$ - projection of a_k onto span$(q_1, q_2, \dots, q_{k-1})$

$$q_k = a_k - [c_1 q_1 + c_2 q_2 + \dots + c_{k-1} q_{k-1}]$$

$$c_i = \langle a_k, q_i \rangle_A$$

Now

$$q_k = \frac{1}{\| q_k \|} q_k$$

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track the $\langle \, , \rangle$ values and dividing by length

values as we do the process..

$$A = [a_1 \; a_2 \; a_3 \dots a_n] = [q_1 \; q_2 \; q_3 \dots q_n] \begin{bmatrix} r_{11} & r_{12} & r_{13} & \dots & r_{1n} \\ 0 & r_{22} & r_{23} & \dots & r_{2n} \\ 0 & 0 & r_{33} & \dots & r_{3n} \\ & 0 & & \ddots & \\ & & & & r_{nn} \end{bmatrix}$$

$$a_1 = r_{11} q_1$$

$$a_2 = r_{12} q_1 + r_{22} q_2$$

$$a_2 - \boxed{r_{12} q_1} = r_{22} q_2$$

proj. of a_2 onto q_1

$$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$$

$$a_3 - (r_{13} q_1 + r_{23} q_2) = r_{33} q_3$$

proj of a_3 on span (q_1, q_2)

$$\langle q_1, q_3 \rangle$$

If you do this $A = QR$

Q: orthogonal matrix

R: upper triangular

Why do this?

$A\mathbb{x} = \mathbb{b}$ as least squares soln.

$A^T A \mathbb{x} = A^T \mathbb{b}$

Normally $\mathbb{x} = (A^T A)^{-1} (A^T \mathbb{b})$

Trick:

① $A = \boxed{QR}$

② $A^T A \mathbb{x} = A^T \mathbb{b}$

$$(QR)^T (QR)\, \mathbb{x} = (QR)^T \mathbb{b}$$

$$R^T \underbrace{Q^T Q}_{= I} R \mathbb{x} = R^T Q^T \mathbb{b}$$

$$R^T R \mathbb{x} = R^T Q^T \mathbb{b}$$

$R\mathbb{x} = Q^T \mathbb{b}$ ← solve by backsolve