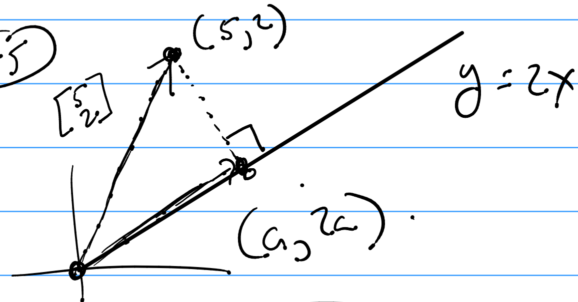
Math 511

Q's/ 5.1 #5



$$d^2 = \boxed{(5-a)^2 + (2-2a)^2}$$ ← minimize this for a. (deriv. = 0)

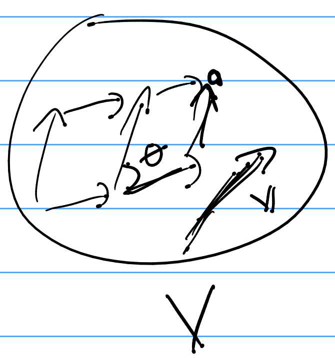
$$D_a\left[(5-a)^2 + (2-2a)^2\right] = -2(5-a) - 4(2-2a)$$

so $-18 + 10a = 0$

$a = 9/5$

or $(9/5,\ 18/5)$

Inner Product Space



$v \in V$ - basis vectors $\{v_1, v_2, \ldots, v_n\}$

- $\dim(V) = n$

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$v = [c]_{\text{basis}}$$ (coordinates of a basis)

Note: on coordinates .. we had change of basis

$B = [b_1, b_2 \, .. \, b_n]$ $\quad V = [c]_B$

$B[c]_B = [c]_E$ (E standard)

Def V_i are non-zero in an inner prod space.

if $\boxed{\langle V_i, V_j \rangle = 0}$ for $i \neq j$

$\rightarrow \{V_1, V_2, .., V_k\}$ are an orthogonal set.

Th^n if $\{V_i\}$ is an orthogonal set

$\rightarrow V_1, V_2, \, . \, , V_k$ are linearly indep.

(so we can use orthogonal sets to start to build up a basis)

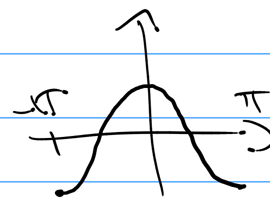
Def an orthogonal set of vectors that are each of length 1 $\rightarrow$ $\boxed{\text{orthonormal set}}$

ex given $V_1, V_2, .., V_k$ are orthogonal

$\rightarrow \frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, .., \frac{V_k}{\|V_k\|}$ is $\boxed{\text{orthonormal}}$

$\boxed{\text{ex}}$ $C[-\pi, \pi]$ $\qquad \langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

given $\{1, \cos x, \cos 2x\}$



① orthogonal? $\boxed{\text{yes}}$

$$\langle 1, \cos x \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x\,dx = 0$$
$$\langle 1, \cos 2x \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos 2x\,dx = 0$$
$$\langle \cos x, \cos 2x \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cos 2x\,dx = 0$$

② orthonormal? $\boxed{\text{NO}}$ $\quad \|v\| = \langle v, v \rangle^{1/2}$

$$\|1\| = \langle 1, 1 \rangle^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} dx\right)^{1/2} = \sqrt{2}$$
$$\|\cos x\| = \langle \cos x, \cos x \rangle^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,dx\right)^{1/2} = 1$$
$$\|\cos 2x\| = \langle \cos 2x, \cos 2x \rangle^{1/2} = \left(\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 2x\,dx\right)^{1/2} = 1$$

③ Make an orthonormal set?

$$\left\{\frac{1}{\sqrt{2}}, \cos x, \cos 2x\right\} \leftarrow \text{orthonormal set}$$

Now using coord. $v \in V$

$$v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

coord in Basis $B = [b_1\ b_2 \cdots b_n] = [c]_B$

Th^m $\mathcal{U} = [u_1, u_2, \dots, u_n]$ is an orthonormal basis of inner prod. space V

if $v = c_1 u_1 + c_2 u_2 + \dots + c_n u_n$

$\rightarrow$ $c_1 = \langle v, u_1 \rangle$, $c_2 = \langle v, u_2 \rangle$, ...

$$v = [c]_{\mathcal{U}} = \begin{bmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \vdots \\ \langle v, u_3 \rangle \end{bmatrix}_{\mathcal{U}}$$

ex $\{ 1/\sqrt{2}, \cos x, \cos 2x \} = \mathcal{U}$

$V = \text{span}(\{ 1/\sqrt{2}, \cos x, \cos 2x \})$

given a $f \in V$

$\rightarrow f = \boxed{c_1}(1/\sqrt{2}) + \boxed{c_2}(\cos 2x) + \boxed{c_3}(\cos 2x)$

$c_1 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (f)(1/\sqrt{2})\, dx$

$c_2 = \langle f, \cos x \rangle$

$c_3 = \langle f, \cos 2x \rangle$

so finding $[\mathbb{c}]_u$ an orthonormal basis

is just $c_i = \langle \mathbb{V}, \mathbb{u}_i \rangle$ ★

Corollary $[\mathbb{a}]_u$, $[\mathbb{b}]_u$

$\rightarrow \langle \mathbb{a}, \mathbb{b} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

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ex $f = 3(1/\sqrt{2}) + 4\cos x - 2\cos 2x = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}_u$

$g = (-1)(1/\sqrt{2}) + 7\cos x + 3\cos 2x = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}_u$

$\langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \left(3/\sqrt{2} + 4\cos x - 2\cos 2x\right)\left(\frac{-1}{\sqrt{2}} + 7\cos x + 3\cos 2x\right)dx$

$= (3)(-1) + (4)(7) + (-2)(3) = \boxed{19}$

Corollary $\mathbb{V} = [\mathbb{c}]_u$

$\rightarrow \|\mathbb{V}\|^2 = \langle \mathbb{V}, \mathbb{V} \rangle = c_1^2 + c_2^2 + \cdots + c_n^2$

(remember $c_i = \langle \mathbb{V}, \mathbb{u}_i \rangle$)

ex $g = -1(1/\sqrt{2}) + 4(\cos x) + -2(\cos 2x)$

$\|g\|^2 = (-1)^2 + (4)^2 + (-2)^2 = 21$

in $\mathbb{R}^n$ given orthonormal basis / orthonormal set

we can collect into $U = [u_1, u_2 \cdots u_n]$

$\boxed{\text{Def}}$ $Q_{n \times n}$ is an $\boxed{\text{orthogonal matrix}}$

if q_i vectors in Q are orthonormal

$$q_i^T q_i = 1 \quad \text{(normal: length} = 1)$$

$$q_i^T q_j = 0 \quad i \neq j \quad \text{(orthogonal part)}$$

$\boxed{\text{Thm}}$

$$Q^T Q = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & 0 & & \ddots & 1 \end{bmatrix} = I$$

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b/c $Q^T Q = I \rightarrow Q^T = Q^{-1}$

Properties of $Q_{n \times n}$ an orthogonal matrix

(1) q_i are an orthonormal basis of $\mathbb{R}^n$

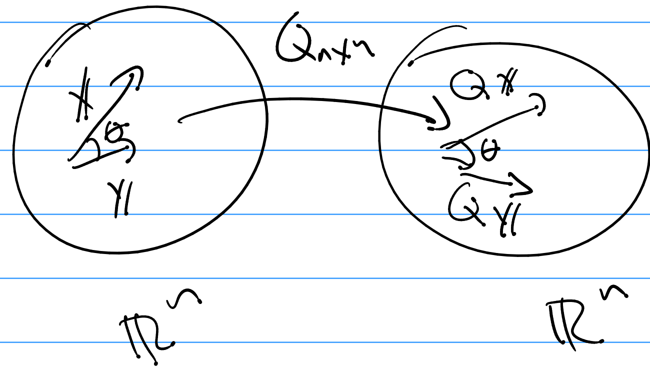
(2) $Q^T Q = I$

(3) $Q^T = Q^{-1}$

(4) $\langle Q\mathbf{x}, Q\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$

(5) $\|Q\mathbf{x}\|_2 = \|\mathbf{x}\|_2$

angles and lengths are preserved by Q.



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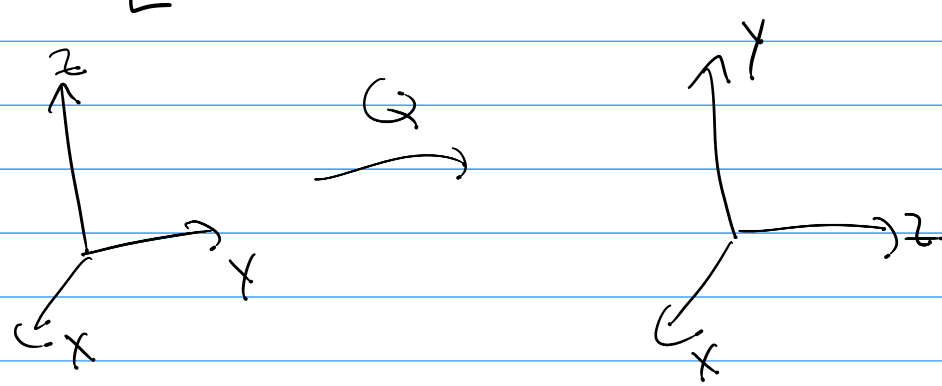
ex) of an orthogonal matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

permutation of I.



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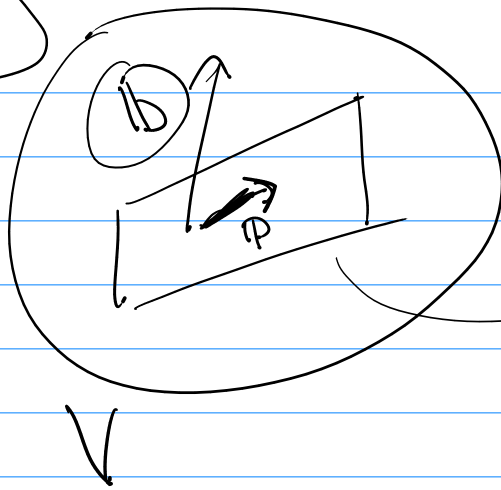
ex) $A\mathbf{x} = \mathbf{b}$ is overdet and has no soln → we look for $\hat{\mathbf{x}}$ the least square soln,

$$A\mathbf{x} = \mathbf{b} \rightarrow A^T A \mathbf{x} = A^T \mathbf{b}$$

if A was orthogonal $(A^T A = I)$ → $\boxed{\hat{\mathbf{x}} = A^T \mathbf{b}}$

Note in 5.3



$$\mathbb{p} = A\hat{x}$$

$$\mathbb{p} = A(A^TA)^{-1}A^T\mathbb{b}$$

→ a subspace

5.5 Th^m $\mathbb{b} \in V$ and give $u_1, u_2, \dots, u_n$ an orthonormal basis for S.

$$\mathbb{p} = c_1 \underline{u_1} + c_2 \underline{u_2} + \dots + c_n \underline{u_n}$$

with $c_i = \langle \mathbb{b}, u_i \rangle$

$\rightarrow \mathbb{p} - \mathbb{b} \in S^\perp$

(2) $\mathbb{p}$ is closest vector in S to $\mathbb{b}$.

$$\rightarrow \boxed{\mathbb{p} = UU^T\mathbb{b}}$$